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GENERALIZED INVERSES OF TOEPLITZ OPERATORS AND INVERSE APPROXIMATION IN H^2

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1. Introduction. Let H^2 (resp. H^∞) be the Hardy space of analytic functions in the open unit disc D with square-integrable (resp. essentially bounded measurable) boundary functions, and let π_k ($k \in N := \{0, 1, \cdots\}$) be the linear subspace of all polynomials with degree at most k. Following Chui [1], we then define, for $f \in H^\infty$, the least-squares inverse in π_k of f as the (unique) polynomial $g = g_k$ such that the L^2-norm on the unit circle C

$$\|1 - fg\|_2 := \left\{(2\pi)^{-1}\int_{-\pi}^{\pi} |1 - f(e^{it})g(e^{it})|^2 dt\right\}^{1/2}$$

is minimal when g runs over π_k. Furthermore, the double least-squares inverse $h_{n,k}$ in π_n of f through π_k is defined as the least-squares inverse in π_n of g_k. Using orthogonal polynomials, Chui [1] proved that each g_k is zero-free in the closed unit disc $\bar{D}$, and that if $f \in \pi_n$ then each $h_{n,k}$ is a very good approximant of f in the same π_n which has no zeros in $\bar{D}$.

Now, let A be a (bounded linear) operator on H^2, $\phi \in H^2$ and consider the equation

$$Ag = \phi\,, \quad g \in H^2\,. \tag{1.1}$$

Then an element $g \in H^2$ which minimizes the norm $\|Ag - \phi\|_2$ is called a least-squares solution of (1.1). It is well-known (cf. [3], [7]) that if A has closed range the least-squares solution with minimum norm is unique and is represented as $A^\dagger\phi$, where $A^\dagger$ stands for the (Moore-Penrose) generalized inverse of A. (The generalized inverse is uniquely determined by the four Penrose identities, $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$ and $(A^\dagger A)^* = A^\dagger A$.)

Suppose that T_f is the Toeplitz operator with symbol $f \in H^\infty$, and that E_k is the orthogonal projection from H^2 onto π_k (as a subspace of H^2). Then the product $T_f E_k$ is of finite rank, and hence has closed range. The solution $(T_f E_k)^\dagger 1 = E_k(T_f E_k)^\dagger 1$ of (1.1) for $A = T_f E_k$, $\phi = 1$ is nothing but the least-squares inverse g_k defined before. Similarly the double least-squares inverse of f is represented as $h_{n,k} = (T_{g_k} E_n)^\dagger 1$. Hence the approximation problem of least-squares and double least-squares inverses is identical to the convergence problem of generalized inverses.

In this paper we study convergence of least-squares and double least-squares inverses, using generalized inverses of Toeplitz operators restricted to finite dimensional subspaces. We extend (or refine) the recent results in [1], and we also settle a conjecture in [1].

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2. Least-squares inverses. Every (non-zero) $f \in H^\infty$ has the inner-outer decomposition $f = uF$, where u is an inner function in H^∞ and F is an outer function in H^∞. Let g_k and G_k be the least-squares inverses in π_k of f and F respectively, that is, $g_k = (T_f E_k)^\dagger 1$ and $G_k = (T_F E_k)^\dagger 1$. Then

LEMMA 2.1. $g_k = \overline{u(0)} G_k$.

PROOF. For the inner function u, we see by [2], [4] that the Toeplitz operator T_u is an isometry and $T_u T_u^*$ is the orthogonal projection from H^2 onto $uH^2 = T_u H^2$. Furthermore, $T_u T_u^* 1 = \overline{u(0)} u$ or $T_u^* 1 = \overline{u(0)} 1$. Using those facts, we can show the desired identity by direct computation. We can, however, show it by the reverse order law on generalized inverses:

$$(T_f E_k)^\dagger = (T_u \cdot T_F E_k)^\dagger = (T_F E_k)^\dagger \cdot T_u^* ,$$

which is obtained from the Penrose identities. (Replace A by $T_f E_k$ and $A^\dagger$ by $(T_F E_k)^\dagger T_u^*$, respectively.) q.e.d.

On the basis of Lemma 2.1 we may restrict the problem on the convergence of least-squares and double least-squares inverses to the case where f is outer.

