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A CONDITION FOR ISOPARAMETRIC HYPERSURFACES OF S^n TO BE HOMOGENEOUS

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1. Introduction. Let M be a connected hypersurface of the n-dimensional sphere S^n of radius 1. $O(n+1)$ acts on S^n as an isometry group. M is said to be homogeneous if it is an orbit of a certain subgroup of $O(n+1)$. M is said to be isoparametric if it has constant principal curvatures. If M is homogeneous then it is isoparametric. E. Cartan investigated the converse problem and he gave an affirmative answer in some special cases ([2], [3], [4], [5]). But, recently, Ozeki and Takeuchi gave examples of isoparametric hypersurfaces which are not homogeneous in [8], using a result of Münzner [7]. On the other hand, homogeneous hypersurfaces of S^n are investigated in detail by Hsiang and Lawson [6] and by Takagi and Takahashi [10].

In the present paper, we give an additional differential geometric condition for isoparametric hypersurfaces of S^n to be homogeneous, using the result to Münzner. Our main results are the following Theorems A and B. To state them, we need some notations. Let $T_1, \cdots, T_r$ and T be tensor fields on a manifold. T is said to be generated by $T_1, \cdots, T_r$ if T is a constant linear combination of tensor fields, each of which is a tensor product of some members of $T_1, \cdots, T_r$ or its contraction. We denote this fact by $T = P(T_1, \cdots, T_r)$. Let M be a Riemannian manifold. Let M_p and M_q be the tangent spaces at $p, q \in M$. Then M_p and M_q are vector spaces with the inner products given by the Riemannian metric. A linear isometry L of M_p onto M_q is extended naturally to an isomorphism of the tensor algebra $T(M_p)$ onto $T(M_q)$, which is denoted also by L. For an oriented hypersurface M of S^n, we denote by G, H, ∇ and $\nabla^m H$ the first and second fundamental forms, the covariant differentiation and the m-th covariant differential, respectively. By G^{-1}, we denote the inner product for 1-forms on M induced naturally from G.

THEOREM A. *Let M be an oriented isoparametric hypersurface of S^n with g distinct principal curvatures. Then, for any $m \geqq g-1$, $\nabla^m H$*

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is generated by $G, G^{-1}, H, \nabla H, \cdots, \nabla^{g-2}H$.

THEOREM B. *Let* M *be a closed isoparametric hypersurface of* S^n *with* g *distinct principal curvatures. Then,* M *is homogeneous if and only if the following condition* (*) *is satisfied*:
(*) *For every* $p, q \in M$, *there exists a linear isometry* $L: M_p \to M_q$ *satisfying* $L(\nabla^m H)_p = (\nabla^m H)_q$ *for each* $m \leqq g-2$.

2. Proof of Theorem A. Let S^n be $\{p \in \boldsymbol{R}^{n+1} \mid \|p\| = 1\}$, M be an oriented hypersurface of S^n and N be a unit vector field on M normal to M and tangent to S^n at every point. Define $\phi: \boldsymbol{R} \times M \to S^n$ by $\phi(\theta, p) = (\cos\theta)p + (\sin\theta)N(p)$ and $\phi_\theta: M \to S^n$ by $\phi_\theta(p) = \phi(\theta, p)$, where we regard $N(p)$ to be in S^n. Let I be an open interval containing 0. To prove the theorem, we may assume I and M are sufficiently small so that $U = \phi(I \times M)$ is open in S^n and that $\phi: I \times M \to U$ is a diffeomorphism. Hence $\phi_\theta(M)$ is a hypersurface of S^n for $\theta \in I$. Define $\theta: U \to \boldsymbol{R}$ by $\theta(\phi(\delta, p)) = \delta$. Then $\phi_\theta(M)$ is a level hypersurface of the function θ. The vector field $N = \operatorname{grad}\theta$ on U is a unit vector field normal to each level hypersurface of θ and tangent to S^n at every point, and $N(\phi(\theta, p)) = -(\sin\theta)p + (\cos\theta)N(p)$ in $\boldsymbol{R}^{n+1}$. For brevity, we denote by $(\, , \,)$ or G the Riemannian metrics of M, S^n and $\boldsymbol{R}^{n+1}$. We denote by D and ∂ the covariant differentiations of S^n and $\boldsymbol{R}^{n+1}$, respectively. Then, $A = -DN$ gives a symmetric transformation of the tangent space U_p at $p \in U$ satisfying $AN = 0$. We call a vector or vector field X on U horizontal if $(X, N) = 0$.

LEMMA 1. *If* X *is horizontal, then* $(D_N A)X = A^2X + X$ *and moreover, for the* m*-th covariant differential* $D^m A$, *there exists a polynomial* $P_m(x)$ *satisfying* $((D^m A)(N, \cdots, N))X = P_m(A)X$.

