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AXIOMS FOR STIEFEL-WHITNEY HOMOLOGY CLASSES OF Z_2-EULER SPACES

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1. Introduction and the statement of results. In [2], Blanton and Schweitzer gave an axiomatic characterization for Stiefel-Whitney classes or Stiefel-Whitney homology classes of smooth manifolds, and raised a question of axiomatic characterizations of these classes for other categories, for example, categories of PL-manifolds, topological manifolds or Euler spaces. In this paper we give an answer to this question for Z_2-Euler spaces (cf. [5], [8]).

Let X and Y be Z_2-Euler spaces and let $\varphi: Y \to X$ be a PL-embedding. We call φ a regular embedding if $\dim X = \dim Y$, $\varphi(Y)$ is closed in X, $\varphi(\mathrm{Int}\, Y) \cap \partial X = \emptyset$ and $\varphi|\mathrm{Int}\, Y$ is an open map, where $\mathrm{Int}\, Y = Y - \partial Y$.

Let H_*^{inf} denote the homology theory of infinite chains. Given a regular embedding $\varphi: Y \to X$, we define a homomorphism $\varphi^\#: H_*^{\mathrm{inf}}(X, \partial X; Z_2) \to H_*^{\mathrm{inf}}(Y, \partial Y; Z_2)$ by $\varphi^\# = (\varphi_*)^{-1} \circ i_*$, where $i_*: H_*^{\mathrm{inf}}(X, \partial X; Z_2) \to H_*^{\mathrm{inf}}(X, X - \varphi(\mathrm{Int}\, Y); Z_2)$ is the homomorphism induced from the identity $i: (X, \partial X) \to (X, X - \varphi(\mathrm{Int}\, Y))$. Note that $\varphi_*: H_*^{\mathrm{inf}}(Y, \partial Y; Z_2) \to H_*^{\mathrm{inf}}(X, X - \varphi(\mathrm{Int}); Z_2)$ is an isomorphism by the excision property. Therefore $\varphi^\#$ is well defined.

Let $\mathscr{E}$ be the category whose objects are Z_2-Euler spaces and whose morphisms are regular embeddings. Let $\mathscr{S}$ be a full subcategory of $\mathscr{E}$. Consider a homology class

$$S_*(X) = S_0(X) + S_1(X) + \cdots + S_n(X) \quad \text{in} \quad H_*^{\mathrm{inf}}(X, \partial X; Z_2),$$

where n is the dimension of X, satisfying the following axioms:

AI. For every object X of $\mathscr{S}$ and every integer $i \geqq 0$, there is a homology class $S_i(X)$ in $H_i^{\mathrm{inf}}(X, \partial X; Z_2)$.

AII. If $\varphi: Y \to X$ is a morphism of $\mathscr{S}$, then $S_*(Y) = \varphi^\# S_*(X)$.

AIII. $S_*(X \times Y) = S_*(X) \times S_*(Y)$ for every objects X, Y of $\mathscr{S}$, such that $X \times Y$ is an object of $\mathscr{S}$.

AIV. For every integer $n \geqq 0$, $S_*(\boldsymbol{P}^n) = s_*(\boldsymbol{P}^n)$, where $s_*(\boldsymbol{P}^n)$ is the Stiefel-Whitney homology class of the n-dimensional real projective space $\boldsymbol{P}^n$.

We call $S_*(X)$ an axiomatic Stiefel-Whitney homology class of X in

$\mathscr{S}$. Since the Stiefel-Whitney homology classes satisfy the axioms ([2], [3], [6]), there exists at least one axiomatic Stiefel-Whitney homology class.

The purpose of this paper is to prove the following theorem:

THEOREM. *Let $\mathscr{S}$ be a full subcategory of $\mathscr{E}$ satisfying the following conditions*:

(1) *All compact $\boldsymbol{Z}_2$-Euler spaces are objects of $\mathscr{S}$.*

(2) *If X is an object of $\mathscr{S}$, so is $X\times[0, 1]$.*

Then, the axiomatic Stiefel-Whitney homology class $S_*(X)$ of X in $\mathscr{S}$ coincides with the Stiefel-Whitney homology class $s_*(X)$.

REMARK. In general, axiomatic Stiefel-Whitney homology classes in the category of $\boldsymbol{Z}_2$-Poincaré-Euler spaces are not unique (cf. [8]). For example, the Poincaré dual of Stiefel-Whitney class $[X]\cap w^*(X)$ and the Stiefel-Whitney homology class $s_*(X)$ both satisfy the axioms.

