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BANACH ALGEBRA RELATED TO DISK POLYNOMIALS

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Introduction. Let $\alpha \geqq 0$, and let m and n be nonnegative integers. *Disk polynomials* $R_{m,n}^{(\alpha)}(z)$ are defined by

$$R_{m,n}^{(\alpha)}(z) = \begin{cases} R_n^{(\alpha,m-n)}(2r^2-1)e^{i(m-n)\phi}r^{m-n} & \text{if } m \geqq n\,, \\ R_m^{(\alpha,n-m)}(2r^2-1)e^{i(m-n)\phi}r^{n-m} & \text{if } m < n\,, \end{cases}$$

where $z = re^{i\phi}$ and $R_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n and of order (α, β) normalized so that $R_n^{(\alpha,\beta)}(1) = 1$.

Denote by $A^{(\alpha)}$ the space of absolutely convergent disk polynomial series on the closed unit disk $\bar{D}$ in the complex plane, that is, the space of functions f on $\bar{D}$ such that

$$f(z) = \sum_{m,n=0}^{\infty} a_{m,n} R_{m,n}^{(\alpha)}(z) \quad \text{with} \quad \sum_{m,n=0}^{\infty} |a_{m,n}| < \infty\,,$$

and introduce a norm in $A^{(\alpha)}$ by

$$\|f\| = \sum_{m,n=0}^{\infty} |a_{m,n}|\,.$$

The space $A^{(\alpha)}$ consists of continuous functions on $\bar{D}$, since if $\sum |a_{m,n}| < \infty$ then the series $\sum a_{m,n}R_{m,n}^{(\alpha)}(z)$ converges uniformly on $\bar{D}$ by the inequality;

(1) $\qquad |R_{m,n}^{(\alpha)}(z)| \leqq 1$ on $\bar{D}$ $\qquad$ (Koornwinder [5; (5.1)]) .

Our purpose is to study some structure of the algebra $A^{(\alpha)}$.

Let $A^{(\alpha,\beta)}$ be the space of absolutely convergent Jacobi polynomial series $f(x) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\beta)}(x)$, $\sum_{n=0}^{\infty} |a_n| < \infty$ on the closed interval $[-1, 1]$. The space $A^{(\alpha,\beta)}$ has the structure of a Banach algebra with pointwise multiplication of functions. This is proved by the nonnegativity of the linearization coefficients of products of Jacobi polynomials (see Gasper [2]) Igari and Uno [3] and Cazzaniga and Meaney [1] studied some structure of the algebra $A^{(\alpha,\beta)}$, that is, the maximal ideal space, Helson sets, spectral synthesis, etc. For the space $A^{(\alpha)}$, we will consider some of these problems. In §§1 and 2, we will show that $A^{(\alpha)}$ is a Banach algebra by the nonnegativity of the linearization coefficients of products of disk polynomials that is proved by Koornwinder [6], and then determine the maximal ideal space of $A^{(\alpha)}$. Moreover, we will show that if $\alpha \geqq 1$ and

z_0 is in the open unit disk D then the singleton $\{z_0\}$ is not a set of spectral synthesis for $A^{(\alpha)}$. In § 3, we will give a characterization of a set of interpolation with respect to $A^{(\alpha)}$, $\alpha > 0$. The structure of $A^{(\alpha)}$ seems simpler than that of the algebra of absolutely convergent Fourier series and is similar to that of the algebra $A^{(\alpha,\beta)}$, but we will use sharper asymptotic formulas and apply delicate calculus.

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1. The Banach algebra $A^{(\alpha)}$. First we mention some properties of disk polynomials $R^{(\alpha)}_{m,n}(z)$ (cf. [5], [6]):

(i) $R^{(\alpha)}_{m,n}(z)$ is a polynomial of degree $m + n$ in x and y, where $z = x + iy$.

