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REMARKS ON THE CLASS OF CONTINUOUS MARTINGALES WITH BOUNDED QUADRATIC VARIATION

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Introduction. Let M be a continuous local martingale with associated increasing process $\langle M\rangle$. Here we shall consider the class H^∞ of all M such that $\langle M\rangle_\infty \in L^\infty$. Clearly $H^\infty \subset BMO$. However, at the present stage, very little is known about H^∞ in the space BMO. Our aim is to show that H^∞ possesses an interesting feature which is connected with various weighted norm inequalities for martingales. Further we remark in passing that $BMO \setminus \bar{H}^\infty \neq \emptyset$ whenever there is an unbounded BMO-martingale.

1. Preliminaries. We shall briefly recall the basic matters which are needed later. Throughout this note we shall work with a fixed probability system $(\Omega, F, P; (F_t))$ which satisfies the usual conditions. Let now M be a local martingale. For any real number a we then denote by $Z^{(a)}$ the process given by the formula $Z_t^{(a)} = \exp(aM_t - a^2\langle M\rangle_t/2)$ $(0 \leqq t \leqq \infty)$. As is well-known, it is a positive local martingale. This implies that $E[Z_T^{(a)}] \leqq 1$ for every stopping time T. For simplicity, set $Z = Z^{(1)}$. Next let $1 < p < \infty$. We say that Z satisfies (A_p) if $\sup_T \| E[(Z_T/Z_\infty)^{1/(p-1)} | F_T] \|_\infty < \infty$ where the supremum is taken over all stopping times T. This is the probablistic version of the Muckenhaupt condition which has often appeared in the literature in connection with weighted norm inequalities for many operators, such as the Hardy-Littlewood maximal operator, the singular integral operators, and many others. Similarly, in the probablistic setting, the condition (A_p) plays an important role in various weighted norm inequalities for martingales.

Now let M be a uniformly integrable martingale, and we set

$$\|M\|_{BMO_p} = \sup_T \| E[|M_\infty - M_T|^p | F_T]^{1/p} \|_\infty \qquad (1 \leqq p < \infty)\,.$$

These norms are mutually equivalent. We say that M belongs to the class BMO if $\|M\|_{BMO_p} < \infty$. Let $d_p(\ ,\ )$ denote the distance on BMO deduced from the norm $\|\ \ \|_{BMO_p}$ by the usual procedure. It is not difficult to see that, if $M \in BMO$, then $Z^{(a)}$ is a uniformly integrable martingale. On the other hand, if $\|M\|_{BMO_1} < 1/4$, then we have

$$(1) \qquad E[\exp\{|M_\infty - M_T|\}|F_T] \leqq \frac{1}{1 - 4\|M\|_{BMO_1}} .$$

Furthermore, if $\|M\|_{BMO_2} < 1$, then

$$(2) \qquad E[\exp\{\langle M\rangle_\infty - \langle M\rangle_T\}|F_T] \leqq \frac{1}{1 - \|M\|^2_{BMO_2}} .$$

These inequalities are the main total to deal with various questions about *BMO*-martingales. They have been obtained in [3] by A. M. Garsia for discrete parameter martingales.

2. Dependence of (A_p) on the distance to H^∞. For a martingale M we define $p(M) = \inf\{p > 1: Z, Z^{(-1)}$ satisfy $(A_p)\}$. Hölder's inequality shows that Z and $Z^{(-1)}$ satisfy (A_p) for $p > p(M)$. Note that $p(M)$ may equal ∞. However, as is shown in [4], Z satisfies (A_p) for some $p > 1$ if and only if $M \in BMO$. This implies that $BMO = \{M: p(M) < \infty\}$. It should be noted that $p(M) \geqq 1$ for $M \in BMO$.

Our first aim is to show that $p(M) \leqq \{d_2(M, H^\infty) + 1\}^2$ for $M \in BMO$. We restate it as follows.

