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SEMISIMPLE DEGREE OF SYMMETRY AND MAPS OF NON-ZERO DEGREE INTO A PRODUCT OF 1-SPHERES AND 2-SPHERES

(Dedicated to Professor Minoru Nakaoka on his sixtieth birthday)

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Introduction. In this note, we shall consider the topological degree of symmetry of a manifold M with a map $f: M \to (S^1)^r \times (S^2)^s$ of non-zero degree. Here the topological degree of symmetry of a manifold is, by definition, the maximum of dimension of compact connected Lie groups which act on the manifold topologically and almost effectively.

This note is motivated by recent works on the degree of symmetry of manifolds with large low homotopy groups or cohomology groups such as $K(\pi, 1)$-manifolds, A_k-manifolds or hyper-aspherical manifolds ([2], [5], [6], [8] or [9]). Moreover the results in this note are generalizations of results in [8].

In the following, we shall consider only topological almost effective action.

We shall prove the following

THEOREM A. *Let M be a closed topological manifold with a map $f: M \to (S^1)^r \times (S^2)^s$ of non-zero degree. Then S^3 is the unique, up to local isomorphism, compact connected simple Lie group which can act on M.*

THEOREM B. *Let M be as in Theorem* A *and G a compact connected Lie group which acts on M. Then G is locally isomorphic to $T^u \times (S^3)^v$, where $v \leqq s$. Moreover if the Euler characteristic of M is non-zero, then we have $u + v \leqq s$.*

A typical example of a manifold as in Theorem A is a connected sum $((S^1)^r \times (S^2)^s) \# L$, where L is a closed topological manifold of dimension $r + 2s$. As for such a manifold we have the following

THEOREM C. *Let L be an orientable closed manifold of dimension $r + 2s$ and M the connected sum $L \# ((S^1)^r \times (S^2)^s)$. Assume an n-dimensional toral group acts on M. Then we have $n \leqq r + s$.*

THEOREM D. *Let M and L be as in Theorem* B *and Theorem* C, *respectively. If L is not a rational homology sphere, then $X = M \# L$*

admits no action of S^3. In other words, the topological semisimple degree of symmetry of X is zero. Here the topological semisimple degree of symmetry of X is, by definition, the maximum of dimension of compact connected semisimple Lie group which acts on X.

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In this note, "manifold" means always "compact connected topological manifold" and we use the following notations;

1. $H_i(X)$ and $H^i(X)$ denote i-th homology and i-th cohomology group of X with rational coefficients, respectively.

2. T^n and T denote n-dimensional and 1-dimensional toral group, respectively and we call a 1-dimensional toral group a torus.

1. Preliminaries. In this section, we shall recall some rusults about the Leray spectral sequence of the orbit map and prove some Propositions which are needed to prove Theorems A, B, C and D. Let G be a compact connected Lie group and act on a compact connected space X. Let $\pi: X \to X/G$ be the orbit map and $\{E_r^{p,q}, d_r\}$ the Leray spectral sequence of π. Then we have $E_2^{p,q} = H^p(X/G, H^q(\pi))$, where $H^q(\pi)$ is the sheaf generated by the presheaf $U^* \to H^q(\pi^{-1}(U^*))$ for open set U^* in X/G. Recall that the stalk of $H^q(\pi)$ at $x^* = \pi(x)$ is $H^q(G(x))$ and the edge homomorphism $e: H^q(X) \to E_2^{0,q}$ is given by $e(a)(x^*) = i_x^*(a)$, where $i_x: G(x) \to X$ is the inclusion (see [1] for the details).

We have the following

PROPOSITION 1 (see [2]). *Let k be the dimension of a principal orbit of the action of G on X. If the action has a singular orbit, then the edge homomorphism $e: H^k(X) \to E_2^{0,k}$ is trivial. In particular, we have $E_\infty^{0,k} = 0$.*

See [8] for the proof.

By the same argument as in Proposition 1, we can prove the following

PROPOSITION 2. *Let k be as in Proposition 1. If there is a point x in X such that $i_x^*: H^k(X) \to H^k(G(x))$ is trivial, then the edge homomorphism $e: H^k(X) \to E_2^{0,k}$ is trivial.*

The following Propositions 3, 4 and 6 are generalizations of results in [8] (see Propositions 3 and 5 in [8]).