PROOF. Let $X(p) \in M_p$ be an eigenvector of A with the eigenvalue $\lambda_0 = \cot\theta_0$, where $\theta_0 \in (0, \pi)$. Then, we have $\phi_\theta X(p) = (\sin(\theta_0 - \theta)/\sin\theta_0)X(p)$ in $\boldsymbol{R}^{n+1}$ and

$$A(\phi_\theta X(p)) = (\cot(\theta_0 - \theta))\phi_\theta X(p) \,. \tag{1.1}$$

Let X be a vector field defined by $X(\phi_\theta(p)) = \phi_\theta X(p)$. Then, we have $AX = (\cot(\theta_0 - \theta))X$ and $D_N X = \partial_N X = -(\cot(\theta_0 - \theta))X$, from which follows, $(D_N A)X = D_N(AX) - A(D_N X) = (\cot^2(\theta_0 - \theta) + 1)X = A^2X + X$. Hence, for any horizontal vector X, we have $(D_N A)X = A^2X + X$, since X is a linear combination of eigenvectors of A. Now, we note that the m-th order derivative of $\cot(\theta_0 - \theta)$ is a polynomial in $\cot(\theta_0 - \theta)$ and that $(D^m A)(N, \cdots, N) = D_N \cdots D_N A$ for each m. Then, the latter assertion is easily seen by induction on m. q.e.d.

A is called the Weingarten map when we regard A as a transformation of horizontal vectors.

LEMMA 2. *Let V and V' be open domains of M, and $\psi: V \to V'$ be an isometry which leaves the Weingarten map A invariant. Then, there exists a unique isometry $\Psi: S^n \to S^n$ satisfying $\Psi|_V = \psi$.*

PROOF. Let $\pi: U \to M$ be the projection defined by $\pi(\phi(\theta, p)) = p$. Define $\Psi: \pi^{-1}(V) \to \pi^{-1}(V')$ by $\Psi(\phi(\theta, p)) = \phi_\theta \circ \psi(p)$. Then $\Psi \circ \phi_\theta = \phi_\theta \circ \psi$ and $\psi \circ A = A \circ \psi$. If $X \in M_p$, then $\phi_\theta X = (\cos\theta)X - (\sin\theta)AX$ and $\phi_\theta \psi X = (\cos\theta)\psi X - (\sin\theta)\psi AX$ in $\boldsymbol{R}^{n+1}$. Hence $\|\phi_\theta X\| = \|\phi_\theta \circ \psi X\|$. On the other hand, we see $\Psi N = N$ by $\Psi \circ \phi_\theta = \phi_\theta \circ \psi$. Thus, $\Psi: \pi^{-1}(V) \to \pi^{-1}(V')$ is an isometry which is extended to an isometry Ψ of S^n, since S^n is simply connected. The uniqueness is obvious. q.e.d.

Now, we assume that M has g distinct constant principal curvatures, that is, Weingarten map A has g distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_g$ at each point, which are constant and have the same multiplicities on M. Let $\lambda_i = \cot\theta_i$, $0 < \theta_1 < \theta_2 < \cdots < \theta_g < \pi$ and m_i be the multiplicity of λ_i. Then, by (1.1), each level hypersurface of θ also has g distinct constant principal curvatures. Münzner proved the following Lemmas 3, 4 and 5 in [7].

LEMMA 3. (i) $\theta_i = \theta_1 + (i-1)\pi/g$, (ii) $m_i = m_{i+2}$, *where* $i + g \equiv i$.

LEMMA 4. *Define the function $f: U \to \boldsymbol{R}$ by $f(q) = \cos(g(\theta_1 - \theta(q)))$. Then $(\operatorname{grad} f, \operatorname{grad} f) = g^2(1 - f^2)$ and $\Delta f = -g(g + n - 1)f + c$, where Δ is the Laplace operator on S^n and $c = (m_2 - m_1)g^2/2$.*

LEMMA 5. *Let $\hat{U} = \{rp \in \boldsymbol{R}^{n+1} \mid r > 0,\ p \in U\}$. Define the function $F: \hat{U} \to \boldsymbol{R}$ by $F(rp) = r^g f(p)$. Then F is a homogeneous polynomial of degree g satisfying $\Delta F = cr^{g-2}$ and $(\operatorname{grad} F, \operatorname{grad} F) = g^2 r^{2g-2}$, where $c = (m_2 - m_1)g^2/2$ and Δ is the Laplace operator on $\boldsymbol{R}^{n+1}$.*

LEMMA 6. *Denote by X the vector field $x^0\partial/\partial x^0 + \cdots + x^n\partial/\partial x^n$ in $\boldsymbol{R}^{n+1}$. Then $\partial_X \partial^k F = (g - k)\partial^k F$, where $x^0, \cdots, x^n$ are Cartesian coordinates of $\boldsymbol{R}^{n+1}$ and $\partial^k F$ denotes the k-th covariant differential.*

PROOF. We note $\partial_X \partial_i = \partial_i \partial_X - \partial_i$, $\partial_i = \partial_{\partial/\partial x^i}$ and $\partial_X F = gF$, since F is a homogeneous polynomial of degree g. Then the lemma easily follows by induction on k. q.e.d.