(ii) Let m_α be the probability measure on $\bar{D}$ defined by

$$dm_\alpha(z) = \frac{\alpha + 1}{\pi}(1 - x^2 - y^2)^\alpha dxdy .$$

Then $\{R^{(\alpha)}_{m,n}\}^\infty_{m,n=0}$ is a complete orthogonal system in $L^2(\bar{D}, m_\alpha)$, that is,

$$\int_{\bar{D}} R^{(\alpha)}_{m,n}(z)R^{(\alpha)}_{k,l}(\bar{z})\ dm_\alpha(z) = h^{(\alpha)-1}_{m,n}\delta_{mk}\delta_{nl} ,$$

where

$$h^{(\alpha)}_{m,n} = \frac{(m + n + \alpha + 1)\Gamma(m + \alpha + 1)\Gamma(n + \alpha + 1)}{(\alpha + 1)\Gamma(\alpha + 1)^2\Gamma(m + 1)\Gamma(n + 1)} ,$$

$\bar{z} = x - iy$ and δ_{mk} is Kronecker's symbol. Moreover, $\hat{f}(m, n) = 0$ for all m, n implies $f = 0$, where

$$\hat{f}(m, n) = \int_{\bar{D}} f(z)R^{(\alpha)}_{m,n}(\bar{z})dm_\alpha(z) .$$

(iii) The linearization coefficients of products are nonnegative, that is,

$$R^{(\alpha)}_{m,n}(z)R^{(\alpha)}_{k,l}(z) = \sum_{p,q} c_{p,q}(m, n; k, l)h^{(\alpha)}_{p,q}R^{(\alpha)}_{p,q}(z)$$

with $c_{p,q}(m, n; p, q) \geqq 0$ [6; Corollary 5.2].

(iv) If $\alpha=0, 1, 2, \cdots$, then disk polynomials are the spherical functions on the sphere $S^{2\alpha+3}$ considered as the homogeneous space $U(\alpha+2)/U(\alpha+1)$.

Let l^1 be the Banach space of absolutely convergent double sequences $b = \{b_{m,n}\}^\infty_{m,n=0}$ with norm $\|b\| = \sum |b_{m,n}|$. Then the space $A^{(\alpha)}$ is a Banach space isometric to l^1 by the mapping $f \mapsto \{\hat{f}(m, n)h^{(\alpha)}_{m,n}\}^\infty_{m,n=0}$ of $A^{(\alpha)}$ onto l^1. We now claim that $A^{(\alpha)}$ is a Banach algebra.

Assume that $f(z) = \sum a_{m,n}R^{(\alpha)}_{m,n}(z)$ and $g(z) = \sum b_{k,l}R^{(\alpha)}_{k,l}(z)$ are in $A^{(\alpha)}$. Then we have

$$f(z)g(z) = \sum_{m,n;k,l} a_{m,n} b_{k,l} R^{(\alpha)}_{m,n}(z) R^{(\alpha)}_{k,l}(z)$$
$$= \sum_{p,q} \{ \sum_{m,n;k,l} a_{m,n} b_{k,l} c_{p,q}(m, n; k, l) \} h^{(\alpha)}_{p,q} R^{(\alpha)}_{p,q}(z)$$

and

$$\|fg\| \leqq \sum_{p,q} \{ \sum_{m,n;k,l} |a_{m,n}| \, |b_{k,l}| \, |c_{p,q}(m, n; k, l) h^{(\alpha)}_{p,q}| \} \leqq \|f\| \, \|g\| ,$$

since $\sum_{p,q} |c_{p,q}(m, n; k, l) h^{(\alpha)}_{p,q}| = 1$ by (iii) and $R^{(\alpha)}_{m,n}(l) = 1$. Thus we have the following:

PROPOSITION 1. *The space $A^{(\alpha)}$ is a commutative Banach algebra with pointwise multiplication of functions.*

2. The maximal ideal space of $A^{(\alpha)}$. Let $\mathfrak{m}$ be the maximal ideal space of $A^{(\alpha)}$. The maximal ideal space is identified with the space of multiplicative linear functionals, that is, nonzero complex homomorphisms. Since the mapping $f \mapsto f(z)$ defines a multiplicative linear functional on $A^{(\alpha)}$, every z in $\bar{D}$ corresponds to a maximal ideal $\iota(z)$ in $\mathfrak{m}$ such that $\tilde{f}(\iota(z)) = f(z)$ for all f in $A^{(\alpha)}$, where $\tilde{f}$ is the Gelfand transform of f. Thus we have a mapping $\iota: z \mapsto \iota(z)$ of $\bar{D}$ into $\mathfrak{m}$.