2. Elementary properties of axiomatic Stiefel-Whitney homology classes. In this section, we consider axiomatic Stiefel-Whitney homology classes in a full subcategory $\mathscr{S}$ of $\mathscr{E}$ such that all compact $\boldsymbol{Z}_2$-Euler spaces are objects of $\mathscr{S}$.

LEMMA 1. *Let X be an object of $\mathscr{S}$ and let $S_*(X)$ be an axiomatic Stiefel-Whitney homology class in $\mathscr{S}$. Then,* (1) $S_n(X)=[X]$, *where* $\dim X=n$ *and* $[X]$ *is the homology class given by the chain of all n-simplexes of a triangulation of X, and* (2) $S_i(\partial X)=\partial S_{i+1}(X)$ *when X is compact.*

PROOF. (1) Let Δ^n be the n-dimensional simplex and let $\iota\colon \Delta^n\to \boldsymbol{P}^n$ be a regular embedding. Then $S_*(\Delta^n)=\iota^\#S_*(\boldsymbol{P}^n)$ by AII. Noting that $\iota^\#\colon H_n(\boldsymbol{P}^n; Z_2)\to H_n(\Delta^n, \partial\Delta^n; Z_2)$ is an isomorphism, we have $S_n(\Delta^n)=[\Delta^n]$ by AIV. By AII and the above, for every regular embedding $c\colon \Delta^n\to X$, we have $c^\# S_n(X)=S_n(\Delta^n)=[\Delta^n]$. Then $S_n(X)=[X]$.

(2) Let $i\colon X\to X\cup(\partial X\times I)$ and $j\colon \partial X\times I\to X\cup(\partial X\times I)$ be the canonical inclusions. Then they are regular embeddings. By AII, we have $S_{i+1}(X)=i^\#S_{i+1}(X\cup(\partial X\times I))$ and $S_{i+1}(\partial X\times I)=j^\#S_{i+1}(X\cup(\partial X\times I))$. Note that $(\times)^{-1}(S_{i+1}(\partial X\times I))=S_i(\partial X)\times S_1(I)$ by AIII, and that $\partial\circ i^\#=p_*\circ(\times)^{-1}\circ j^\#$, where $\times\colon H_*^{\inf}(\partial X; \boldsymbol{Z}_2)\times H_*(I, \{0, 1\}; \boldsymbol{Z}_2)\to H_*^{\inf}(\partial X\times I, \partial X\times\{0, 1\}; \boldsymbol{Z}_2)$ is the cross product and $p\colon \partial X\times I\to\partial X$ is the projection. Thus $\partial S_{i+1}(X)=S_i(\partial X)$. q.e.d.

Define a homomorphism $S\colon \mathfrak{B}_*(A, B)\to H_*(A, B; Z_2)$ by $S(\varphi, X)=\varphi_*S_*(X)$. Here $\mathfrak{B}_*(A, B)$ is the bordism group of compact $\boldsymbol{Z}_2$-Euler spaces. (See [8].) The following lemma shows that S is well defined:

LEMMA 2. *Let $S_*(X)$ be an axiomatic Stiefel-Whitney homology class of X in $\mathscr{S}$. Let $\varphi\colon (X, \partial X) \to (A, B)$ be in $\mathfrak{B}_n(A, B)$. Suppose that $(\varphi, X) = 0$ in $\mathfrak{B}_n(A, B)$. Then $\varphi_* S_*(X) = 0$ in $H_*(A, B; \boldsymbol{Z}_2)$.*

PROOF. Let (Φ, W) be a cobordism of (φ, X). Then the inclusion $\iota\colon X \to \partial W$ is a regular embedding. Put $U = \partial W - \iota(\operatorname{Int} X)$. If we denote by i, j the identity and the inclusion respectively, we have a commutative diagram:

$$\begin{array}{ccccc}
H_{i+1}(W, \partial W; \boldsymbol{Z}_2) & \xrightarrow{i_* \circ \partial} & H_i(\partial W, U; \boldsymbol{Z}_2) & \xrightarrow{j_*} & H_i(W, U; \boldsymbol{Z}_2) \\
\partial \downarrow & \nearrow i_* & \downarrow \iota_* & & \downarrow \Phi_* \\
H_i(\partial W; \boldsymbol{Z}_2) & \xrightarrow[\iota^\#]{} & H_i(X, \partial X; \boldsymbol{Z}_2) & \xrightarrow[\varphi_*]{} & H_i(A, B; \boldsymbol{Z}_2)\ ,
\end{array}$$

where the upper sequence is exact. Now $S_*(\partial W) = \partial S_*(W)$ by (2) of Lemma 1 and $S_*(X) = \iota^\# S_*(\partial W)$ by AII. Therefore $\varphi_* S_*(X) = 0$. q.e.d.