PROPOSITION 3. *Let M be a closed manifold with a map $f: M \to (S^1)^r \times (S^2)^s$ of non-zero degree. Assume $K = SU(2)$ acts on M with a* *finite principal isotropy subgroup. Then there is a point x in M whose isotropy subgroup is a torus.*

PROOF. Assume the contrary. Then we have $H^i(K(x)) = 0$ for $i = 1, 2$ and for every point x in M. Then the Vietoris-Begle Theorem shows that $\pi^*: H^i(M/K) \to H^i(M)$ is an isomorphism for $i = 1, 2$, where $\pi: M \to M/K$ is the orbit map. It follows from the existence of f that there are elements $a_1, \cdots, a_r \in H^1(M)$ and $b_1, \cdots, b_s \in H^2(M)$ such that the cup product $a_1 \cdots a_r b_1 \cdots b_s$ is non-zero. The above argument shows that all a_i's and b_j's are in the image of π^*. Put $a_i = \pi^*(a'_i)$ and $b_j = \pi^*(b'_j)$ for $i = 1, \cdots, r$ and $j = 1, \cdots, s$. Then we have

$$0 \neq a_1 \cdots a_r b_1 \cdots b_s = \pi^*(a'_1 \cdots a'_r b'_1 \cdots b'_s) = 0 ,$$

since $a'_1 \cdots a'_r b'_1 \cdots b'_s \in H^{\dim M}(M/K) = 0$, which is a contradiction.

PROPOSITION 4. *Let M be a closed manifold with a map $f: M \to (S^1)^r \times (S^3)^s$ of non-zero degree. Assume $G = SU(3)$ or $Sp(2)$ acts on M with a finite principal isotropy subgroup. Then there is a singular orbit.*

To prove this Proposition, we need the following Lemma.

LEMMA 5. *Let X be a closed manifold. Assume a compact connected simple Lie group G acts on X almost freely, i.e. all isotropy subgroups are finite. Then we have $\dim E_2^{0,3} \leqq 1$, where $\{E_r^{p,q}, d_r\}$ is the Leray spectral sequence of the orbit map $\pi: X \to X/G$.*

PROOF. It follows from the argument of the proof of Theorem 1 in [3] that the sheaf $H^q(\pi)$ is locally constant, i.e. for every point x^* in X/G, there is a neighborhood U^* of x^* such that the restriction $H^q(\pi)|U^*$ is isomorphic to the product $U^* \times H^q(G(x))$. Thus we have $E_2^{0,3}$ = the vector space of all sections of $H^3(\pi) \subseteqq H^3(G(x)) \cong Q$. q.e.d.

PROOF OF PROPOSITION 4. Assume the contrary. Consider the Leray spectral sequence $\{E_r^{p,q}, d_r\}$ of the orbit map $\pi: M \to M/G$. It follows from the assumption that $H^i(G(x)) = 0$ for $i = 1, 2$ and for every point x in M. This shows that $E_2^{p,1} = E_2^{p,2} = 0$ and that $\pi^*: H^i(M/G) \to H^i(M)$ is an isomorphism for $i = 1, 2$. Hence we have the following exact sequence;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_\infty^{3,0} & \longrightarrow & H^3(M) & \longrightarrow & E_\infty^{0,3} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \cong \pi^*} & & & & & & \\ & & H^3(M/G) \; . & & & & & & \end{array}$$

It follows from Lemma 5 that $\dim E_\infty^{0,3} \leqq 1$, which implies that $\dim H^3(M/G) = \dim E_\infty^{3,0} \geqq \dim H^3(M) - 1$. Let $a_1, \cdots, a_r \in H^1(M)$ and $b_1, \cdots, b_s \in H^3(M)$ such that $a_1 \cdots a_r b_1 \cdots b_s \neq 0$. We may assume that $b_1, \cdots, b_{s-1}$ are in $\operatorname{Im} \pi^*$. Since $H^i(M/G) = H^i(M)$ via π^* for $i = 1, 2$, we have

$$0 \neq a_1 \cdots a_r b_1 \cdots b_{s-1} = \pi^*(a_1' \cdots a_r' b_1' \cdots b_{s-1}') = 0 ,$$

because $a_1' \cdots a_r' b_1' \cdots b_{s-1}' \in H^{r+3(s-1)}(M/G)$ and $\dim M/G \leqq \dim M - 8$, which is a contradiction. q.e.d.