THEOREM 1. *Let $1 < p < \infty$. If $d_2(M, H^\infty) < \sqrt{p} - 1$, then Z and $Z^{(-1)}$ satisfy (A_p).*

PROOF. Let $b(M)$ denote the supremum of the set of b for which $\sup_T \|E[\exp\{b^2(\langle M\rangle_\infty - \langle M\rangle_T\})|F_T]\|_\infty < \infty$. First we claim

$$(3) \qquad \frac{1}{\sqrt{2}\, d_2(M, H^\infty)} \leqq b(M) .$$

To show this, let $0<b<1/\{\sqrt{2}\, d_2(M, H^\infty)\}$. Then $b<1/(\sqrt{2}\|M-N\|_{BMO_2})$ for some $N \in H^\infty$. Since $\langle M\rangle_t - \langle M\rangle_s \leqq 2\{(\langle M-N\rangle_t - \langle M-N\rangle_s) + (\langle N\rangle_t - \langle N\rangle_s)\}$ for $s \leqq t$ and $\langle N\rangle_\infty \leqq C$ for some constant C, we find applying (2)

$$\begin{aligned} E[\exp\{b^2(\langle M\rangle_\infty - \langle M\rangle_T)\}|F_T] &\leqq e^{2b^2C} E[\exp\{2b^2(\langle M - N\rangle_\infty - \langle M - N\rangle_T)\}|F_T] \\ &\leqq \frac{e^{2b^2C}}{1 - 2b^2\|M - N\|^2_{BMO_2}} . \end{aligned}$$

This means that $b \leqq b(M)$, so that (3) holds. We take this opportunity to remark that it is not difficult to extend (3) to right continuous martingales.

Now let $r = \sqrt{p} + 1$. Then the exponent conjugate to r is $s = (\sqrt{p} + 1)/\sqrt{p}$. Thus, applying Hölder's inequality we find

$$E\left[\left(\frac{Z_T}{Z_\infty}\right)^{1/(p-1)}\middle|F_T\right] = E\left[\exp\left\{-\frac{1}{p-1}(M_\infty - M_T) - \frac{r}{2(p-1)^2}(\langle M\rangle_\infty - \langle M\rangle_T)\right\}\right.$$

$$\times \exp\left\{\frac{1}{2s(\sqrt{p}-1)^2}(\langle M\rangle_\infty - \langle M\rangle_T)\right\}\Big| F_T\Big]$$

$$\leqq E\left[\frac{Z_\infty^{(\alpha)}}{Z_T^{(\alpha)}}\Big| F_T\right]^{1/r} E\left[\exp\left\{\frac{1}{2(\sqrt{p}-1)^2}(\langle M\rangle_\infty - \langle M\rangle_T)\right\}\Big| F_T\right]^{1/s},$$

where $\alpha = -1/(\sqrt{p}-1)$. The first conditional expectation on the right hand side is equal to 1, because $Z^{(\alpha)}$ is a uniformly integrable martingale. On the other hand, if $d_2(M, H^\infty) < \sqrt{p}-1$, then $b(M) > 1/\{\sqrt{2}(\sqrt{p}-1)\}$ by (3), so that the second conditional expectation is bounded by some constant C_p. The same conclusion holds for $Z^{(-1)}$. Thus the proof is complete.

The converse statement in the theorem is not true. We give an example below.

EXAMPLE 1. Let G^o be the class of all topological Borel sets in $R_+ = [0, \infty[$ and S be the identity mapping of R_+ onto R_+. We define a probability measure $d\mu$ on R_+ such that $\mu(S > t) = e^{-t}$. Let G be the completion of G^o with respect to $d\mu$, and similarly G_t the completion of the Borel field generated by $S \wedge t$, where $x \wedge y = \min\{x, y\}$. Clearly S is a stopping time over (G_t). We now construct in the usual way a probability system $(\Omega, F, P; (F_t))$ by taking the product of the system $(R_+, G, \mu; (G_t))$ with another system $(\Omega', F', P'; (F'_t))$ which carries a one dimensional Brownian motion $B = (B_t)$ starting at 0. Then S is also a stopping time over (F_t), so that the process M given by $M_t = B_{t\wedge S}$ is a continuous martingale. As $\langle M\rangle_t = t\wedge S$, we find that $\|M\|_{BMO_2} = 1$. Next let $2 < p < (1 + 1/\sqrt{p})^2$ Then $1 < 1/\{2(\sqrt{p}-1)^2\}$, and so