LEMMA 7. *$D^{g+1}f$ is generated by $f, D^1f, D^2f, \cdots, D^{g-1}f$ and G, where $D^m f$ denotes the m-th covariant differential.*

PROOF. Let $S^n = \{(x^0, x^1, \cdots, x^n) \in \boldsymbol{R}^{n+1} \mid \sum_{i=0}^n (x^i)^2 = 1\}$. We may regard

$(x^1, \cdots, x^n)$ as a local coordinate system around $p \in U \subset S^n$. Then the function f is given by the function F as follows: $f(x^1, \cdots, x^n) = F((1 - (x^1)^2 - \cdots - (x^n)^2)^{1/2}, x^1, \cdots, x^n)$. x^0 is a function given by $x^0(x^1, \cdots, x^n) = (1 - (x^1)^2 - \cdots - (x^n)^2)^{1/2}$. Here, we need a notation. Let T be a covariant tensor field of degree k on S^n. We denote by $T(i_k \cdots i_2 i_1)$ the component $T_{i_k \cdots i_2 i_1} = T(\partial/\partial x^{i_k}, \cdots, \partial/\partial x^{i_2}, \partial/\partial x^{i_1})$ with respect to the basis $\partial/\partial x^1, \cdots, \partial/\partial x^n$. Then we have

$$(1.2) \quad G(ji) = \delta_{ji} + x^j x^i/(x^0)^2 , \quad G^{-1}(dx^j, dx^i) = \delta_{ji} - x^j x^i , \quad D_{\partial/\partial x^j}(\partial/\partial x^i) = \sum_{h=1}^{n} x^h G(ji)\partial/\partial x^h , \quad (Dx^0)(i) = -x^i/x^0 , \quad (D^2x^0)(ji) = -x^0 G(ji) .$$

We use the same notation as above to denote a component of a covariant tensor field T on $\boldsymbol{R}^{n+1}$ with respect to the basis $\partial/\partial x^0, \partial/\partial x^1, \cdots, \partial/\partial x^n$. In this case, $(\partial_Y T)(i_k \cdots i_2 i_1) = \partial_Y (T(i_k \cdots i_2 i_1))$ for any vector field Y, which may be written as $\partial_Y T\ (i_k \cdots i_2 i_1)$. We denote $\partial_{\partial/\partial x^j} T$ by $\partial_j T$. Then, $\partial_j T\ (i_k \cdots i_2 i_1) = \partial T\ (j i_k \cdots i_2 i_1)$. We note $\partial^m T\ (j_m \cdots j_1; i_k \cdots i_1)$ is symmetric in every pair of indices $j_m, \cdots, j_1$.

By Lemma 5, it is sufficient to prove

$$(1.3) \quad (D^k f)(i_k \cdots i_1) = \sum_{s=0}^{k} \sum_{\sigma} \partial^k F(\underbrace{0 \cdots 0}_{s} i_{\sigma(k)} \cdots i_{\sigma(s+1)}) x^{i_{\sigma(s)}} \cdots x^{i_{\sigma(1)}}/(-x^0)^s + P(f, D^1 f, \cdots, D^{k-2} f, G)(i_k \cdots i_1) ,$$

where σ runs through the permutations of order k satisfying $\sigma(k) > \cdots > \sigma(s+1)$ and $\sigma(s) > \cdots > \sigma(1)$. We prove it by induction on k. It is trivial for $k = 1$. Hence, we assume (1.3) for $1, \cdots, k$. Then, by (1.2), we have

$$(D^{k+1} f)(i_{k+1} i_k \cdots i_1) = \partial((D^k f)(i_k \cdots i_1))/\partial x^{i_{k+1}} - \sum_{t=1}^{k} \sum_{u=1}^{n} x^u G(i_{k+1} i_t)(D^k f)(i_k \cdots i_{t+1} u i_{t-1} \cdots i_1) = \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)} ,$$