We shall prove the following

PROPOSITION 6. *Let M be as in Proposition 4. Assume the group $K = SU(2)$ acts on M almost freely. Then $H^*(M)$ is isomorphic to $H^*(M/K) \otimes H^*(S^3)$ as rings. In particular, for every point x in M, the homomorphism $i_x^*: H^3(M) \to H^3(K(x))$ is non-trivial.*

PROOF. Consider the Leray spectral sequence $\{E_r^{p,q}, d_r\}$ of the orbit map $\pi: M \to M/K = M^*$. Assume the edge homomorphism $e: H^3(M) \to E_2^{0,3}$ is not surjective. Then it follows from Lemma 5 that e is trivial, which implies that $E_\infty^{0,3} = 0$. It is easy to see that $\pi^*: H^i(M^*) \to H^i(M)$ is an isomorphism for $i = 1, 2$ and 3. The same argument as in Proposition 4 leads a contradiction. This proves that e is surjective. Since the map $\pi: M \to M^*$ behaves as if it were a fiber bundle in rational coefficients, the argument of the Leray-Hirsch Theorem shows that $H^*(M)$ is isomorphic to $H^*(M^*) \otimes H^*(S^3)$ as rings (see also [2]). The second part follows from Proposition 2 and the fact that e is surjective. q.e.d.

Let M be as in Theorem A. As in [8], we shall construct a principal T^s-bundle $\tilde{M}$ over M as follows. Let $N_i = (S^1)^r \times (S^3)^i \times (S^2)^{s-i} (i = 0, \cdots, s)$. Put $N_0 = N$ and $N_s = \tilde{N}$. Consider N_{i+1} as a principal T-bundle over N_i $(i = 0, \cdots, s-1)$. Let M_1 be the pull-back of the bundle $N_1 \to N$ by the map f and $f_1: M_1 \to N_1$ the bundle map covering f. It is easy to see that f_1 is a map of non-zero degree. Inductively we can construct a sequence of manifolds $M_0 = M, M_1, \cdots, M_s = \tilde{M}$ and a sequence of maps $f_0 = f, f_1, \cdots, f_s = \tilde{f}$ such that $f_i: M_i \to N_i$ is a map of non-zero degree and $p_i: M_i \to M_{i-1}$ is a principal T-bundle which is a pull-back of $q_i: N_i \to N_{i-1}$ by f_{i-1} for $i = 1, \cdots, s$. Put $p = p_1 \circ p_2 \circ \cdots \circ p_s$ and $q = q_1 \circ q_2 \circ \cdots \circ q_s$. It follows from a result in [7] (Theorem 4.1 in [7]) that every action of T^n on M can be lifted over M_i. Let $a_{i1}, \cdots, a_{ir} \in H^1(N_i)$, $b_{i1}, \cdots, b_{is-i} \in H^2(N_i)$ and $c_{i1}, \cdots, c_{ii} \in H^3(N_i)$ be generators of $H^*(N_i)$ for $i = 0, \cdots, s$. Put $a_i = f^*(a_{0i})$ and $b_j = f^*(b_{0j})$. These notations are used in the following sections.

We shall prove the following

PROPOSITION 7. *Let M be as in Theorem* A. *Assume T^n acts on M and there is a point x in M such that the homomorphism $ev_*^x\colon \pi_1(T^n, e) \to \pi_1(M, x)$, where $ev^x\colon T^n \to M$ defined by $ev^x(t) = tx$, is trivial. Then we have $n \leqq s$.*

PROOF. Note that the homomorphism $ev_*^x\colon \pi_1(T^n, e) \to \pi_1(M, x)$ is trivial for every point x in M, because of pathwise connectedness of M. Since the action of T^n on M is lifted over $\tilde{M}$, we have the following diagram;

$$\begin{array}{ccccc} & & \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{N} \\ & {}^{ev^{\tilde{x}}}\nearrow & \downarrow p & & \downarrow q \\ T^n & \xrightarrow{ev^x} & M & \xrightarrow{f} & N \end{array}$$