Now, let f be outer and let $h \in H^2$. Then from the density of fH^2 in H^2, we can find a sequence $\{l_k\}$ in H^2 such that

$$\|f l_k - h\|_2 \to 0 \quad \text{as} \quad k \to \infty .$$

We may assume that $l_k \in \pi_k$ for each k, so that

$$\|f \cdot (T_f E_k)^\dagger h - h\|_2 \leqq \|f l_k - h\|_2 \to 0 ,$$

that is,

$$f \cdot (T_f E_k)^\dagger h \to h \quad \text{as} \quad k \to \infty . \tag{2.1}$$

Hence, if $1/f \in H^\infty$, then

$$(T_f E_k)^\dagger h \to (1/f) h = T_{1/f} h \quad \text{as} \quad k \to \infty .$$

This implies

$$(2.2) \qquad (T_f E_k)^\dagger \to T_{1/f} \quad \text{strongly as} \quad k \to \infty .$$

REMARK. Concerning properties of least-squares inverses, Chui [1] proved the following fact, using orthogonal polynomials.

PROPOSITON. *For each k, the least-squares inverses g_k in π_k of $f \in H^\infty$ is zero-free if $f(0) \neq 0$.*

We want to show this fact more directly, using generalized inverses: For simplicity we assume $\|f\|_2 = 1$. First we easily see that $g_0 = (1, f) = \overline{f(0)} \neq 0$ Next, to compute g_1, let $g_1 = a + b(z - c)$, where $c = (zf, f)$. (Note $|c| < 1$.) Then, using

$$\|(z - c)f\|_2^2 = 1 - c\bar{c} , \quad (f, (z - c)f) = 0 \quad \text{and} \quad (1, (z - c)f) = -\bar{c}\overline{f(0)} ,$$

we have

$$\begin{aligned} \|1 - fg_1\|_2^2 &= \|(af - 1) + b(z - c)f\|_2^2 = \|af - 1\|_2^2 + 2\,\mathrm{Re}(bcf(0)) + b\bar{b}(1 - c\bar{c}) \\ &= \|af - 1\|_2^2 + (1 - c\bar{c})|b + \bar{c}(1 - c\bar{c})^{-1}\overline{f(0)}|^2 - c\bar{c}(1 - c\bar{c})^{-1}|f(0)|^2 . \end{aligned}$$

Hence, from the minimality of the norm $\|1 - fg_1\|_2$, we have $a = g_0 = \overline{f(0)}$ and $b = -\bar{c}(1 - c\bar{c})^{-1}\overline{f(0)}$, that is,

$$g_1 = (1 - c\bar{c})^{-1}\overline{f(0)}(1 - \bar{c}z) .$$

Hence clearly g_1 is zero-free in $\bar{D}$. Finally, to see the assertion of the proposition for $k \geqq 2$, observe that g_k has degree at least one. Assume that α is a zero of g_k, and put $\phi = f \cdot g_k/(z - \alpha)$. Then $\phi \in H^\infty$, and $\phi(0) \neq 0$ which is seen from $g_k(0) \neq 0$, or from

$$\begin{aligned} |1 - f(0)g_k(0)| &= |(1, 1 - fg_k)| \leqq \|1 - fg_k\|_2 \\ &\leqq \|1 - fg_0\|_2 = (1 - |f(0)|^2)^{1/2} < 1 . \end{aligned}$$

Hence, by the previous argument we see the least-squares inverse $(T_\phi E_1)^\dagger 1$ in π_1 of ϕ is zero-free in $\bar{D}$. Now by the uniqueness of the least-squares inverse, we see that $z - \alpha = (T_\phi E_1)^\dagger 1$. Hence $\alpha \notin \bar{D}$. This implies that g_k has no zeros in $\bar{D}$. q.e.d.

3. Convergence of double least-squares inverses. On the uniform perturbation of generalized inverses we know by [8] that

$$\|B^\dagger - A^\dagger\| \leqq 3\max\{\|B^\dagger\|^2, \|A^\dagger\|^2\}\|B - A\| ,$$

where A and B are operators with closed range. From this inequality we can show the following fact (cf. [5], [6]):

LEMMA 3.1. *Let A, A_k $(k \in N)$ be operators with closed range, and* *suppose that* $A_k \to A$ *uniformly as* $k \to \infty$. *Then* $A_k^\dagger \to A^\dagger$ *uniformly if (and only if)* $\sup_k \|A_k^\dagger\| < \infty$.