THEOREM 1. *The maximal ideal space $\mathfrak{m}$ of the algebra $A^{(\alpha)}$ is homeomorphic to the closed unit disk $\bar{D}$ by the mapping ι and the Gelfand transform $\tilde{f}$ of f in $A^{(\alpha)}$ is given by $\tilde{f}(\iota(z)) = f(z)$ for z in $\bar{D}$.*

LEMMA 1. *Let $\alpha \geqq 0$, $0 < \theta < \pi$ and $\rho > 1$. Then there exist positive constants C and K not depending on β, n which satisfy the following: If n and β are positive integers such that $n > K\beta$, then*

$$\left(\cos^{\beta}\frac{\theta}{2}\right) R^{(\alpha,\beta)}_n(\cos\theta) = \left\{(\theta/\sin\theta)^{1/2}\left(\sin^{-\alpha}\frac{\theta}{2}\right) J_\alpha(N\theta) + R\right\} \Big/ \binom{n+\alpha}{n}$$

where $|R| \leqq C\rho^{\beta}(n - K\beta)^{-1}$, $N = n + (\alpha+\beta+1)/2$, $\binom{p}{n} = p(p-1)\cdots(p-n+1)/n!$ and J_α is the Bessel function of the first kind of order α.

This lemma is essentially the asymptotic formula of Szegö [7; Satz II], but gives an estimate of the error term with respect to the parameter β which we need for our purpose. We omit the proof since it follows from term by term application of Szegö's method.

LEMMA 2. *Let $\alpha \geqq 0$, $0 < \theta < \pi$, $\sigma > 1$ and k be positive integers. Then there exist positive integers λ and μ such that $\sigma^{2\mu/(\alpha+1/2)} > \lambda > 1$ and*

$$\limsup_{k\to\infty} |\cos(N\theta + \gamma)| > 0 ,$$

where $N = \lambda^k + (\alpha + 2\mu k + 1)/2$, $\gamma = -\alpha\pi/2 - \pi/4$.

PROOF. Since

$$\cos(N\theta + \gamma) = \mathrm{Re}[e^{i\{(\alpha+1)\theta/2+\gamma\}}e^{i(\lambda^k+\mu k)\theta}] ,$$

it suffices to show that there exist positive integers λ and μ such that $\sigma^{2\mu/(\sigma+1/2)} > \lambda > 1$ and that the sequence $\{e^{i(\lambda^k+\mu k)\theta}\}_{k=1}^{\infty}$ has more than two accumulation points. Put $\theta = 2\pi\eta$. First we suppose that η is a rational number, and write $\eta = q/p$, where positive integers p and q are relatively prime. Since $0 < \eta < 1/2$, we have $p > 2$. Let $\lambda = p$ and let μ be a positive integer such that $\sigma^{2\mu/(\alpha+1/2)} > p$ and μ and p are relatively prime. Then $\{e^{i(\lambda^k+\mu k)\theta}\}_{k=1}^{\infty}$ has p accumulation points. Next we suppose that η is irrational. Let λ and μ be integers such that $\sigma^{2\mu/(\alpha+1/2)} > \lambda > 1$ and let the accumulation points of $\{e^{i(\lambda^k+\mu k)\theta}\}_{k=1}^{\infty}$ be $\{\xi_\nu\}$. Assume that $\mathrm{Card}\{\xi_\nu\} = Q < \infty$. We write

$$e^{i(\lambda^{k+1}+\lambda\mu k)\theta} = e^{i\{\lambda^{k+1}+\mu(k+1)\}\theta}e^{i(\lambda-1)\mu k\theta}e^{-i\mu\theta} .$$

The accumulation points of $\{e^{i(\lambda^{k+1}+\lambda\mu k)\theta}\}_{k=1}^{\infty}$ are $\xi_1^\lambda, \xi_2^\lambda, \cdots, \xi_Q^\lambda$. On the other hand, $\{e^{i(\lambda-1)\mu k\theta}\}_{k=1}^{\infty}$ is dense in the unit circle $|z| = 1$, since η is irrational. This contradicts the finiteness of $\{\xi_\nu\}$. q.e.d.

PROOF OF THEOREM 1. Since two different points in $\bar{D}$ separate functions in $A^{(\alpha)}$, the mapping ι is one to one from $\bar{D}$ into $\mathfrak{m}$. It follows from the definition of the Gelfand topology that the mapping ι is continuous. Since $\bar{D}$ and $\mathfrak{m}$ are compact, it suffices to show that ι is surjective.