where

$$\text{(I)} = \sum_{s=0}^{k} \sum_{\sigma} [\partial^{k+1} F(\underbrace{0 \cdots 0}_{s} i_{k+1} i_{\sigma(k)} \cdots i_{\sigma(s+1)}) - \partial^{k+1} F(\underbrace{00 \cdots 0}_{s+1} i_{\sigma(k)} \cdots i_{\sigma(s+1)}) x^{i_{k+1}}/x^0] \times x^{i_{\sigma(s)}} \cdots x^{i_{\sigma(1)}}/(-x^0)^s ,$$

$$\text{(II)} = \sum_{s=1}^{k} \sum_{\sigma} \partial^k F(\underbrace{0 \cdots 0}_{s} i_{\sigma(k)} \cdots i_{\sigma(s+1)}) \times \sum_{t=1}^{s} [\partial(-x^{i_{\sigma(t)}}/x^0)/\partial x^{i_{k+1}}] \times x^{i_{\sigma(s)}} \cdots x^{i_{\sigma(t+1)}} x^{i_{\sigma(t-1)}} \cdots x^{i_{\sigma(1)}}/(-x^0)^{s-1} ,$$

$$\text{(III)} = -\sum_{s=0}^{k-1}\sum_{\sigma}\sum_{t=s+1}^{k}\sum_{u=1}^{n} x^u G(i_{k+1}i_{\sigma(t)})$$
$$\times \partial^k F(\underbrace{0\cdots 0}_{s} i_{\sigma(k)}\cdots i_{\sigma(t+1)} u i_{\sigma(t-1)}\cdots i_{\sigma(s+1)}) \times x^{i_{\sigma(s)}}\cdots x^{i_{\sigma(1)}}/(-x^0)^s ,$$

$$\text{(IV)} = -\sum_{s=1}^{k}\sum_{\sigma}\partial^k F(\underbrace{0\cdots 0}_{s} i_{\sigma(k)}\cdots i_{\sigma(s+1)} \times \sum_{t=1}^{s}\sum_{u=1}^{n} x^u G(i_{k+1}i_{\sigma(t)})(-x^u/x^0)$$
$$\times x^{i_{\sigma(s)}}\cdots x^{i_{\sigma(t+1)}}x^{i_{\sigma(t-1)}}\cdots x^{i_{\sigma(1)}}/(-x^0)^{s-1} ,$$

$$\text{(V)} = P(f, D^1 f, \cdots, D^{k-1} f, G)(i_{k+1}i_k\cdots i_1) .$$

Then,

$$\text{(I)} = \sum_{s=0}^{k+1}\sum_{\tau}\partial^{k+1}F(\underbrace{0\cdots 0}_{s} i_{\tau(k+1)}i_{\tau(k)}\cdots i_{\tau(s+1)}) \times x^{i_{\tau(s)}}\cdots x^{i_{\tau(1)}}/(-x^0)^s ,$$

where $\tau(k+1) > \tau(k) > \cdots > \tau(s+1)$ and $\tau(s) > \cdots > \tau(1)$. By (1.2),

$$\text{(II)} + \text{(IV)} = \sum_{s=1}^{k}\sum_{\sigma}\partial^k F(\underbrace{0\cdots 0}_{s} i_{\sigma(k)}\cdots i_{\sigma(s+1)}) \times \sum_{t=1}^{s}(D^2 x^0)(i_{k+1}i_{\sigma(t)})$$
$$\times x^{i_{\sigma(s)}}\cdots x^{i_{\sigma(t+1)}}x^{i_{\sigma(t-1)}}\cdots x^{i_{\sigma(1)}}/(-x^0)^{s-1}$$
$$= -\sum_{s=1}^{k}\sum_{\sigma} x^0\partial_0\partial^{k-1}F(\underbrace{0\cdots 0}_{s-1} i_{\sigma(k)}\cdots i_{\sigma(s+1)})$$
$$\times \sum_{t=1}^{s}[x^{i_{\sigma(s)}}\cdots x^{i_{\sigma(t+1)}}x^{i_{\sigma(t-1)}}\cdots x^{i_{\sigma(1)}}/(-x^0)^{s-1}] \times G(i_{k+1}i_{\sigma(t)}) .$$

$$\text{(III)} = -\sum_{s=1}^{k}\sum_{\rho}\sum_{r=s}^{k}\sum_{u=1}^{n} x^u\partial_u\partial^{k-1}F(\underbrace{0\cdots 0}_{s-1} i_{\rho(k)}\cdots i_{\rho(r+1)}i_{\rho(r-1)}\cdots i_{\rho(s)})$$
$$\times [x^{i_{\rho(s-1)}}\cdots x^{i_{\rho(1)}}/(-x^0)^{s-1}]G(i_{k+1}i_{\rho(r)}) ,$$

where $\rho(k) > \cdots > \rho(s)$ and $\rho(s-1) > \cdots > \rho(1)$. Let X be as in Lemma 6. Then, (II) + (IV) + (III) is equal to

$$-\sum_{s=1}^{k}\sum_{\rho}\sum_{r=s}^{k}\partial_X\partial^{k-1}F(\underbrace{0\cdots 0}_{s-1} i_{\rho(k)}\cdots i_{\rho(r+1)}i_{\rho(r-1)}\cdots i_{\rho(s)})$$
$$\times [x^{i_{\rho(s-1)}}\cdots x^{i_{\rho(1)}}/(-x^0)^{s-1}]G(i_{k+1}i_{\rho(r)}) .$$