$$E\left[\exp\left\{\frac{1}{2(\sqrt{p}-1)^2}(\langle M\rangle_\infty - \langle M\rangle_t)\right\}\Big| F_t\right] = \infty \quad \text{on} \quad \{t < S\}$$

This means that $b(M) \leqq 1/\{\sqrt{2}(\sqrt{p}-1)\}$, so that $\sqrt{p}-1 \leqq d_2(M, H^\infty)$ by (3). On the other hand, from the definition of the conditional expectation it follows that

$$E\left[\left(\frac{Z_t}{Z_\infty}\right)^{1/(p-1)}\Big| F_t\right] = I_{\{S\leqq t\}} + \int_0^\infty \exp\left\{\left(\frac{1}{2(p-1)^2} - 1\right)x\right\}dx I_{\{S>t\}}\,.$$

This is finite or not according as $p > 2$ or $1 < p \leqq 2$. The same may be said of $Z^{(-1)}$. Namely $p(M) = 2$. Thus the converse is not true.

3. Further remarks on H^∞ and L^∞. In this section let L^∞ denote the class of all bounded martingales. Of course $L^\infty \subset BMO$, but they are not identical. Moreover there is no relation of inclusion between L^∞ and

H^∞. Now, for $M \in BMO$ let $a(M)$ denote the supremum of the set of a for which

$$\sup_T \| E[\exp\{a | M_\infty - M_T |\} | F_T] \|_\infty < \infty .$$

By using the Schwarz inequality we find

$$E[\exp\{a | M_\infty - M_T |\} | F_T] \leqq 2E[\exp\{2a^2(\langle M \rangle_\infty - \langle M \rangle_T)\} | F_T]^{1/2} .$$

Thus we have $b(M) \leqq \sqrt{2}\, a(M)$ for $M \in BMO$. In 1981 Emery proved ([2]):

$$\text{(4)} \qquad \frac{1}{4d_1(M, L^\infty)} \leqq a(M) \leqq \frac{4}{d_1(M, L^\infty)} .$$

However, Varopoulos had already obtained these inequalities for Brownian martingales (see [7]). On the other hand, Dellacherie, Meyer and Yor proved in [1] that $BMO \setminus \overline{L^\infty} \neq \emptyset$ whenever $BMO \neq L^\infty$, and, at the same time, they conjectured that H^∞ must be dense in BMO. Three years later, contrary to their expectations, Pavlov gave a counterexample in a certain discrete parameter case ([5]). In Section 2 we have just given another counterexample. Furthermore, noticing $b(M) \leqq \sqrt{2}\, a(M)$ and combining (4) with (3) we derive $H^\infty \subset \overline{L^\infty}$. Thus we have the following:

THEOREM 2. *If $BMO \neq L^\infty$, then $BMO \setminus \overline{H^\infty} \neq \emptyset$.*

In this connection, it is necessary to know whether or not $\overline{H^\infty} = \overline{L^\infty}$. We demonstrate below that there is a bounded martingale which does not belong to $\overline{H^\infty}$.

EXAMPLE 2. Let $\tau = \min\{t: |B_t| = 1\}$, and let M denote the process B stopped at τ. Then M is a bounded martingale. However, since $\lim_{t\to\infty} \exp(\pi^2 t/8)P(\tau > t) = \pi/4$ (see Proposition 8.4 in [6]), we easily find that $E[\exp(b^2\langle M \rangle_\infty)] = \infty$ for $b > \pi/(2\sqrt{2})$. This means that $b(M) \leqq \pi/2\sqrt{2}$. Consequently $d_2(M, H^\infty) \geqq 2/\pi$ by (3). We remark in passing that in this case $p(M) \geqq 1 + 4/\pi^2$.