Now, assume that $f \in H^\infty$ be outer as in Section 2. Write $S_k = T_{g_k}$, $V_{n,k} = (S_k E_n)^\dagger$ and $P_{n,k} = (S_k E_n)(S_k E_n)^\dagger$ for simplicity. ($g_k = (T_f E_k)^\dagger 1$). Then, as one more key fact for our discussion we have:

LEMMA 3.2. *For each* $n \in N$, *the set* $\{V_{n,k}; k \in N\}$ *is bounded, and its limit points (weak, strong and uniform topologies are the same in this case) consist of all operators of the form* $T_f W$, *where* W *runs over the set of weak limit points of the set* $\{P_{n,k}; k \in N\}$.

PROOF. Since the L^2-norm and L^∞-norm are equivalent on the finite dimensional subspace π_n of L^∞, it follows from (2.1) that for $h \in \pi_n$ with $\|h\|_2 \leqq 1$, $\|(1 - f g_k)h\|_2 \to 0$ as $k \to \infty$, or equivalently, that

$$R_{n,k} := (1 - T_f S_k)E_n \to 0 \quad \text{(uniformly) as} \quad k \to \infty .$$

Hence, for sufficiently large k the operator $1 - R_{n,k}$ is invertible and $(1 - R_{n,k})^{-1} \to 1$ as $k \to \infty$. Now, since $(S_k E_n) V_{n,k} = P_{n,k}$ and $(1 - E_n) V_{n,k} = 0$, we have

$$(1 - R_{n,k}) V_{n,k} = \{1 - (1 - T_f S_k)E_n\} V_{n,k} = T_f P_{n,k} .$$

Hence

$$V_{n,k} = (1 - R_{n,k})^{-1} T_f P_{n,k}$$

and the assertion follows.

THEOREM 3.3 (cf. [1, Theorem 4.1]). *If* $1/f \in H^\infty$, *then*

$$\lim_{k\to\infty} V_{n,k} = (T_{1/f} E_n)^\dagger \quad for \quad n \in N .$$

PROOF. By (2.2), $g_k \to 1/f$, so that $S_k E_n \to T_{1/f} E_n$. Hence, since $\|(S_k E_n)^\dagger\| = \|V_{n,k}\|$ is bounded by Lemma 3.2, we see, from Lemma 3.1,

$$V_{n,k} = (S_k E_n)^\dagger \to (T_{1/f} E_n)^\dagger . \qquad \text{q.e.d.}$$

If (the outer function) f is in π_m, then the double least-squares inverse $h_{m,k}$ in π_m of f converges to f as $k \to \infty$ by [1, Theorem 2.1]. The following result extends this fact.

THEOREM 3.4. *If* $f \in \pi_m$, *then*

$$\lim_{k\to\infty} V_{m+n,k} E_n = T_f E_n \quad for \quad n \in N .$$

PROOF. Since $fh \in \pi_{m+n}$ for $h \in \pi_n$, we see that

$$\|g_k \cdot V_{m+n,k} h - h\|_2 \leqq \|g_k f h - h\|_2 = \|(g_k f - 1)h\|_2 \to 0$$

as $k \to \infty$ (cf. Proof of Lemma 3.2). This implies that $S_k V_{m+n} E_n \to E_n$ or

$$T_f S_k V_{m+n,k} E_n \to T_f E_n \quad \text{as} \quad k \to \infty \ .$$

On the other hand, since $\| V_{m+n,k} \|$ is bounded (Lemma 3.2) we have

$$\| T_f S_k V_{m+n,k} E_n - V_{m+n,k} E_n \| \leqq \| (T_f S_k - 1) E_{m+n} \| \, \| V_{m+n,k} \| \to 0 \quad \text{as} \quad k \to \infty \ .$$

Hence we conclude that $V_{m+n,k} \to T_f E_n$. q.e.d.

The following theorem shows that the conjecture raised in [1, p. 157] is true.

THEOREM 3.5. *If* $f = \prod_{j=1}^{m} (z - \alpha_j) p$, *where* $|\alpha_j| = 1$ $(j = 1, 2, \cdots, m)$ *and* $p \in H^\infty$ *is outer, then*

$$\lim_{k \to \infty} V_{n,k} = 0 \quad \text{for} \quad n = 0, 1, \cdots, m - 1 \ .$$

PROOF. Take $h \in H^2$ and any non-zero limit point l of the bounded set $\{V_{n,k} h; k \in N\}$. Then the point l belongs to π_n, and is of the form $T_f W h$ for some operator W on H^2 (Lemma 3.2). Hence,

$$l / \prod_{j=1}^{m} (z - \alpha_j) = p \cdot W h \in H^2 \ .$$

But this is possible only when $n \geqq m$.

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