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LUSIN FUNCTIONS AND NONTANGENTIAL MAXIMAL FUNCTIONS IN THE H^p THEORY ON THE PRODUCT OF UPPER HALF-SPACES

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1. Introduction. In this note, we will give a proof of the L^p norm equivalence between the Lusin area integral $A(u)$ and the nontangential maximal function $N(u)$ of a biharmonic function u defined on the product space $\boldsymbol{D} = \boldsymbol{R}_+^{n_1+1} \times \boldsymbol{R}_+^{n_2+1}$, where $\boldsymbol{R}_+^{n_i+1} = \boldsymbol{R}^{n_i} \times (0, \infty)$ $(i = 1, 2)$.

We will use the following notations. We write

$$(x^{(1)}, y_1; x^{(2)}, y_2) = (x_1^{(1)}, \cdots, x_{n_1}^{(1)}, y_1; x_1^{(2)}, \cdots, x_{n_2}^{(2)}, y_2)$$

for the point of $\boldsymbol{R}^{n_1+1} \times \boldsymbol{R}^{n_2+1}$, where $(x^{(i)}, y_i) \in \boldsymbol{R}^{n_i+1}$, $x^{(i)} = (x_1^{(i)}, \cdots, x_{n_i}^{(i)}) \in \boldsymbol{R}^{n_i}$, and $y_i \in \boldsymbol{R}$ $(i = 1, 2)$. We also write $(x^{(1)}, y_1; x^{(2)}, y_2) = (x, y)$, where $x = (x^{(1)}, x^{(2)}) \in \boldsymbol{R}^N$ $(N = n_1 + n_2)$, and $y = (y_1, y_2) \in \boldsymbol{R}^2$. Let $\boldsymbol{R}_+^{n_i+1} = \{(x^{(i)}, y_i) \in \boldsymbol{R}^{n_i+1}: y_i > 0\}$ $(i = 1, 2)$ and $\boldsymbol{D} = \boldsymbol{R}_+^{n_1+1} \times \boldsymbol{R}_+^{n_2+1}$.

Let $u(x, y)$ be a biharmonic function on $\boldsymbol{D}$, that is, u is twice continuously differentiable and $\Delta_i u = 0$ on $\boldsymbol{D}$ $(i = 1, 2)$, where

$$\Delta_i = \sum_{j=1}^{n_i} (\partial/\partial x_j^{(i)})^2 + (\partial/\partial y_i)^2$$

is the Laplacian in the $(x^{(i)}, y_i)$ variable. For $a = (a_1, a_2)$, $a_1 > 0$, $a_2 > 0$, and $x = (x^{(1)}, x^{(2)}) \in \boldsymbol{R}^N$, we define a product cone $\Gamma_a(x)$ by

$$\Gamma_a(x) = \{(t^{(1)}, y_1; t^{(2)}, y_2) \in \boldsymbol{D}: |t^{(1)} - x^{(1)}| < a_1 y_1, |t^{(2)} - x^{(2)}| < a_2 y_2\} . \tag{1.1}$$

We say that $u \in H^p(\boldsymbol{D})$ $(0 < p < \infty)$ if its nontangential maximal function

$$N_a(u) = \sup\{|u(t, y)|: (t, y) \in \Gamma_a(x)\} \tag{1.2}$$

belongs to the Lebesgue space $L^p(\boldsymbol{R}^N)$. It is known that this definition is independent of a. The Lusin area integral of a biharmonic function u is defined by

$$A_a(u)(x) = \left(\int_{\Gamma_a(x)} |\nabla_1 \nabla_2 u(t, y)|^2 y_1^{1-n_1} y_2^{1-n_2} dt dy\right)^{1/2} , \tag{1.3}$$

where $|\nabla_1 \nabla_2 u|^2 = \sum_{j=1}^{n_1+1} \sum_{k=1}^{n_2+1} |\partial^2/(\partial x_j^{(1)} \partial x_k^{(2)}) u|^2$ with $\partial/\partial x_{n_1+1}^{(1)} = \partial/\partial y_1$, $\partial/\partial x_{n_2+1}^{(2)} = \partial/\partial y_2$. We write $A_{(1,1)}(u) = A(u)$, $N_{(1,1)}(u) = N(u)$, and $\Gamma_{(1,1)}(x) = \Gamma(x)$.

The main purpose of this note is to give proofs of the inequalities

$$\|A(u)\|_p \leqq c_p \|N(u)\|_p \tag{1.4}$$

and

$$\|N(u)\|_p \leqq c_p \|A(u)\|_p\ , \tag{1.5}$$

for $u \in H^p(\boldsymbol{D})$. Gundy and Stein [7] showed the inequalities (1.4) and (1.5) for $u \in H^p(\boldsymbol{R}_+^2 \times \boldsymbol{R}_+^2)$, $0<p<\infty$ (see also Gundy [6]). We will give a simpler proof of the inequality (1.4). In order to prove the inequality (1.5), we will introduce H^p spaces of conjugate biharmonic functions in §2. Our result is stated as Theorem (2.5) in §2.

In this note, tne letter c will denote a positive constant, which need not be the same at each occurrence, and CE denotes the complement of a set E.