Finally we remark that the distance in BMO to L^∞ affects the truth of the condition (A_p) in the following sense.

THEOREM 3. *Let $M \in BMO$. If $d_1(M, L^\infty) \geqq 8(\sqrt{p} - 1)$, then $p(M) \geqq p$.*

PROOF. It suffices to prove the contraposition. For this purpose, let $p > p(M)$. Set $u = 2\sqrt{r}/(\sqrt{r} + 1)$ for r with $p(M) < r < p$. Then the exponent conjugate to u is $v = 2\sqrt{r}/(\sqrt{r} - 1)$. By using the Hölder inequality we find

$$
\begin{aligned}
&E\left[\exp\left\{-\frac{1}{2(\sqrt{r}-1)}(M_\infty - M_T)\right\}\Big| F_T\right] \\
&\quad = E\left[\exp\left\{-\frac{1}{u(r-1)}(M_\infty - M_T) + \frac{1}{2u(r-1)}(\langle M\rangle_\infty - \langle M\rangle_T)\right\}\right. \\
&\quad\quad \times \exp\left\{\left(-\frac{1}{2(\sqrt{r}-1)} + \frac{1}{u(r-1)}\right)(M_\infty - M_T)\right. \\
&\quad\quad\quad \left.\left. - \frac{1}{2u(r-1)}(\langle M\rangle_\infty - \langle M\rangle_T)\right\}\Big| F_T\right] \\
&\quad \leqq E\left[\left(\frac{Z_T}{Z_\infty}\right)^{1/(r-1)}\Big| F_T\right]^{1/u} E\left[\frac{Z_\infty^{(\alpha)}}{Z_T^{(\alpha)}}\Big| F_T\right]^{1/v},
\end{aligned}
$$

where $\alpha = -1/(\sqrt{r}-1)$. Since Z (and also $Z^{(-1)}$) satisfies (A_r) by the definition of $p(M)$, the first conditional expectation on the right hand side is bounded by some constant C_r. Furthermore the second conditional expectation is equal to 1, because $Z^{(\alpha)}$ is a uniformly integrable martingale, and we may note that the same estimation holds with M replaced by $-M$. Then from the definition of $a(M)$ it follows at once that $a(M) \geqq 1/\{2(\sqrt{r}-1)\}$. Therefore, using the right-hand side of (4), we obtain

$$d_1(M, L^\infty) \leqq 8(\sqrt{r}-1) < 8(\sqrt{p}-1)\,.$$

This completes the proof.

Example 2 shows that the converse statement in this theorem is not true.

References

[1] C. Dellacherie, P. A. Meyer and M. Yor, Sur certaines propriétés des espaces de Banach H^1 et *BMO*, Sém. de Prob. XII, Lecture Notes in Math. 649, Springer-Verlag, 1978, 98–113.

[2] M. Emery, Le théorème de Garnett-Jones d'après Varopoulos, Sém. de Prob. XV, Lecture Notes in Math. 850, Springer-Verlag 1981, 278–284.

[3] A. M. Garsia, Martingale inequalities, Seminar Notes on Recent Progress, New York, Benjamin 1973.

[4] N. Kazamaki, A characterization of *BMO*-martingales, Sém. de Prob. X, Lecture Notes in Math. 511, Springer-Verlag 1976, 536–538.

[5] I. V. Pavlov, A counterexample to the conjecture that H^∞ is dense in *BMO*, Th. of Prob. and its Appl., vol. 25, no. 1 (1980), 152–155.

[6] S. C. Port and C. J. Stone, Brownian Motion and Classical Potential Theory, Academic Press 1978.

[7] N. Th. Varopoulos, A probablistic proof of the Garnett-Jones theorem on *BMO*, Pacific J. Math., 90 (1980), 201–221.

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