where $x = p(\tilde{x})$. It follows from the assumption and the above diagram that $(ev^x)^*\colon H^1(M) \to H^1(T^n)$ and $(ev^x)^* f^*\colon H^2(N) \to H^2(T^n)$ are trivial, because $q^*(b_{0j}) = 0$ for $j = 1, \cdots, s$. Thus $(ev^x)^* f^*\colon H^k(N) \to H^k(T^n)$ is trivial for any k. This implies that $i_x^* f^*\colon H^k(N) \to H^k(T^n(x))$ is trivial for every point x and k. Now consider the Leray spectral sequence $\{E_r^{p,q}, d_r\}$ of the orbit map $\pi\colon M \to M/T^n = M^*$. Since $i_x^*\colon H^k(M) \to H^k(T^n(x))$ is trivial on $\mathrm{Im}\{f^*\colon H^k(N) \to H^k(M)\}$, the edge homomorphism $e\colon H^k(M) \to E_2^{0,k}$ is trivial on $\mathrm{Im}\, f^*$ and hence every element of $\mathrm{Im}\, f^*$ has filtration $\geqq 1$ (this implies that $\mathrm{Im}\, f^* \subseteqq J^{1,k-1}$, where $H^k(M) = J^{0,k} \supset J^{1,k-1} \supset \cdots \supset J^{k,0}$). In particular, $a_i \in J^{1,0}$ $(i = 1, \cdots, r)$ and $b_j \in J^{1,1} (j = 1, \cdots, s)$. Hence $a_1 \cdots a_r b_1 \cdots b_s$ has filtration $\geqq r + s$. If $\dim M^* < r + s$, then $E_r^{p,q} = 0$ for $p \geqq r + s$, which means that $a_1 \cdots a_r b_1 \cdots b_s$ is zero. This shows that $\dim M^*$ must be greater than $r + s - 1$. Thus we have $\dim M^* = r + 2s - n \geqq r + s$ and hence $n \leqq s$. q.e.d.

REMARK. As shown in the proof of Proposition 7, we can replace the hypothesis that $ev_*^x\colon \pi_1(T^n, e) \to \pi_1(M, x)$ is trivial by the statement that $(f \circ ev^x)_*\colon \pi_1(T^n, e) \to \pi_1(N, f(x))$ is trivial.

COROLLARY. *If the action of T^n has a fixed point, then we have $n \leqq s$.*

This follows from Proposition, because ev_*^x is trivial for a fixed point x.

2. Proof of Theorem A. In this section, we shall prove Theorem A. To prove it, it is sufficient to show that $G = SU(3)$ or $Sp(2)$ cannot act on M, because the exceptional group G_2 and simple Lie group of rank $\geqq 3$ contain $SU(3)$ or $Sp(2)$. Since the case of $Sp(2)$ is completely parallel to the case of $SU(3)$, we shall prove only that $SU(3)$ cannot act on M.

From now on, we assume M admits an action ϕ of $G=SU(3)$. Let $K=SU(2)$ be the standard subgroup of $SU(3)$ and $\psi: K\times M \to M$ the restriction of the action ϕ. Let M_i be as before and ϕ_i (resp. ψ_i) the lifting of ϕ (resp. ψ) over M_i. Put $\phi_s=\tilde{\phi}$ and $\psi_s=\tilde{\psi}$.

First we shall prove the following

LEMMA 8. *The action ψ has a finite principal isotropy subgroup.*

PROOF. Assume the contrary. Then every isotropy subgroup contains a torus and hence the center C of K is contained in every isotropy subgroup. This implies that every isotropy subgroup of the action of G contains C and C is contained in the ineffective kernel of the action ϕ. Hence C is contained in the center of G, which is easily seen to be impossible. q.e.d.

We shall prove the following

PROPOSITION 9. *The action $\tilde{\psi}$ is almost free.*

To prove this, we need the following two Lemmas.

LEMMA 10. *There is a point x in M such that the homomorphism $i_x^* f^*: H^2(N) \to H^2(K(x))$ is non-trivial.*

PROOF. Assume the contrary. It follows that the edge homomorphism $e: H^2(M) \to E_2^{0,2}$ of the Leray spectral sequence of the orbit map $\pi: M \to M/K$ is trivial on the $\operatorname{Im} f^*$. This implies that $f^*(H^2(N))$ is contained in the kernel of e, which equals $\pi^*(H^2(M/K))$. Since $\pi^*: H^1(M/K) \to H^1(M)$ is an isomorphism by the Vietoris-Begle Theorem, $a_1 \cdots a_r b_1 \cdots b_s$ is zero, which is a contradiction. q.e.d.