2. H^p spaces of conjugate biharmonic functions and the theorem. Let $u_{jk}(x, y)$ $(j = 1, 2, \cdots, n_1 + 1$ and $k = 1, 2, \cdots, n_2 + 1)$ be $(n_1 + 1)\times(n_2 + 1)$ biharmonic functions on $\boldsymbol{D}$ which satisfy the following generalized Cauchy-Riemann equations:

$$\sum_{j=1}^{n_1+1} \partial u_{jk} / \partial x_j^{(1)} = 0\ , \quad \partial u_{jk} / \partial x_i^{(1)} = \partial u_{ik} / \partial x_j^{(1)}\ , \tag{2.1}$$

$1 \leqq i, j \leqq n_1 + 1$, $k = 1, 2, \cdots, n_2 + 1$, and

$$\sum_{k=1}^{n_2+1} \partial u_{jk} / \partial x_k^{(2)} = 0\ , \quad \partial u_{jk} / \partial x_l^{(2)} = \partial u_{jl} / \partial x_k^{(2)}\ , \tag{2.2}$$

$1 \leqq k, l \leqq n_2 + 1$, $j = 1, 2, \cdots, n_1 + 1$, where $\partial/\partial x_{n_1+1}^{(1)} = \partial/\partial y_1$ and $\partial/\partial x_{n_2+1}^{(2)} = \partial/\partial y_2$. Let $F(x, y)$ be the $(n_1 + 1)\times(n_2 + 1)$ matrix-valued function whose (j, k)-component is $u_{jk}(x, y)$ for $1 \leqq j \leqq n_1 + 1$ and $1 \leqq k \leqq n_2 + 1$: $F(x, y) = (u_{jk}(x, y))$. We call F a system of conjugate biharmonic functions. Let $|F| = (\sum_{j=1}^{n_1+1} \sum_{k=1}^{n_2+1} |u_{jk}|^2)^{1/2}$ and let $p_0 = \max((n_1 - 1)/n_1, (n_2 - 1)/n_2)$.

DEFINITION (2.3). Let $p_0 < p < \infty$, and let F be a system of conjugate biharmonic functions on $\boldsymbol{D}$. We say that $F \in H_A^p(\boldsymbol{D})$, if

$$\sup_{y_1, y_2>0} \left(\int_{\boldsymbol{R}^N} |F(x, y)|^p dx\right)^{1/p} < \infty\ . \tag{2.4}$$

We write $\|F\|_p$ for the left hand side of the above inequality.

$H^p(\boldsymbol{D})$ spaces are characterized in terms either of the area integral or H_A^p spaces. In fact, we have the following theorem.

THEOREM 2.5. *Let $u(x, y)$ be a biharmonic function on $\boldsymbol{D}$ and let $p_0<p<2$. Then the following three properties are equivalent.*

(1) $N(u) \in L^p(\boldsymbol{R}^N)$.

(2) $u(x, y_1, y_2) \to 0$ *as* $y_1 + y_2 \to \infty$ *and* $A(u) \in L^p(\boldsymbol{R}^N)$.

(3) *There exists* $F = (u_{jk}) \in H_A^p(\boldsymbol{D})$ *such that* $u = u_{LM}$ $(L = n_1 + 1, M = n_2 + 1)$.

Moreover, we have

$$\|A(u)\|_p \leqq c_p \|N(u)\|_p \tag{2.6}$$

and

$$\|N(u)\|_p \leqq c_p \|A(u)\|_p \,, \tag{2.7}$$

where c_p *is a constant independent of* u.

REMARK (2.8). The inequalities (2.6) and (2.7) were shown in Gundy and Stein [7] for $u \in H^p(\boldsymbol{R}_+^2 \times \boldsymbol{R}_+^2)$.

REMARK (2.9). Theorem (2.5) is stated only for $p, p_0 < p < 2$, for simplicity, but in view of the one-variable theory in Fefferman and Stein [4], Theorem (2.5) is also valid for all $p, 0<p<\infty$, if we introduce appropriate H_A^p spaces for $p \leqq p_0$.

3. Proof of Theorem (2.5), **I.** In this section and §4, we prove the implication (1) $\Rightarrow$ (2) in Theorem (2.5).

Assume that $N(u) \in L^p$. Then by Lemma (6.3) in §6, we have that $u(x, y) \to 0$ as $y_1 + y_2 \to \infty$. In order to show that $A(u) \in L^p$, we will prove the following.

PROPOSITION (3.1). *Let* $u \in H^p(\boldsymbol{D})$ $(0 < p < 2)$. *Then we have*

$$|\{x \in \boldsymbol{R}^N \colon A(u)(x) > \alpha\}| \leqq c\alpha^{-2} \|N(u) \wedge \alpha\|_2^2 \tag{3.2}$$

for all $\alpha > 0$, *where* $|\cdot|$ *denotes the Lebesgue measure and c is a constant independent of* u *and* α.

The analogue of Proposition (3.1) for the bidisc was shown in Gundy and Stein [7] (see also Gundy [6]). We give a simpler proof. By Proposition (3.1) and a well-known argument, we obtain $\|A(u)\|_p \leqq c_p \|N(u)\|_p$ $(0<p<2)$ (see Fefferman and Stein [4, p. 165]). Thus we only have to prove Proposition (3.1).