By Lemma 6, (II) + (III) + (IV)

$$= -\sum_{s=1}^{k}\sum_{\rho}\sum_{r=s}^{k}(g-k+1)\partial^{k-1}F(\underbrace{0\cdots 0}_{s-1} i_{\rho(k)}\cdots i_{\rho(r+1)}i_{\rho(r-1)}\cdots i_{\rho(s)})$$
$$\times [x^{i_{\rho(s-1)}}\cdots x^{i_{\rho(1)}}/(-x^0)^{s-1}]G(i_{k+1}i_{\rho(r)})$$
$$= -(g-k+1)\sum_{t=1}^{k}\left[\sum_{s=0}^{k-1}\sum_{\delta}\partial^{k-1}F(\underbrace{0\cdots 0}_{s} i_{\delta\gamma(k-1)}\cdots i_{\delta\gamma(s+1)})\right.$$
$$\left.\times x^{i_{\delta\gamma(s)}}\cdots x^{i_{\delta\gamma(1)}}/(-x^0)^s\right]G(i_{k+1}i_t) ,$$

where $\gamma(s) = s$ for $s = 1, \cdots, t-1$, $\gamma(s) = s+1$ for $s = t, \cdots, k-1$ and δ runs through the permutations of $\{\gamma(1), \cdots, \gamma(k-1)\}$ satisfying $\delta\gamma(k-1) > \cdots > \delta\gamma(s+1)$ and $\delta\gamma(s) > \cdots > \delta\gamma(1)$. By the induction hypothesis, we have (II) + (III) + (IV)

$$= -(g-k+1)\sum_{t=1}^{k}(D^{k-1}f)(i_k \cdots i_{t+1}i_{t-1} \cdots i_1)G(i_{k+1}i_t)$$
$$+ P(f, D^1 f, \cdots, D^{k-3}f, G)(i_{k+1}i_k \cdots i_1) .$$

This completes the proof. q.e.d.

We denote also by H the covariant tensor field of degree 2 on U defined by $H(X, Y) = (AX, Y)$.

LEMMA 8. *$D^{g-1}H$ is generated by G, $D\theta$, H, DH, $\cdots$ and $D^{g-2}H$ along each level hypersurface of θ.*

PROOF. We note $H = -D^2\theta$, $f = \cos(g(\theta_1 - \theta))$ and $\theta = \theta_1 - (1/g)\cos^{-1} f$ on U. Hence we have

$$D^m f = (df/d\theta)D^m\theta + \cdots + (d^m f/d\theta^m)(D\theta)^m$$

for every m. Conversely, we have

$$-D^{g-1}H = D^{g+1}\theta = (d\theta/df)D^{g+1}f + \cdots + (d^{g+1}\theta/df^{g+1})(Df)^{g+1} .$$

Then, by Lemma 7, we have the assertion. q.e.d.

We denote by $\mathscr{F}$ the set of all C^∞ functions on U and by $\mathscr{X}_H$ the set of all C^∞ horizontal vector fields on U. For $X, Y \in \mathscr{X}_H$, we denote by $\nabla_X Y$ the horizontal part of $D_X Y$. Then $D_X Y = \nabla_X Y + H(X, Y)N$. Along M, this ∇ coincides with the covariant differentiation. Every C^∞ covariant tensor field T of degree k on U is regarded as a field $\mathscr{X}_H \times \cdots \times \mathscr{X}_H \to \mathscr{F}$, which is denoted also by T. We define a field $\nabla T: \mathscr{X}_H \times \cdots \times \mathscr{X}_H \to \mathscr{F}$ by

$$(\nabla T)(X_{k+1}, X_k, \cdots, X_1) = X_{k+1}(T(X_k, \cdots, X_1)) - \sum_{i=1}^{k} T(X_k, \cdots, \nabla_{X_{k+1}}X_i, \cdots, X_1) .$$

Let us consider a field $T: \mathscr{X}_H \times \cdots \times \mathscr{X}_H \to \mathscr{F}$ defined as follows:

$$T(X_k, \cdots, X_i, \cdots, X_1) = (D^m H)(\cdots, X_k, \cdots, N, \cdots, X_i, \cdots, N, \cdots, X_1, \cdots) ,$$

where N appears $m-k+2$ times in $(D^m H)(\cdots\cdots)$. We call such a T fundamental field of type (m, k).