3. Stiefel-Whitney classes of block bundles. Let $\xi = (E(\xi), A, \iota)$ be an n-block bundle (cf. [10]) over a locally compact polyhedron A where $\iota\colon A \to E(\xi)$ is the inclusion. Let $\bar{E}(\xi)$ be the total space of the sphere bundle associated with ξ. We shall define a homomorphism $e_\xi\colon \mathfrak{B}_*(E(\xi), \bar{E}(\xi)) \to \boldsymbol{Z}_2$ (cf. [8]), where $\mathfrak{B}_*(E(\xi), \bar{E}(\xi))$ is the bordism group of compact $\boldsymbol{Z}_2$-Euler spaces. We need the following:

TRANSVERSALITY THEOREM (Rourke and Sanderson [10]). *Let M and N be PL-manifolds. Suppose that $f\colon (M, \partial M) \to (N, \partial N)$ is a locally flat proper embedding and that X is a closed subpolyhedron in N. If $f(\partial M) \cap X = \emptyset$ or if $(\partial N, \partial N \cap X)$ is collared in (N, X) and $\partial N \cap X$ is block transverse to $f|\partial M\colon \partial M \to \partial N$, then there exists an embedding $g\colon M \to N$, ambient isotopic to f relative to ∂N such that X is block transverse to g.*

Let R be a regular neighborhood of A embedded properly in $\boldsymbol{R}^\alpha$ for α sufficiently large (cf. [7]). Let $i\colon A \subset R$ be the inclusion and let $p\colon R \to A$ be a retraction. Let $p^*\xi = (E(p^*\xi), R, \iota_R)$ be the induced bundle (cf. [10].) Then there exist bundle maps $(\bar{i}, i)\colon (E(\xi), A) \to (E(p^*\xi), R)$ and $(\bar{p}, p)\colon (E(p^*\xi), R) \to (E(\xi), A)$. For each (φ, X) in $\mathfrak{B}_*(E(\xi), \bar{E}(\xi))$, there exists an embedding $\tilde{\varphi}\colon (X, \partial X) \to (E(p^*\xi), \bar{E}(p^*\xi))$ such that $\tilde{\varphi} \simeq i \circ \varphi$. By the transversality theorem, we may assume that $\tilde{\varphi}(X)$ is block transverse to $\iota_R\colon R \to E(p^*\xi)$. Let $Y = \tilde{\varphi}^{-1}(\iota_R(R))$. Then Y is a closed $\boldsymbol{Z}_2$-Euler space. We write $e_\xi(\varphi, X)$ for the modulo 2 Euler number $e(Y)$ of Y.

We need the following lemma to prove Lemma 6:

LEMMA 3. *Let $S_*(X)$ be an axiomatic Stiefel-Whitney homology class* *of X in $\mathscr{S}$. Let $\nu=(E, M, \iota)$ be the normal block bundle of a proper embedding from a compact triangulated differentiable manifold M into $\boldsymbol{R}_{+}^{\alpha}$ for α sufficiently large. Then $\langle U_{\nu} \cup (\iota^{*})^{-1} w^{*}(M), \varphi_{*} S_{*}(X)\rangle = e_{\nu}(\varphi, X)$ for each (φ, X) in the bordism group $\mathfrak{B}_{*}(E, \bar{E})$ of compact $\boldsymbol{Z}_2$-Euler spaces, where $\bar{E}$ is the total space of the sphere bundle associated with ν.*

This lemma is a consequence of the following two lemmas, which we merely state without proof. For, if we note AI, $\cdots$, AIV, and Lemmas 1, 2, the proofs given in Matsui [8] for the case of Stiefel-Whitney homology classes can be applied without any change to the present situation by simply replacing therein s_{*} by S_{*}.