Before we give a proof of Proposition (3.1), we introduce the iterated Poisson integral of a function defined on $\boldsymbol{R}^N$. For $(x, y) = (x^{(1)}, y_1; x^{(2)}, y_2) \in \boldsymbol{D}$, the iterated Poisson kernel $P_y(x)$ is defined by $P_y(x) = P_1(x^{(1)}, y_1) P_2(x^{(2)}, y_2)$, where

$$P_i(x^{(i)}, y_i) = c_{n_i} y_i / (|x^{(i)}|^2 + y_i^2)^{(n_i+1)/2}$$

is the Poisson kernel associated with the upper half space $\boldsymbol{R}_+^{n_i+1}$ $(i = 1, 2)$ (see [11]). For $f \in L^p(\boldsymbol{R}^N)$ $(1 \leqq p \leqq \infty)$, we define the iterated Poisson integral of f by

$$(3.3) \qquad P(f)(x, y) = P_y * f(x) = \int_{\boldsymbol{R}^N} f(x-t)P_y(t)dt\ .$$

It is easy to see that $P(f)(x, y)$ is a biharmonic function on $\boldsymbol{D}$.

Now we begin the proof of Proposition (3.1). We may assume that u is a real-valued function. We first assume that u is the iterated Poisson integral of $f \in L^2(\boldsymbol{R}^N)$: $u(x, y) = P_y * f(x)$. Let $\alpha > 0$ and let

$$(3.4) \qquad E = \{x \in \boldsymbol{R}^N \colon N(u)(x) \leqq \alpha\}\ .$$

We need the following lemmas on the iterated Poisson integral of the characteristic function of E.

LEMMA (3.5). *Let E be the same as in* (3.4). *Let $v(x, y) = P(\chi_E)(x, y)$, where χ_E is the characteristic function of E. Then there exists a positive constant δ_0 not depending on E such that $0 < \delta_0 < 2^{-5}$ and*

$$\sup\{|u(x, y)| \colon (x, y) \in S\} \leqq \alpha\ ,$$

where $S = \{(x, y) \in \boldsymbol{D} \colon v(x, y) \geqq 1 - 2\delta_0\}$.

LEMMA (3.6). *Let δ_0 be the same as in Lemma* (3.5). *Put $\delta = \delta_0/4$. Let E be the same as in* (3.4). *Then there exists a subset E^* of E such that*

$$(3.7) \qquad \inf_{x \in E^*} \inf\{P(\chi_E)(t, y) \colon (t, y) \in \Gamma(x)\} \geqq 1 - \delta$$

and

$$(3.8) \qquad |CE^*| \leqq c|CE|\ ,$$

where c is a constant independent of f and α.

Lemma (3.5) follows from the definition of E. Lemma (3.6) is essentially given in [7]. We omit the proof.

We continue to prove Proposition (3.1). Let δ_0 and δ be the same as in Lemmas (3.5) and (3.6), respectively. Let ϕ and ψ be non-negative C^∞ functions on $\boldsymbol{R}$ such that $\phi(r) = 1$ if $r \geqq 1 - \delta$, $\phi(r) = 0$ if $r \leqq 1 - 2\delta$, and $\psi(r) = 1$ if $r \geqq 1 - \delta_0$, $\psi(r) = 0$ if $r \leqq 1 - 2\delta_0$. Note that

$$(3.9) \qquad |\phi'(r)|^2 \leqq c\phi(r)$$

$$(3.10) \qquad |\psi'(r)|^2 \leqq c\psi(r)$$

for all $r \in \boldsymbol{R}$ and

$$(3.11) \qquad \inf_{r \in T} \psi(r) > 0 \quad (T = \{r \in \boldsymbol{R} \colon |\phi'(r)| + |\phi''(r)| > 0\})$$

(see [8]). Then by (3.7), we have

$$\int_{E^*} A^2(u)(x)dx \leqq c\int y_1 y_2 \phi(v)|\nabla_1\nabla_2 u|^2 dxdy\ , \tag{3.12}$$

where $v = P(\chi_E)$. We write I for the integral on the right hand side of (3.12). We modify I. For $\varepsilon > 0$, wet set

$$I_\varepsilon = \int y_1 y_2 \phi(v_\varepsilon(x, y))|\nabla_1\nabla_2 u_\varepsilon(x, y)|^2 dxdy\ , \tag{3.13}$$

where $v_\varepsilon(x, y) = P(\chi_E)(x, y_1+\varepsilon, y_2+\varepsilon)$ and $u_\varepsilon(x, y) = u(x, y_1+\varepsilon, y_2+\varepsilon)$ $(y_1, y_2 \geqq 0)$.

For a continuously differentiable function $g(x, y)$ on $\bar{D}$, let

$$|\nabla_i g(x, y)| = \left(\sum_{j=1}^{n_i} |(\partial/\partial x_j^{(i)})g(x, y)|^2 + |(\partial/\partial y_i)g(x, y)|^2\right)^{1/2}$$

for $i = 1, 2$. Then, applying Green's theorem with respect to $(x^{(1)}, y_1)$, we obtain

$$\begin{aligned}\int y_1 y_2 \Delta_1(\phi(v_\varepsilon)|\nabla_2 u_\varepsilon|^2)dxdy &= \int y_2 \phi(v_\varepsilon(x, 0, y_2))|\nabla_2 u_\varepsilon(x, 0, y_2)|^2 dxdy_2 \\ &= J_\varepsilon,\ \text{say}\ .\end{aligned} \tag{3.14}$$