LEMMA 9. *Let T be a fundamental field of type (m, k). Then, T is generated by G, G^{-1}, H, ∇H, $\cdots$, $\nabla^{k-3}H$ and $\nabla^{k-2}H$.*

PROOF. Let T_1, T_2 and T_3 be fundamental fields defined by

$$T_1(X_2, X_1) = (D^m H)(N, \cdots, N; X_2, X_1)$$
$$T_2(X_1) = (D^m H)(N, \cdots, N; N, X_1)$$
$$T_3 = (D^m H)(N, \cdots, N; N, N) .$$

Then, by Lemma 1, $T_1 = P(H, G, G^{-1})$, $T_2 = 0$ and $T_3 = 0$. Let S and S' be fundamental fields of type (m, k) defined by

$$S(X_k, \cdots, X_i, \cdots X_1) = (D^m H)(\cdots, N, X_i, \cdots)$$
$$S'(X_k, \cdots, X_i \cdots, X_1) = (D^m H)(\cdots, X_i, N, \cdots) ,$$

where we transposed only N and X_i in $(D^m H)(\cdots\cdots)$. Then, by the Ricci formula, $S' - S$ is generated by G and fundamental fields of types $(m-2, j)$, $j \leqq k$, as S^n has the constant curvature. Repeating the transpositions as above, we arrive at one of the fundamental fields T, T' and T'' defined as follows:

$$T(X_k, \cdots, X_1) = (D^m H)(X_k, \cdots, X_3, N, \cdots, N; X_2, X_1)$$
$$T'(X_k, \cdots, X_1) = (D^m H)(X_k, \cdots, X_2, N, \cdots, N; N, X_1)$$
$$T''(X_k, \cdots, X_1) = (D^m H)(X_k, \cdots, X_1, N, \cdots, N; N, N) .$$

$T - S$, $T' - S$ or $T'' - S$ is generated by G and fundamental fields of types $(m-2, j)$, $j \leqq k$. Hence it is sufficient to prove the assertion only for the fields T, T' and T''. We prove it by induction on m. We assume it is valid for $0, 1, \cdots, m$. Let T be a fundamental field of type $(m+1, k+1)$ defined by

$$T(X_{k+1}, X_k, \cdots, X_1) = (D^{m+1} H)(X_{k+1}, X_k, \cdots, X_3, N, \cdots, N; X_2, X_1) .$$

But the right hand term is written as follows:

$$\begin{aligned}
& X_{k+1}((D^m H)(X_k, \cdots, X_3, N, \cdots, N; X_2, X_1)) \\
& \quad - \sum_{i=3}^{k} (D^m H)(X_k, \cdots, D_{X_{k+1}} X_i, \cdots, X_3, N, \cdots, N; X_2, X_1) \\
& \quad - \sum (D^m H)(X_k, \cdots, X_3, N, \cdots, D_{X_{k+1}} N, \cdots, N; X_2, X_1) \\
& \quad - (D^m H)(X_k, \cdots, X_3, N, \cdots, N; D_{X_{k+1}} X_2, X_1) \\
& \quad - (D^m H)(X_k, \cdots, X_3, N, \cdots, N; X_2, D_{X_{k+1}} X_1) \\
& = X_{k+1}((D^m H)(X_k, \cdots, X_3, N, \cdots, N; X_2, X_1)) \\
& \quad - \sum_{i=3}^{k} (D^m H)(X_k, \cdots, \nabla_{X_{k+1}} X_i, \cdots, X_3, N, \cdots, N; X_2, X_1) \\
& \quad - (D^m H)(X_k, \cdots, X_3, N, \cdots, N; \nabla_{X_{k+1}} X_2, X_1) \\
& \quad - (D^m H)(X_k, \cdots, X_3, N, \cdots, N; X_2, \nabla_{X_{k+1}} X_1) \\
& \quad - \sum_{i=3}^{k} H(X_{k+1}, X_i)(D^m H)(X_k, \cdots, N, \cdots, X_3, N, \cdots, N; X_2, X_1)
\end{aligned}$$

$$+ \sum (D^m H)(X_k, \cdots, X_3, N, \cdots, AX_{k+1}, \cdots, N; X_2, X_1)$$
$$- H(X_{k+1}, X_2)(D^m H)(X_k, \cdots, X_3, N, \cdots, N; N, X_1)$$
$$- H(X_{k+1}, X_1)(D^m H)(X_k, \cdots, X_3, N, \cdots, N; X_2, N) .$$