LEMMA 4. *Let $S_{*}(X)$ be an axiomatic Stiefel-Whitney homology class of X in $\mathscr{S}$. Let $\xi=(E, A, \iota)$ be an n-block bundle over a locally compact polyhedron A. Then there exists a unique cohomology class $\Phi(\xi)$ in $H^{*}(E, \bar{E}; \boldsymbol{Z}_2)$ satisfying $\langle \Phi(\xi), \varphi_{*} S_{*}(X)\rangle = e_{\xi}(\varphi, X)$ for each (φ, X) in $\mathfrak{B}_{*}(E, \bar{E})$.*

Let $\xi=(E, X, \iota)$ be a block bundle. Let $\Phi(\xi)$ be the cohomology class as in Lemma 4. Define $\tilde{w}(\xi)$ by $\tilde{w}(\xi)=\iota^{*}\circ(U_{\xi}\cup)^{-1}\Phi(\xi)$, where $\iota^{*}\circ(U_{\xi}\cup)^{-1}\colon H^{*}(E, \bar{E}; \boldsymbol{Z}_2)\rightarrow H^{*}(X; Z_2)$ is the Thom isomorphism of ξ. Then we have the following:

LEMMA 5. *If ξ is the block bundle induced by a vector bundle over a locally compact polyhedron X, the cohomology class $\tilde{w}(\xi)$ coincides with the dual Stiefel-Whitney class $\bar{w}(\xi)$ of $w^{*}(\xi)$.*

PROOF OF LEMMA 3. Since ν is induced from a vector bundle, we have $\langle U_{\nu} \cup (\iota^{*})^{-1}\bar{w}(\nu), \varphi_{*} S_{*}(X)\rangle = e_{\nu}(\varphi, X)$ by Lemmas 4 and 5. On the other hand, there holds $w^{*}(M)=\bar{w}(\nu)$. Thus $\langle U_{\nu} \cup (\iota^{*})^{-1} w^{*}(M), \varphi_{*} S_{*}(X)\rangle = e_{\nu}(\varphi, X)$. q.e.d.

4. Proof of Theorem. Let X be an n-dimensional $\boldsymbol{Z}_2$-Euler space. Then there exists a proper PL-embedding $\varphi\colon (X, \partial X)\rightarrow(\boldsymbol{R}_{+}^{\alpha}, \partial \boldsymbol{R}_{+}^{\alpha})$ for α sufficiently large. (See Hudson [7].) Suppose that R is a regular neighborhood of X in $\boldsymbol{R}_{+}^{\alpha}$. Put $\tilde{R}=R\cap\partial\boldsymbol{R}_{+}^{\alpha}$ and $\bar{R}=\mathrm{cl}(\partial R-\tilde{R})$. Regard φ as a proper PL-embedding from $(X, \partial X)$ to $(R, \tilde{R})$. We also call $(R; \tilde{R}, \bar{R}; \varphi)$ a regular neighborhood of X in $\boldsymbol{R}_{+}^{\alpha}$. We will define a homomorphism $e_{\varphi}\colon \mathfrak{N}_{*}(R, \bar{R})\rightarrow Z_2$ as in [8], where $\mathfrak{N}_{*}(R, \bar{R})$ is the unoriented differentiable bordism group. Let $f\colon(M, \partial M)\rightarrow(R, \bar{R})$ be in $\mathfrak{N}_{*}(R, \bar{R})$. Then there exists an PL-embedding $g\colon(M, \partial M)\rightarrow(R\times D^{\beta}, \bar{R}\times D^{\beta})$ for β sufficiently large, such that $g\simeq f\times\{0\}$ and that $(\varphi\times\mathrm{id})(X\times D^{\beta})$ is block transverse to g by the transversality theorem. Let $Y=(\varphi\times\mathrm{id})^{-1}\circ g(M)$. Then Y

is a closed $\boldsymbol{Z}_2$-Euler space. We write $e_\varphi(f, M)$ for the modulo 2 Euler number $e(Y)$ of Y.

LEMMA 6. *Let X be an object of $\mathscr{S}$. Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of X in $\boldsymbol{R}^a_+$. Then $\langle([R]\cap)^{-1}\circ\varphi_* S_*(X),\ f_*([M]\cap w^*(M))\rangle = e_\varphi(f, M)$ for each (f, M) in $\mathfrak{N}_*(R, \bar{R})$.*

PROOF. (i) First we prove the lemma in the case where $f\colon (M, \partial M)\to (R, \bar{R})$ is a PL-embedding with a normal block bundle $\xi = (E, M, f_E)$ and φ is transverse to ξ. Let $\Phi = ([R]\cap)^{-1}\circ\varphi_* S_*(X)$. Since $[E]\cap U_\xi = (f_E)_*[M]$ and $j_E\circ f_E = f$, where U_ξ is the Thom class of ξ and $j_E\colon E\to R$ is an inclusion, we get

$$\langle\Phi,\ f_*([M]\cap w^*(M))\rangle = \langle U_\xi\cup(f_E^*)^{-1}w^*(M),\ [E]\cap j_E^*\Phi\rangle\ .$$