On the other hand, we have

$$\begin{aligned}&|\Delta_1(\phi(v_\varepsilon)|\nabla_2 u_\varepsilon|^2) - \phi''(v_\varepsilon)|\nabla_1 w_\varepsilon|^2|\nabla_2 u_\varepsilon|^2 - 2\phi(v_\varepsilon)|\nabla_1\nabla_2 u_\varepsilon|^2| \\ &\quad\leqq c|\phi'(v_\varepsilon)|\,|\nabla_1 w_\varepsilon|\,|\nabla_2 u_\varepsilon|\,|\nabla_1\nabla_2 u_\varepsilon|\ ,\end{aligned} \tag{3.15}$$

where $w_\varepsilon(x, y) = P(\chi_{CE})(x, y_1+\varepsilon, y_2+\varepsilon)$. Put

$$K_\varepsilon = \int y_1 y_2 \psi(v_\varepsilon)|\nabla_1 w_\varepsilon|^2|\nabla_2 u_\varepsilon|^2 dx\,dy\ . \tag{3.16}$$

Then by (3.9), (3.11), (3.14), and (3.15), using Schwarz's inequality, we obtain

$$I_\varepsilon \leqq c(J_\varepsilon + K_\varepsilon + I_\varepsilon^{1/2}K_\varepsilon^{1/2})\ . \tag{3.17}$$

We need the following two inequalities (3.18) and (3.19), whose proofs we will give in §4.

$$J_\varepsilon \leqq c\left(\int_{R^N} \phi(v_\varepsilon(x, 0))u_\varepsilon^2(x, 0)dx + \alpha^2\int_{R^N} w_\varepsilon^2(x, 0)dx\right)\ , \tag{3.18}$$

$$K_\varepsilon \leqq c\alpha^2\int_{R^N} w_\varepsilon^2(x, 0)dx\ , \tag{3.19}$$

where c is a constant independent of ε, u, and α.

By (3.17), (3.18), and (3.19), we have

$$(3.20)\qquad I_\varepsilon \leqq c\left(\int \phi(v_\varepsilon(x, 0))u_\varepsilon^2(x, 0)dx + \alpha^2\int w_\varepsilon^2(x, 0)dx\right).$$

We can easily see that $\int \phi(v_\varepsilon(x, 0))u_\varepsilon^2(x, 0)dx \to \int_E f^2(x)dx$ and $\int w_\varepsilon^2(x, 0)dx \to |CE|$ as $\varepsilon \to 0$. Thus letting $\varepsilon \to 0$ in (3.20), we have

$$(3.21)\qquad I \leqq c\left(\int_E N^2(u)(x)dx + \alpha^2|CE|\right).$$

By (3.21), (3.12), and (3.8), we conclude that

$$\begin{aligned} |\{x \in \boldsymbol{R}^N : A(u)(x) > \alpha\}| &\leqq |CE^*| + \alpha^{-2}\int_{E^*} A^2(u)(x)dx \\ &\leqq c\left(|CE| + \alpha^{-2}\int_E N^2(u)(x)dx\right) \\ &= c\alpha^{-2}\|N(u) \wedge \alpha\|_2^2 . \end{aligned}$$

Now we remove the restriction that u is the iterated Poisson integral of an L^2 function. Assume that $N(u) \in L^p(\boldsymbol{R}^N)$ $(0 < p < 2)$. Let $\eta > 0$ and let $u_\eta(x, y) = u(x, y_1 + \eta, y_2 + \eta)$. By Lemmas (6.3) and (6.6) in §6, it is easy to see that u_η is the iterated Poisson integral of an L^2 function. Therefore by what we have proved, we have

$$(3.22)\qquad |\{x \in \boldsymbol{R}^N : A(u_\eta)(x) > \alpha\}| \leqq c\alpha^{-2}\|N(u_\eta) \wedge \alpha\|_2^2 .$$

Letting $\eta \to 0$ in (3.22), we obtain $|\{x \in \boldsymbol{R}^N : A(u)(x) > \alpha\}| \leqq c\alpha^{-2}\|N(u) \wedge \alpha\|_2^2$. This completes the proof of Proposition (3.1).

4. Proof of (3.18) and (3.19). In this section, we prove (3.18) and (3.19) in §3.

Proof of (3.18). Applying Green's theorem, we find

$$(4.1)\qquad \int y_2\Delta_2(\phi(v_\varepsilon(x, 0, y_2))u_\varepsilon^2(x, 0, y_2))dx\,dy_2 = \int \phi(v_\varepsilon(x, 0))u_\varepsilon^2(x, 0)dx .$$

On the other hand, we have

$$(4.2)\qquad \begin{aligned} &|\Delta_2(\phi(v_\varepsilon)u_\varepsilon^2) - \phi''(v_\varepsilon)|\nabla_2 w_\varepsilon|^2 u_\varepsilon^2 - 2\phi(v_\varepsilon)|\nabla_2 u_\varepsilon|^2| \\ &\quad \leqq c|\phi'(v_\varepsilon)u_\varepsilon|\,|\nabla_2 w_\varepsilon|\,|\nabla_2 u_\varepsilon| . \end{aligned}$$