Let χ be a multiplicative linear functional on $A^{(\alpha)}$. Since the norm of a multiplicative linear functional is at most 1, we have $|\chi(R_{1,0}^{(\alpha)})| \leqq 1$ and $|\chi(R_{0,1}^{(\alpha)})| \leqq 1$. Pick points $se^{i\phi}$ and $te^{i\psi}$ in $\bar{D}$ such that $\chi(R_{1,0}^{(\alpha)}) = se^{i\phi}$ and $\chi(R_{0,1}^{(\alpha)}) = te^{i\psi}$. By the identity

$$R_{1,0}^{(\alpha)}R_{0,1}^{(\alpha)} = \frac{\alpha+1}{\alpha+2}R_{1,1}^{(\alpha)} + \frac{1}{\alpha+2} \qquad \text{(cf. Szegö [8])} ,$$

we have

$$st\, e^{i(\phi+\psi)} = \frac{\alpha+1}{\alpha+2}\chi(R_{1,1}^{(\alpha)}) + \frac{1}{\alpha+2} . \tag{2}$$

Let $A_0^{(\alpha)}$ be the closed subalgebra of $A^{(\alpha)}$ generated by the set $\{R_{n,n}^{(\alpha)}\}_{n=0}^{\infty}$. Then $A_0^{(\alpha)}$ is identified with the algebra $A^{(\alpha,0)}$ of absolutely convergent Jacobi polynomial series of order $(\alpha, 0)$. The maximal ideal space of $A^{(\alpha,0)}$ is identified with the closed interval $[-1, 1]$ and the Gelfand transfrom of f in $A^{(\alpha,0)}$ is $f(\cdot)$ [3; Theorem 1]. Thus, restricting χ to $A_0^{(\alpha)}$ we have a unique point r such that $0 \leqq r \leqq 1$ and $\chi(R_{1,1}^{(\alpha)}) = R_{1,1}^{(\alpha)}(r)$. Since $R_{1,1}^{(\alpha)}(r) = \{(\alpha+2)(2r^2-1)+\alpha\}/2(\alpha+1)$, (2) implies that $\psi = -\phi$ and $st = r^2$.

Next we show that $s = t$. By the identities $R_{m,n}^{(\alpha)} = (R_{1,0}^{(\alpha)})^{m-n}R_n^{(\alpha,m-n)}$

$(2R_{1,0}^{(\alpha)}R_{0,1}^{(\alpha)}-1)$ for $m \geqq n$ and $=(R_{0,1}^{(\alpha)})^{n-m}R_m^{(\alpha,n-m)}(2R_{1,0}^{(\alpha)}R_{0,1}^{(\alpha)}-1)$ for $m \leqq n$, we have

$$(3)\qquad \chi(R_{m,n}^{(\alpha)}) = \begin{cases} (se^{i\phi})^{m-n}R_n^{(\phi,m-n)}(2st-1) & \text{for } m \geqq n\,, \\ (te^{i(-\phi)})^{n-m}R_m^{(\alpha,n-m)}(2st-1) & \text{for } m \leqq n\,. \end{cases}$$

Since $|\chi(R_{m,n}^{(\alpha)})| \leqq 1$ for all m, n, we have

$$(4)\qquad s^{m-n}|R_n^{(\alpha,m-n)}(2st-1)| \leqq 1$$

for $m \geqq n$ and

$$t^{n-m}|R_m^{(\alpha,n-m)}(2st-1)| \leqq 1$$

for $m \leqq n$. If we show that (4) implies $s \leqq t$, we have $s = t$ by symmetry. The condition (4) with $t = 0$ implies $s = 0$ by the equality $R_n^{(\alpha,m-n)}(-1) = (-1)^n\binom{m}{n}\Big/\binom{n+\alpha}{n}$. Suppose that $t \neq 0$. Put $\cos\theta = 2st-1$, $0 \leqq \theta < \pi$. Then the condition (4) is equivalent to

$$(5)\qquad \left(\cos\frac{\theta}{2}\Big/t\right)^{m-n}\left(\cos^{m-n}\frac{\theta}{2}\right)|R_n^{(\alpha,m-n)}(\cos\theta)| \leqq 1$$

for $m \geqq n$. If $\theta = 0$, we have obviously $t = 1$ and $s = 1$. If $0 < \theta < \pi$, we put $\sigma = t^{-1}\cos(\theta/2)$ and $\beta = m - n$. Suppose that $\sigma > 1$, and choose λ and μ as in Lemma 2. Let ρ be a positive constant such that $\rho > 1$ and $\rho^{2\mu} < \lambda^{1/2}$. By Lemma 1 with this ρ and a well known asymptotic formula