The above equality says $T = \nabla S + R$, where S is a fundamental field of type (m, k) defined by $S(X_k, \cdots, X_1) = (D^m H)(X_k, \cdots, X_3, N, \cdots, N; X_2, X_1)$ and R is a field generated by fundamental fields of types (n, j), where $n \leqq m$ and $j \leqq k+1$. Hence, by assumption, $T = P(G, G^{-1}, H, \cdots, \nabla^{k-1}H)$. The proofs for T' and T'' are similar to the above. So we omit them, noting

$$(D^m H)(N, \cdots, N, X, N, \cdots, N; N, N) = 0 \quad \text{for} \quad X \in \mathscr{X}_H . \qquad \text{q.e.d.}$$

LEMMA 10. *Regard $D^m H$ as a fundamental field of type $(m, m+2)$. Then $D^m H = \nabla^m H + R$, where R is a field generated by $G, G^{-1}, H, \nabla H, \cdots, \nabla^{m-2}H$.*

PROOF. Note that

$$\begin{aligned}(D^m H)&(X_{m+2}, \cdots, X_3; X_2, X_1)\\ &= (\nabla D^{m-1}H)(X_{m+2}, \cdots, X_3; X_2, X_1)\\ &\quad + \sum_{i=1}^{m+1} H(X_{m+2}, X_i)(D^{m-1}H)(X_{m+1}, \cdots, X_{i+1}, N, X_{i-1}, \cdots, X_3; X_2, X_1) .\end{aligned}$$

Then, we get the assertion by Lemma 9 and induction on m. q.e.d.

By Lemmas 8 and 10, we complete the proof of Theorem A.

3. Proof of Theorem B. First, we note the following result due to Münzner ([7]).

LEMMA 11. *Let M be a connected closed isoparametric hypersurface of S^n with g distinct principal curvatures. Then, the function f in Lemma 4 is extended to a unique analytic function on S^n denoted also by f such that $M = f^{-1}(t_1)$, $t_1 = \cos(g\theta_1) \in (-1, 1)$.*

REMARK. (i) In particular, M is oriented by

$$N = g^{-1}(1 - f^2)^{-1/2} \operatorname{grad} f .$$

Hence, we can define $H, A, \cdots$ over M.

(ii) Let $\phi\colon \boldsymbol{R} \times M \to S^n$ be as in Section 2. Define $\Phi\colon (-1, 1) \times M \to S^n$ by $\Phi(t, p) = \phi((\cos^{-1} t_1 - \cos^{-1} t)/g, p)$. Then, $f(\Phi(t, p)) = t$, $U = \Phi((-1, 1) \times M)$ is open in S^n and $\Phi\colon (-1, 1) \times M \to U$ is a diffeomorphism.

Let $\pi\colon \tilde{M} \to M$ be the universal covering. Then, by the pull back, $\tilde{M}$ has the structure $\tilde{G}, \tilde{H}, \tilde{A}, \tilde{\nabla}, \cdots$ which are briefly denoted also by $G, H, A, \nabla, \cdots$.

LEMMA 12. *Under the condition* (*) *in Theorem B,* $\tilde{M}$ *admits a transitive group of isometries leaving* A *invariant.*

PROOF. Since our proof is quite similar to that in Singer [9] and Ambrose and Singer [1], we only give a sketch. Let B be the orthonormal frame bundle over $\tilde{M}$. Let $j_g, \cdots, j_1$ be a sequence of g integers, where $1 \leqq j_g, \cdots, j_1 \leqq n-1$. Let $\rho[j_g \cdots j_1]: B \to \boldsymbol{R}^{g-1}$ be a mapping defined by

$$(\rho[j_g \cdots j_1])(b) = (H(j_2 j_1), \nabla H(j_3 j_2 j_1), \cdots, \nabla^{g-2} H(j_g \cdots j_3 j_2 j_1)) ,$$