Set

$$L_\varepsilon = \int y_2\psi(v_\varepsilon(x, 0, y_2))u_\varepsilon^2(x, 0, y_2)|\nabla_2 w_\varepsilon(x, 0, y_2)|^2 dx\,dy_2 .$$

By (4.1), (4.2), (3.9), (3.11), and Schwarz's inequality, we obtain

$$(4.3)\qquad J_\varepsilon \leqq c\left(\int \phi(v_\varepsilon(x, 0))u_\varepsilon^2(x, 0)dx + L_\varepsilon + J_\varepsilon^{1/2}L_\varepsilon^{1/2}\right).$$

By Lemma (3.5) and the definition of ψ, we have

$$L_\varepsilon \leqq c\alpha^2\int y_1|\nabla_2 w_\varepsilon(x, 0, y_2)|^2 dxdy_2 = c\alpha^2\int w_\varepsilon^2(x, 0)dx \ . \tag{4.4}$$

By (4.3) and (4.4), we obtain (3.18). This completes the proof of (3.18).

PROOF OF (3.19). By Green's theorem, Lemma (3.5) and the definition of ψ, we have

$$\begin{aligned}&\int y_1 y_2 \Delta_2(\psi(v_\varepsilon)|\nabla_1 w_\varepsilon|^2 u_\varepsilon^2)dxdy \\ &\quad\leqq \alpha^2\int y_1\psi(v_\varepsilon(x, y_1, 0))|\nabla_1 w_\varepsilon(x, y_1, 0)|^2 dxdy_1 \leqq c\alpha^2\int w_\varepsilon^2(x, 0)dx \ .\end{aligned}\tag{4.5}$$

On the other hand, we have

$$\begin{aligned}&|\Delta_2(\psi(v_\varepsilon)|\nabla_1 w_\varepsilon|^2 u_\varepsilon^2) - \psi''(v_\varepsilon)|\nabla_2 w_\varepsilon|^2|\nabla_1 w_\varepsilon|^2 u_\varepsilon^2 \\ &\qquad -2\psi(v_\varepsilon)|\nabla_1\nabla_2 w_\varepsilon|^2 u_\varepsilon^2 - 2\psi(v_\varepsilon)|\nabla_1 w_\varepsilon|^2|\nabla_2 u_\varepsilon|^2| \\ &\quad\leqq c(|\psi'(v_\varepsilon)u_\varepsilon^2|\,|\nabla_1 w_\varepsilon|\,|\nabla_2 w_\varepsilon|\,|\nabla_1\nabla_2 w_\varepsilon| \\ &\qquad + |\psi'(v_\varepsilon)u_\varepsilon|\,|\nabla_1 w_\varepsilon|^2|\nabla_2 w_\varepsilon|\,|\nabla_2 u_\varepsilon| \\ &\qquad + |\psi(v_\varepsilon)u_\varepsilon|\,|\nabla_1 w_\varepsilon|\,|\nabla_2 u_\varepsilon|\,|\nabla_1\nabla_2 w_\varepsilon|) \ .\end{aligned}\tag{4.6}$$

Let $M_\varepsilon = \int y_1 y_2|\nabla_1 w_\varepsilon|^2|\nabla_2 w_\varepsilon|^2 dx\,dy$, and $N_\varepsilon = \int y_1 y_2|\nabla_1\nabla_2 w_\varepsilon|^2 dx\,dy$. By (4.5), (4.6), Lemma (3.5), (3.10) and the definition of ψ, using Schwarz's inequality, we find

$$\begin{aligned}K_\varepsilon \leqq c\Big(&\alpha^2\int w_\varepsilon^2(x, 0)dx + \alpha^2 M_\varepsilon + \alpha^2 N_\varepsilon + \alpha^2 M_\varepsilon^{1/2}N_\varepsilon^{1/2} \\ &+\alpha K_\varepsilon^{1/2}M_\varepsilon^{1/2} + \alpha K_\varepsilon^{1/2}N_\varepsilon^{1/2}\Big) \ .\end{aligned}\tag{4.7}$$

We easily see that $N_\varepsilon = c\int w_\varepsilon^2(x, 0)dx$ and M_ε is bounded by $c\int w_\varepsilon^2(x, 0)dx$ (cf. [7]). Since K_ε is finite, by (4.7) we conclude that

$$K_\varepsilon \leqq c\alpha^2\int w_\varepsilon^2(x, 0)dx \ .$$

This completes the proof of (3.19).

REMARK (4.8). *Let* $G^{(i)}(x^{(i)}) = \exp(-|x^{(i)}|^2)$ and $H_j^{(i)}(x^{(i)}) = (\partial/\partial x_j^{(i)})G^{(i)}(x^{(i)})$, $j = 1, 2, \cdots, n_i$ $(i = 1, 2)$. For $(x, y) = (x^{(1)}, x^{(2)}; y_1, y_2) \in \boldsymbol{D}$, let

$$G_y(x) = y_1^{-n_1}y_2^{-n_2}G^{(1)}(x^{(1)}/y_1)G^{(2)}(x^{(2)}/y_2)$$

and

$$H_y^{(j,k)}(x) = y_1^{-n_1}y_2^{-n_2}H_j^{(1)}(x^{(1)}/y_1)H_k^{(2)}(x^{(2)}/y_2) \ .$$

For $f \in L^2(\boldsymbol{R}^N)$ and $(x, y) \in \boldsymbol{D}$, we set $F(x, y) = G_y * f(x)$, $K_{j,k}(x, y) = H_y^{(j,k)} * f(x)$. We define a maximal function f^* and a square function $S(f)$ for $f \in L^2(\boldsymbol{R}^N)$ by

$$f^*(x) = \sup\{|F(t, y)|: (t, y) \in \Gamma(x)\}$$

and

$$S(f)(x) = \left(\int_{\Gamma(x)} \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} |K_{j,k}(t, y)|^2 y_1^{-n_1-1} y_2^{-n_2-1} dt\, dy\right)^{1/2} .$$

We can prove the following theorem in the same way as Proposition (3.1).