Now, we have the following commutative diagram:

$$\begin{array}{ccccc}
(X_E, \partial X_E) & \xrightarrow{\ j_X\ } & (X, X - j(\mathrm{Int}\, X_E)) & \xleftarrow{\ j\ } & (X, \partial X) \\
\Big\downarrow{\scriptstyle \varphi_E} & & \Big\downarrow{\scriptstyle \tilde{\varphi}_E} & & \Big\downarrow{\scriptstyle \varphi} \\
(E, \bar{E}) & \xrightarrow[\ j_E\ ]{} & (R, \tilde{\tilde{R}}) & \xleftarrow[\ i\ ]{} & (R, \tilde{R})\ ,
\end{array}$$

where $X_E = \varphi^{-1}(E)$, $\tilde{\tilde{R}} = \mathrm{cl}(R - j_E(E))$, and where i, j and j_X are inclusions. If we note $(j_E)_*[E] = i_*[R]$, then $[E]\cap j_E^*\Phi = ((j_E)_*^{-1}\circ i_*[R])\cap j_E^*\Phi = (j_E)_*^{-1}\circ i_*([R]\cap\Phi) = (j_E)_*^{-1}\circ i_*\circ\varphi_* S_*(X) = (j_E)_*^{-1}\circ(\tilde{\varphi}_E)_*\circ j_* S_*(X)$. Since $j_X\colon X_E\to X$ is a regular embedding and $S_*(X_E) = j_X^! S_*(X) = (j_X)_*^{-1}\circ j_* S_*(X)$ by AII, we have $[E]\cap j_E^*\Phi = (\varphi_E)_* S_*(X_E)$. Thus $\langle\Phi,\ f_*([M]\cap w^*(M))\rangle = \langle U_\xi\cup (f_E^*)^{-1}w^*(M),\ (\varphi_E)_* S_*(X_E)\rangle$. We have $\langle\Phi, f_*([M]\cap w^*(M))\rangle = e_\xi(\varphi_E, X_E)$ by Lemma 3 and also $e_\varphi(f, M) = e_\xi(\varphi_E, X_E)$ in the view of the definitions of e_φ and e_ξ. Therefore, $\langle\Phi, f_*([M]\cap w^*(M))\rangle = e_\varphi(f, M)$.

(ii) We now consider the case where $f\colon (M, \partial M)\to(R, \bar{R})$ is not an embedding.

Let (f, M) be in $\mathfrak{N}_*(R, \bar{R})$. Then there exists a PL-embedding $g\colon (M, \partial M)\to(R\times D^\beta, \bar{R}\times D^\beta)$ for β sufficiently large, such that $g\simeq f\times\{0\}$ and $(\varphi\times\mathrm{id})(X\times D^\beta)$ is block transverse to g by the transeversality theorem. Here $X\times D^\beta$ is an object of $\mathscr{S}$ in view of the property (2) of $\mathscr{S}$. From the previous result (i), it now follows:

$$\langle([R\times D^\beta]\cap)^{-1}\circ(\varphi\times\mathrm{id})_* S_*(X\times D^\beta),\ g_*([M]\cap w^*(M))\rangle = e_\varphi(f, M)\ .$$

However $S_*(X\times D^\beta) = S_*(X)\times S_*(D^\beta)$ by AIII and $S_*(D^\beta) = [D^\beta]$ by (1) of Lemma 1. Hence we get $\langle([R]\cap)^{-1}\circ\varphi_* S_*(X), f_*([M]\cap w^*(M))\rangle = e_\varphi(f, M)$.

q.e.d.

LEMMA 7 (See [8]). *Let* (A, B) *be a pair of polyhedra and let* $\Phi \in H^*(A, B; \boldsymbol{Z}_2)$. *If* $\langle \Phi, f_*([M] \cap w^*(M)) \rangle = 0$ *for every* (f, M) *in* $\mathfrak{N}_*(A, B)$, *then* $\Phi = 0$.

PROOF OF THEOREM. Noting that $s_*(X)$ is an axiomatic Stiefel-Whitney homology class, we have by Lemma 6, $\langle ([R] \cap)^{-1} \circ \varphi_* s_*(X), f_*([M] \cap w^*(M)) \rangle = e_\varphi(f, M)$ for each (f, M) in $\mathfrak{N}_*(R, \bar{R})$. By Lemmas 6 and 7, we also have $([R] \cap)^{-1} \circ \varphi_* S_*(X) = ([R] \cap)^{-1} \circ \varphi_* s_*(X)$. Therefore $S_*(X) = s_*(X)$. q.e.d.

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