Choose x in M such that $i_x^* f^*$ is non-trivial. We may assume $i_x^*(b_1) \neq 0$. In fact, we have $i_x^*(a_i a_j) = i_x^*(a_i) i_x^*(a_j) = 0$, because $H^1(K(x)) = 0$. Consider the lifting ψ_1 and put $p_1(x_1) = x$. Then we have the following

LEMMA 11. *The homomorphism $i_{x_1}^*: H^3(M_1) \to H^3(K(x_1))$ is non-trivial.*

PROOF. Since $i_x^*(b_1) \neq 0$, $K(x) = S^2$ and $p_1^{-1}(K(x)) \to K(x)$ is a non-trivial principal T-bundle, which implies $p_1^{-1}(K(x)) = K(x_1)$. Then Lemma follows from the following commutative diagram;

$$\begin{array}{ccccc} H^3(N) & \longrightarrow & H^3(N_1) & \longrightarrow & H^2(N) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f_1^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ H^3(M) & \longrightarrow & H^3(M_1) & \longrightarrow & H^2(M) \\ {\scriptstyle i_x^*}\downarrow & & {\scriptstyle i_{x_1}^*}\downarrow & & {\scriptstyle i_x^*}\downarrow \\ H^3(K(x)) & \longrightarrow & H^3(K(x_1)) & \longrightarrow & H^2(K(x)) \end{array},$$

where each horizontal sequence is the Gysin sequence. q.e.d.

Now we shall prove Proposition 9. Assume the contrary. Then $\tilde{\psi}$ has a singular orbit and hence ψ_1 has a singular orbit. Since ψ_1 has a finite principal isotropy subgroup, the edge homomorphism $e_1 \colon H^3(M_1) \to E_2^{0,3}$ of the Leray spectral sequence of the orbit map $\pi_1 \colon M_1 \to M_1/K$ is trivial, because of Proposition 1. Therefore the homomorphism $i_{y_1}^* \colon H^3(M_1) \to H^3(K(y_1))$ is trivial for every point y_1 in M_1, which contradicts Lemma 11. This completes the proof. q.e.d.

It follows from Propositions 6 and 9 that, for every point x in $\tilde{M}$, the homomorphism $i_x^* \colon H^3(\tilde{M}) \to H^3(K(x))$ is non-trivial. Consider the following commutative diagram;

$$\begin{array}{ccc} H^3(\tilde{M}) & \xrightarrow{\ j_x^*\ } & H^3(G(x)) \\ & {\scriptstyle i_x^*}\searrow \quad \swarrow{\scriptstyle k^*} & \\ & H^3(K(x)) . & \end{array}$$

By the above argument, we can conclude that the homomorphism j_x^* is non-trivial for every point x in $\tilde{M}$, which implies that $H^3(G(x)) \neq 0$. It follows from a result in [8] (Proposition 8 in [8]) that G_x is finite for every point x in $\tilde{M}$, which contradicts Proposition 4. This completes the proof of Theorem A.

REMARK. One can prove Proposition 8 in [8] for $Sp(2)$ by a similar way as in [8].

3. Proof of Theorem B. In this section, we shall prove Theorem B. Let M be as in Theorem B. If a compact connected Lie group G acts on M, then G must be locally isomorphic to $T^u \times (S^3)^v$, because of Theorem A. Now we shall prove $v \leqq s$. Let $G_1 = (S^3)^v$ and T^v a maximal torus of G_1. Since $ev_*^x \colon \pi_1(T^v, e) \to \pi_1(G_1, e) \to \pi_1(M, x)$ is trivial, it follows from Proposition 7 that we have $v \leqq s$. The last part of Theorem B follows from Corollary to Proposition 7. This completes the proof of Theorem B.