THEOREM (4.9). *If $f \in L^2(\boldsymbol{R}^N)$, then we have*

$$|\{x \in \boldsymbol{R}^N: S(f)(x) > \alpha\}| \leqq c\alpha^{-2} \| f^* \wedge \alpha \|_2^2$$

for all $\alpha > 0$, where c is a constant independent of f and α.

5. Proof of Theoreom (2.5), II. In this section, we prove the implication $(3) \Rightarrow (1)$ in Theorem (2.5).

LEMMA (5.1). *Let $f \in L^p(\boldsymbol{R}^N)$ $(1 < p < \infty)$. Let $u(x, y) = P_y * f(x)$. Then we have*

$$\| N(u) \|_p \leqq c_p \| f \|_p ,$$

where c_p is a constant independent of f.

This is well-known. See [10] and [11].

LEMMA (5.2). *Let $F \in H_A^p(\boldsymbol{D})$ $(p_0 < p < \infty)$. We define a maximal function F^* by*

$$F^*(x) = \sup\{|F(t, y)|: (t, y) \in \Gamma(x)\} .$$

Then we have

$$\| F^* \|_p \leqq c_p \| F \|_p , \tag{5.3}$$

where c_p is a constant indendent of F.

PROOF. We first note that $|F|^{p_0}$ is bisubharmonic, i.e., subharmonic in each of the variables $(x^{(i)}, y_i)$ $(i = 1, 2)$. This follows from Stein [10, p. 217], because F is a system of conjugate biharmonic functions. Thus by the same argument as in Stein and Weiss [11, pp. 116–117], there exists $g \in L^q(\boldsymbol{R}^N)$ $(q = p/p_0)$ such that

$$|F(x, y)|^{p_0} \leqq P_y * g(x) \tag{5.4}$$

and

$$\| g \|_q^q \leqq c \| F \|_p^p . \tag{5.5}$$

Since $q = p/p_0 > 1$, by (5.4), (5.5) and Lemma (5.1), we have $\|F^*\|_p \leqq c_p \|F\|_p$. This completes the proof.

Now we prove the implication (3) $\Rightarrow$ (1). Suppose that $F = (u_{jk}) \in H_A^p(\boldsymbol{D})$, $u_{LM} = u$ ($L = n_1 + 1$, $M = n_2 + 1$) as in (3) of Theorem (2.5). By Lemma (5.2), we have $\|N(u)\|_p \leqq \|F^*\|_p \leqq c_p \|F\|_p$. This completes the proof.

6. Lemmas for biharmonic functions. In this section, we give six lemmas on biharmonic functions, which will be used in the proof of the implications (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) in Theorem (2.5). We omit the proofs, since we can prove these lemmas by the same argument as in the proofs of the corresponding one-variable results in [4], [10] and [11], or by repeated use of the one-variable results.

LEMMA (6.1) (cf. Stein [10, p. 90]). *Let u be a biharmonic function on $\boldsymbol{D}$. For $x \in \boldsymbol{R}^N$, let*

$$g(u)(x) = \left(\int_0^\infty \int_0^\infty |\nabla_1 \nabla_2 u(x, y)|^2 y_1 y_2 dy_1 dy_2\right)^{1/2} . \tag{6.2}$$

Suppose that $A_a(u)(x) < \infty$. Then we have

$$g(u)(x) \leqq c A_a(u)(x) ,$$

where c is a constant independent of u.

LEMMA (6.3) (cf. Fefferman and Stein [4, p. 173]). *Let u be a biharmonic function on $\boldsymbol{D}$ and let $0 < p < \infty$. Suppose that*

$$I_p(u) = \sup_{y_1, y_2 > 0} \left(\int_{\boldsymbol{R}^N} |u(x, y)|^p dx\right)^{1/p} < \infty . \tag{6.4}$$

Then we have

$$\sup_{x \in \boldsymbol{R}^N} |u(x, y)| \leqq c_p I_p(u) y_1^{-n_1/p} y_2^{-n_2/p} ,$$

where c_p is a constant independent of u and y.

LEMMA (6.5) (cf. Fefferman and Stein [4, p. 166]). *Let u be a biharmonic function on $\boldsymbol{D}$. For $\varepsilon > 0$ and $\delta > 0$, let $u_{\varepsilon,\delta}(x, y) = u(x, y_1 + \varepsilon, y_2 + \delta)$ ($(x, y) \in \boldsymbol{D}$). Let $a = (a_1, a_2)$, $b = (b_1, b_2)$. Suppose that $0 < a_1 < b_1$, $0 < a_2 < b_2$. Then if $A_b(u)(x) < \infty$, we have*

$$A_a(u_{\varepsilon,\delta})(x) \leqq c A_b(u)(x) ,$$

where c is a constant independent of u, ε and δ.