where $b = (q; Y_1, \cdots, Y_{n-1})$ and $\nabla^{m-2} H(j_m \cdots j_3 j_2 j_1) = (\nabla^{m-2} H)(Y_{j_m}, \cdots, Y_{j_3}; Y_{j_2}, Y_{j_1})$. Let $\rho = \{\rho[j_g \cdots j_1]\}$ be the finite sequence of all such $\rho[j_g \cdots j_1]$'s. Then ρ can be regarded as a mapping of B to $\boldsymbol{R}^{g-1} + \cdots + \boldsymbol{R}^{g-1}$. Let $\bar{C} = \{b \in B \mid \rho(b) = \rho(a)\}$, where $a = (p; X_1, \cdots, X_{n-1})$ is a fixed element of B satisfying $H(X_j, X_i) = \lambda_j \delta_{ji}$. Let C be the component of $\bar{C}$ containing a. Under the condition (*), C is a subbundle of B with the structure group K, where K is the component of the group $\bar{K} = \{h \in O(n-1) | \rho(ah) = \rho(a)\}$ containing the identity. Let (ω_{ij}) and (ω_i) be the Riemannian connection form and the canonical form. Let E_i and E_{ij} be the vector fields dual to ω_i and ω_{ij}. Let $\mathfrak{o}(n-1)$ and $\mathfrak{k}$ be the Lie algebra of $O(n-1)$ and K. A bi-invariant metric of $O(n-1)$ gives the orthogonal decomposition $\mathfrak{o}(n-1) = \mathfrak{k} + \mathfrak{m}$. Let γ be the orthogonal projection of $\mathfrak{o}(n-1)$ onto $\mathfrak{k}$, $(\phi_{ij}) = \gamma(\omega_{ij})$ and $\tau_{ij} = \phi_{ij} - \omega_{ij}$. Then (ϕ_{ij}) defines a connection form of C and $\tau_{ij}(E_k)$ is constant on C for each i, j and k. Here, we used the fact that $d\rho(E_k(b))$ has the expression

$$\{\cdots; \nabla H(k j_2 j_1), \cdots, \nabla^{g-1} H(k j_g \cdots j_2 j_1); \cdots\} ,$$

which is the same for every $b \in C$, by Theorem A. Then, on C, $d\omega_i$ and $d\phi_{ij}$ are constant linear combinations of wedge products of ω_i and ϕ_{ij}. Here, we note that the curvature form (Ω_{ij}) of the Riemannian connection is written as $\Omega_{ij} = (\lambda_i \lambda_j + 1)(\omega_i \wedge \omega_j)$ on C. Though C is not always simply connected, it has the group structure such that $\{\omega_i\}$ and some members of $\{\phi_{ij}\}$ give the Maurer Cartan form, and C acts on $\tilde{M}$ as transitive group of isometries; Frame $b = (q; Y_1, \cdots, Y_{n-1})$ corresponds to an isometry ψ such that $\psi(X_i) = Y_i$, where $a = (p; X_1, \cdots, Y_{n-1})$ is the fixed frame. Hence it is obvious that $\psi \circ A = A \circ \psi$. q.e.d.

By Lemma 12, M is locally homogeneous, that is, for every $p, q \in M$, there exist neighborhoods V and V' of p and q in M, respectively and an isometry $\psi: V \to V'$ leaving A invariant. Then, by Lemma 2, there exists an isometry $\Psi: S^n \to S^n$ such that $\Psi|_V = \psi$. To prove Theorem B, it is sufficient to show $\Psi(M) = M$. But it is obvious by Lemma 11 and the fact that M and $\Psi(M)$ are closed isoparametric hypersurfaces and that $M\cap\Psi(M)$ contains the open subset V'. The necessity of (*) is also obvious.

REMARK. Münzner ([7]) proved that, for a connected isoparametric hypersurface of S^n, g is 1, 2, 3, 4 or 6.

REFERENCES

[1] W. AMBROSE AND I. M. SINGER, On homogeneous Riemannian manifolds, Duke Math. J. 25 (1958), 647–669.

[2] E. CARTAN, Familles de surfaces isoparamétriques dans les espaces à courbure constante, Ann. di Mat. 17 (1938), 177–191.

[3] E. CARTAN, Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques, Math. Zeit. 45 (1939), 335–367.

[4] E. CARTAN, Sur quelques familles remarquables d'hypersurfaces, C. R. Congrès Math. Liège, 1939, 30–41.

[5] E. CARTAN, Sur des familles remarquables d'hypersurfaces isoparamétriques sphériques à 5 et 9 dimensions, Atti Fond. Volta 9 (1939), 443–461.

[6] W. Y. HSIANG AND H. B. LAWSON, Minimal submanifolds of low cohomogeneity, J. Diff. Geom. 5 (1971), 1–38.

[7] H. F. MÜNZNER, Isoparametrische Hyperflächen in Sphären, I, II, Math. Ann. 251 (1980), 57–71; 256 (1981), 215–232.

[8] H. OZEKI AND M. TAKEUCHI, On some types of isoparametric hypersurfaces in spheres, I, II, Tôhoku Math. J. 27 (1975), 515–559; 28 (1976), 7–55.

[9] I. M. SINGER, Infinitesimally homogeneous spaces, Comm. Pure Appl. Math. 13 (1960), 685–697.

[10] R. TAKAGI AND T. TAKAHASHI, On the principal curvatures of homogeneous hypersurfaces in a sphere, Diff. Geom. in honor of K. Yano, Kinokuniya, Tokyo, 1972, 469–481.

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