4. Proof of Theorem C. Consider the case where the fundamental group $\pi_1(L)$ is non-trivial. If $\dim M = 2$, the result is well known. Hence we may assume $\dim M \geqq 3$. Then $\pi_1(M)$ has trivial center, because $\pi_1(M) = \pi_1(L) * \pi_1((S^1)^r)$. Since the image of the homomorphism $ev_*^x \colon \pi_1(T^n, e) \to \pi_1(M)$ is contained in the center of $\pi_1(M)$ (see [4] section 4), ev_*^x is trivial. It follows from Proposition 7 that $n \leqq s$. Next consider the case where $\pi_1(L)$ is trivial. Let $K =$ kernel of $ev_*^x \colon \pi_1(T^n, e) \to \pi_1(M) = \pi_1((S^1)^r)$ and put $t = \operatorname{rk} K$. It is clear that $n - t \leqq r$. We can decompose T^n as a product $T^n = T^t \times T^{n-t}$ such that $ev_*^x\colon \pi_1(T^t, e) \to \pi_1(M)$ is trivial. It follows from Proposition 7 that $t \leqq s$. This implies that $n \leqq r + s$ and completes the proof of Theorem C.

5. Proof of Theorem D. Let M and L be as in Theorem D and $X = M \# L$. Let $g\colon M \to N = (S^1)^r \times (S^2)^s$ be a map of non-zero degree and $c\colon X \to M$ the collapsing map. Then the composition $g \circ c$ has non-zero degree. As before, we can construct a T^s-bundle $\tilde{X}$ over X and a map $\tilde{f}\colon \tilde{X} \to \tilde{N} = (S^1)^r \times (S^3)^s$ of non-zero degree. We have the following diagram of fibre bundles and bundle maps;

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\tilde{c}} & \tilde{M} & \xrightarrow{\tilde{g}} & \tilde{N} \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{c} & M & \xrightarrow{g} & N . \end{array}$$

where $\tilde{M}$ is the T^s-bundle over M constructed from g and $\tilde{f} = \tilde{g}\tilde{c}$.

We have the following observations.

(1) $\tilde{X}$ is homeomorphic to the space

$$(\tilde{M} - \operatorname{int} D^m \times T^s) \underset{S^{m-1}\times T^s}{\cup} (L - \operatorname{int} D^m) \times T^s ,$$

where $m = \dim M$.

(2) Consider the following commutative diagram;

$$\begin{array}{ccccc} & & H^k(\tilde{X}, (L - \operatorname{int} D^m)\times T^s) & = & H^k(\tilde{M} - \operatorname{int} D^m \times T^s, S^{m-1}\times T^s) \\ & & j_0^* \downarrow \quad \swarrow r^* & & \downarrow j_1^* \\ H^k(\tilde{X}, \tilde{M} - \operatorname{int} D^m \times T^s) & \longrightarrow & H^k(\tilde{X}) & \xrightarrow{i_1^*} & H^k(\tilde{M} - \operatorname{int} D^m \times T^s) \\ \downarrow \cong & \nearrow q^* & \downarrow i_0^* & & \downarrow i_2^* \\ H^k((L - \operatorname{int} D^m)\times T^s, S^{m-1}\times T^s) & \longrightarrow & H^k((L - \operatorname{int} D^m)\times T^s) & \xrightarrow{i_3^*} & H^k(S^{m-1}\times T^s) . \end{array}$$

Here the vertical and horizontal sequences are exact and q and r are collapsing maps: $\tilde{X} \to \tilde{X}/(\tilde{M} - \operatorname{int} D^m \times T^s)$ and $\tilde{X} \to \tilde{X}/(L - \operatorname{int} D^m)\times T^s$, respectively and the other maps are inclusions. Then it follows that $\operatorname{Im} \tilde{f}^*$ is contained in $\operatorname{Im} r^* = \operatorname{Ker} i_0^*$.

(3) Let $r = \min. \{r'; H^{r'}(L) \neq 0\}$. Since L is not a rational homology sphere, we have $1 \leqq r \leqq m-1$. Choose elements $a' \in H^r(L)$ and $b' \in H^{m-r}(L)$ such that $a'b' \neq 0$. Note that $a' \times [T^s] \in H^{s+r}((L - \operatorname{int} D^m)\times T^s)$ and $b' \times 1 \in H^{m-r}((L - \operatorname{int} D^m)\times T^s)$ are in $\operatorname{Ker} i_3^*$. Then there exist a and b in $H^*(\tilde{X})$ such that $i_0^*(a) = a' \times [T^s]$ and $i_0^*(b) = b' \times 1$. Then we have $ab \neq 0$.

These observations are slight generalizations of results in [8] (Observations (1), (2) and (3) in [8]). We omit their proof.