LEMMA (6.6) (cf. Stein [10], Stein and Weiss [11]). *Let u be a biharmonic function on $\boldsymbol{D}$. Suppose that*

$$\sup_{y_1, y_2 > 0} \int_{\boldsymbol{R}^N} |u(x, y)|^2 dx < \infty .$$

Then there exists $f \in L^2(\boldsymbol{R}^N)$ *such that* $u(x, y) = P_y * f(x)$ *for all* $(x, y) \in \boldsymbol{D}$.

LEMMA (6.7) (cf. Stein [10, p. 143]). *Let* u *be a biharmonic function on* $\boldsymbol{D}$. *Suppose that there exist positive constants* c_0, ε *and* δ *such that*

$$\sup_{x \in \boldsymbol{R}^N} |u(x, y)| \leqq c_0 y_1^{-\varepsilon} y_2^{-\delta}$$

for all $y_1, y_2 > 0$. *Then we have*

$$\sup_{x \in \boldsymbol{R}^N} |(\partial/\partial x_j^{(1)}) u(x, y)| \leqq c y_1^{-\varepsilon-1} y_2^{-\delta} , \tag{6.8}$$

$$\sup_{x \in \boldsymbol{R}^N} |(\partial/\partial x_k^{(2)}) u(x, y)| \leqq c y_1^{-\varepsilon} y_2^{-\delta-1} \tag{6.9}$$

for $j = 1, 2, \cdots, n_1 + 1$; $k = 1, 2, \cdots, n_2 + 1 (\partial/\partial x_{n_1+1}^{(1)} = \partial/\partial y_1, \partial/\partial x_{n_2+1}^{(2)} = \partial/\partial y_2)$ *and for all* $y_1, y_2 > 0$, *where* c *is a constant independent of* y_1 *and* y_2.

LEMMA (6.10) (cf. Stein [10, p. 213]). *Let* $u_{jk} (j = 1, 2, \cdots, n_1 + 1$; $k = 1, 2, \cdots, n_2 + 1)$ *be* $(n_1 + 1)(n_2 + 1)$ *biharmonic functions on* $\boldsymbol{D}$ *which satisfy the generalized Cauchy-Riemann equations* (2.1) *and* (2.2). Let $a = (a_1, a_2)$ *and* $b = (b_1, b_2)$ *be the same as in Lemma* (6.5). *Suppose that* $A_b(u_{LM})(x) < \infty$ $(L = n_1 + 1, M = n_2 + 1)$. *Then*

$$A_a(u_{jk})(x) \leqq c A_b(u_{LM})(x)$$

for $j = 1, 2, \cdots, n_1 + 1$ *and* $k = 1, 2, \cdots, n_2 + 1$, *where* c *is a constant independent of* u_{jk}.

7. Proof of Theorem (2.5), III. In this section, we prove the implication $(2) \Rightarrow (1)$ in Theorem (2.5). We first assume that u is the iterated Poisson integral of an L^2 function. For $j = 1, 2, \cdots, n_1 + 1$ and $k = 1, 2, \cdots, n_2 + 1$, we define $u_{jk}(x, y)$ by

$$u_{jk}(x, y) = \int_{y_1}^{\infty} \int_{y_2}^{\infty} (\partial^2/\partial x_j^{(1)} \partial x_k^{(2)}) u(x, h_1, h_2) dh_1 dh_2 , \tag{7.1}$$

where $\partial/\partial x_{n_1+1}^{(1)} = \partial/\partial h_1$ and $\partial/\partial x_{n_2+1}^{(2)} = \partial/\partial h_2$. It is easy to see that u_{jk} is the iterated Poisson integral of an L^2 function. Note that $u_{LM} = u$ $(L = n_1 + 1, M = n_2 + 1)$.

Let F be the $(n_1 + 1) \times (n_2 + 1)$ matrix-valued function whose (j, k)-component is given by $u_{jk}(x, y)$ for $j = 1, 2, \cdots, n_1 + 1$ and $k = 1, 2, \cdots, n_2 + 1$. By the definition of u_{jk}, it is obvious that F is a system of conjugate biharmonic functions. In order to prove that $F \in H_A^p(\boldsymbol{D})$ $(p_0 < p)$, we need the following lemma.

LEMMA (7.2). *Let $v(x, y)$ be the iterated Poisson integral of an L^2 function. Then we have, for $0 < p < \infty$,*

$$\sup_{y_1, y_2 > 0} \int_{R^N} |v(x, y)|^p dx \leqq c \| A_a(v) \|_p^p ,$$

where c is a constant independent of v.

We can prove this lemma as in [7, Lemm 1] using the one-variable result in [4]. We omit the proof.