Now we assume $K = SU(2)$ acts on X. Consider the action $\tilde{\psi}$ of K over $\tilde{X}$ which is a lift of the action ψ of K on X. We have the following

PROPOSITION 12. *The action $\tilde{\psi}$ is almost free.*

PROOF. First we shall prove that $\tilde{\psi}$ has a finite principal isotropy subgroup. Assume the contrary. Then a principal isotropy subgroup $H_{\tilde{\psi}}$ is a torus or the normalizer N_T of a torus T. If $H_{\tilde{\psi}} = N_T$, then $H^*(\tilde{X}) = H^*(\tilde{X}/K)$ via the homomorphism $\tilde{\pi}^*$ induced by the orbit map $\tilde{\pi}: \tilde{X} \to \tilde{X}/K$, because $H^i(K(\tilde{x})) = 0$ for $i \geqq 1$ and $\tilde{x} \in \tilde{X}$. This is easily seen to be a contradiction. Thus $H_{\tilde{\psi}}$ is a torus and it is easy to see that a principal isotropy subgroup H_{ψ} is also a torus. If ψ has an exceptional orbit or singular orbit, then it follows from Proposition 2 that $E_\infty^{0,2} = 0$, where $\{E_r^{p,q}, d_r\}$ is the Leray spectral sequence of the orbit map $\pi: X \to X/K$. This implies $H^2(X/K) = E_\infty^{2,0} = H^2(X)$. Since $H^1(X/K) = E_\infty^{1,0} = H^1(X)$ by the Vietoris-Begle Theorem, it is easy to lead a contradiction. Thus ψ and $\tilde{\psi}$ have a unique orbit type S^2. If there is a point x in X such that the homomorphism $i_x^*: H^2(X) \to H^2(K(x))$ is trivial, then this holds for every point x in X, which implies that the edge homomorphism $e: H^2(X) \to E_2^{0,2}$ is trivial. This is a contradiction as shown above. Thus the homomorphism i_x^* is not trivial for every point x in X. We may assume $i_x^* f^*(b_{01}) \neq 0$. Consider the T-bundle $p_1: X_1 \to X$, which is the pull-back of $N_1 \to N$ by f. It is easy to see that $p_1^{-1}(K(x))$ is K-invariant and equals S^3. Since the action ψ_1 has a torus as a principal isotropy subgroup, ψ_1 must have a fixed point, which is a contradiction. This shows that ψ and hence $\tilde{\psi}$ has a finite principal isotropy subgroup. The proof of the fact that $\tilde{\psi}$ is almost free is completely parallel to the proof of the Proposition 9. q.e.d.

It follows from Proposition 6 that $H^*(\tilde{X})$ is isomorphic to $H^*(\tilde{X}/K) \otimes H^*(S^3)$ as rings. We have the following observation.

(4) There is an element $\tilde{w}$ in $H^3(\tilde{X})$ such that $\tilde{w}$ is contained in $\operatorname{Im} \tilde{f}^*$, but not in $\operatorname{Im} \tilde{\pi}^*$.

In fact, we have the following exact sequence;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_\infty^{3,0} & \longrightarrow & H^3(\tilde{X}) & \longrightarrow & E_\infty^{0,3} & \longrightarrow & 0 \\ & & \uparrow \cong & & & & & & \\ & & H^3(\tilde{X}/K) \,. & & & & & & \end{array}$$

Let $\tilde{w} \in H^3(\tilde{X})$ be the element corresponding to a generator of $H^3(S^3)$. Since $\operatorname{Im}\{f^*: H^3(\tilde{N}) \to H^3(\tilde{X})\} \not\subset \operatorname{Im}\{\pi^*: H^3(\tilde{X}/K) \to H^3(\tilde{X})\}$, we can choose

$\tilde{w}$ in $\operatorname{Im} \tilde{f}^*$.

It follows from observation 2 that $i_0^*(\tilde{w}) = 0$. Since $H^*(\tilde{X}) = H^*(\tilde{X}/K) \oplus \tilde{w} H^*(\tilde{X}/K)$ and $i_0^*(\tilde{w}) = 0$, a and b can be chosen from $\operatorname{Im} \tilde{\pi}^*$, in other words, $a = \tilde{\pi}^*(a'')$ and $b = \tilde{\pi}^*(b'')$, where a'' and b'' are in $H^*(\tilde{X}/K)$. This implies that $ab = \tilde{\pi}^*(a''b'') = 0$, which is a contradiction. Thus we have completed the proof of Theorem D.

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