Since u_{jk} is the iterated Poisson integral of an L^2 function, by Lemmas (6.10) and (7.2), we have that $F \in H_A^p(\boldsymbol{D})$ and $\| F \|_p \leqq c \| A_{(1/2,1/2)}(u) \|_p$. Thus by Lemma (5.2), we obtain

$$\| N(u) \|_p \leqq c_p \| A_{(1/2,1/2)}(u) \|_p \quad (p_0 < p) . \tag{7.3}$$

Now we remove the restriction that u is the iterated Poisson integral of an L^2 function. Let u be a biharmonic function on $\boldsymbol{D}$ such that $u(x, y) \to 0$ as $y_1 + y_2 \to \infty$ and $A(u) \in L^p(\boldsymbol{R}^N)$. For $\varepsilon > 0$ and $K > 0$ ($\varepsilon < K$), let

$$\begin{aligned} u_{\varepsilon K}(x, y) = {} & u(x, y_1 + \varepsilon, y_2 + \varepsilon) - u(x, y_1 + \varepsilon, y_2 + K) \\ & - u(x, y_1 + K, y_2 + \varepsilon) + u(x, y_1 + K, y_2 + K) . \end{aligned}$$

Then, by Schwarz's inequality and Lemma (6.1), we have

$$\begin{aligned} | u_{\varepsilon K}(x, y) | & \leqq \int_{y_1+\varepsilon}^{y_1+K} \int_{y_2+\varepsilon}^{y_2+K} |(\partial^2/\partial h_1 \partial h_2) u(x, h_1, h_2)| \, dh_1 dh_2 \\ & \leqq \log(K/\varepsilon) \Big(\int_0^\infty \int_0^\infty |(\partial^2/\partial h_1 \partial h_2) u|^2 h_1 h_2 dh_1 dh_2 \Big)^{1/2} \\ & = \log(K/\varepsilon) g(u)(x) \leqq c_{\varepsilon, K} A(u)(x) . \end{aligned}$$

Thus we have

$$\sup_{y_1, y_2 > 0} \int_{R^N} | u_{\varepsilon K}(x, y) |^p dx < \infty .$$

Since $p \leqq 2$, by Lemmas (6.3) and (6.6), we see that $u_{\varepsilon K}$ is the iterated Poisson integral of an L^2 function. By (7.3), we have

$$\| N(u_{\varepsilon K}) \|_p \leqq c \| A_{(1/2,1/2)}(u_{\varepsilon K}) \|_p .$$

Thus by Lemma (6.5), we obtain

$$\| N(u_{\varepsilon K}) \|_p \leqq c \| A_{(2/3,2/3)}(u) \|_p .$$

Since $u_{\varepsilon K}(x, y) \to u(x, y)$ as $\varepsilon \to 0$ and $K \to \infty$, we conclude, by Fatou's lemma,

$$\| N(u) \|_p \leqq \liminf_{\varepsilon \to 0, \ K \to \infty} \| N(u_{\varepsilon K}) \|_p \leqq c \| A_{(2/3,2/3)}(u) \|_p . \tag{7.4}$$

This completes the proof.

8. Proof of Theorem (2.5), IV. In this section, we prove the implication (2) $\Rightarrow$ (3) in Theorem (2.5). Suppose that $u(x, y) \to 0$ as $y_1 + y_2 \to \infty$ and $A(u) \in L^p(\boldsymbol{R}^N)$. we have already proved in §7 that $N(u) \in L^p(\boldsymbol{R}^N)$. Thus by Lemma (6.3), there are constants $\varepsilon > 0$ and $\delta > 0$ such that

$$\sup_{x \in \boldsymbol{R}^N} |u(x, y)| \leqq c y_1^{-\varepsilon} y_2^{-\delta} . \tag{8.1}$$

This allows us to define u_{jk} ($j = 1, 2, \cdots, n_1 + 1$; $k = 1, 2, \cdots, n_2 + 1$) by

$$u_{jk}(x, y) = \int_{y_1}^{\infty}\int_{y_2}^{\infty} (\partial^2/\partial x_j^{(1)} \partial x_k^{(2)}) u(x, h_1, h_2) dh_1 dh_2 \tag{8.2}$$

($\partial/\partial x_{n_1+1}^{(1)} = \partial/\partial h_1$, $\partial/\partial x_{n_2+1}^{(2)} = \partial/\partial h_2$), because the integral on the right hand side of (8.2) converges by Lemma (6.7) and (8.1). Note that $u_{LM} = u$ ($L = n_1 + 1$, $M = n_2 + 1$). Let F be the $(n_1 + 1) \times (n_2 + 1)$ matrix-valued function whose (j, k)-component is given by $u_{jk}(x, y)$ for $j = 1, 2, \cdots, n_1 + 1$ and $k = 1, 2, \cdots, n_2 + 1$. As in §7, it is easy to see that F is a system of conjugate biharmonic functions.

By the definition of u_{jk} in (8.2) and Lemma (6.10), we see that $u_{jk}(x, y) \to 0$ as $y_1 + y_2 \to \infty$ and

$$\| A_{(2/3, 2/3)}(u_{jk}) \|_p \leqq c \| A(u) \|_p .$$

By (7.4), we obtain

$$\| N(u_{jk}) \|_p \leqq c \| A_{(2/3, 2/3)}(u_{jk}) \|_p \leqq c \| A(u) \|_p .$$

Therefore

$$\sup_{y_1, y_2 > 0} \int_{\boldsymbol{R}^N} | F(x, y) |^p dx \leqq c \sum_{j=1}^{n_1+1} \sum_{k=1}^{n_2+1} \| N(u_{jk}) \|_p^p \leqq c \| A(u) \|_p^p .$$

Thus we have proved that $F \in H_A^p(\boldsymbol{D})$. This completes the proof.

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