$$J_\alpha(z) = \sqrt{2/\pi z}\cos(z+\gamma) + O(z^{-3/2})$$

as $z \to \infty$, where $\gamma = -\alpha\pi/2 - \pi/4$, we have

$$\begin{aligned}&\sigma^\beta\cos^\beta\frac{\theta}{2}R_n^{(\alpha,\beta)}(\cos\theta)\\ &\quad = \sigma^\beta\binom{n+\alpha}{n}^{-1}N^{-1/2}\left[(2/\pi\sin\theta)^{1/2}\left(\sin^{-\alpha}\frac{\theta}{2}\right)\{\cos(N\theta+\gamma)+R'\} + N^{-1/2}R\right]\end{aligned}$$

for $n > K\beta$, where $N = n + (\alpha+\beta+1)/2$, $|R| \leqq C\rho^\beta(n-K\beta)^{-1}$, and $|R'| \leqq C'(N\theta)^{-1}$ for $N\theta \geqq 1$ with a positive constant C' not depending on N and θ. Put $n = \lambda^k$ and $\beta = 2\mu k$, and let $k \to \infty$. Then $R' \to 0$, $N^{-1/2}R \to 0$ and $\sigma^\beta\binom{n+\alpha}{n}^{-1}N^{-1/2} \to \infty$, and thus

$$\limsup_{k\to\infty}\sigma^\beta\left(\cos^\beta\frac{\theta}{2}\right)|R_n^{(\alpha,\beta)}(\cos\theta)| = \infty$$

by Lemma 2. This contradicts the condition (5). Thus we have $\sigma = t^{-1}\cos(\theta/2) \leqq 1$. This implies $s \leqq t$ since $st = \cos^2(\theta/2) \leqq t^2$.

By (3) and $s = t = r$, we have

$$\chi(R_{m,n}^{(\alpha)}) = \begin{cases} (re^{i\phi})^{m-n} R_n^{(\alpha,m-n)}(2r^2 - 1) & \text{for } m \geqq n , \\ (re^{i(-\phi)})^{n-m} R_m^{(\alpha,n-m)}(2r^2 - 1) & \text{for } m \leqq n . \end{cases}$$

Thus for every $f = \sum a_{m,n} R_{m,n}^{(\alpha)}$ in $A^{(\alpha)}$ we have

$$\chi(f) = \sum a_{m,n} R_{m,n}^{(\alpha)}(z_0) = f(z_0) ,$$

where $z_0 = re^{i\phi}$. The proof is complete.

By the Wiener-Lévy theorem we have the following:

COROLLARY. *Suppose that* $\alpha \geqq 0$,

$$f(z) = \sum_{m,n=0}^{\infty} a_{m,n} R_{m,n}^{(\alpha)}(z) , \qquad \sum_{m,n=0}^{\infty} |a_{m,n}| < \infty ,$$

and F *is a holomorphic function on an open set containing the range of* f. *Then*

$$F(f(z)) = \sum_{m,n=0}^{\infty} b_{m,n} R_{m,n}^{(\alpha)}(z) \quad \textit{with} \quad \sum_{m,n=0}^{\infty} |b_{m,n}| < \infty .$$

By Theorem 1 the algebra $A^{(\alpha)}$ is semisimple. Repeating integrations by parts we may show that the infinitely differentiable functions on a neighborhood of $\bar{D}$ belong to $A^{(\alpha)}$. This implies that *the Banach algebra* $A^{(\alpha)}$ *is regular.*

Let E be a closed subset of $\bar{D}$. Denote by $I(E)$ the closed ideal in $A^{(\alpha)}$ consisting of all f in $A^{(\alpha)}$ such that $f = 0$ on E, and by $J(E)$ the ideal of all f in $A^{(\alpha)}$ such that $f = 0$ on a neighborhood of E. If $J(E)$ is dense in $I(E)$ then E is called a *set of spectral synthesis* for $A^{(\alpha)}$. By an argument similar to that used for Schwartz's example in the Euclidean space $\boldsymbol{R}^3$ (cf., also, [1]), we have:

THEOREM 2. *If* $\alpha \geqq 1$ *and* z_0 *is in the open unit disk* D, *then* $\{z_0\}$ *is not a set of spectral synthesis for* $A^{(\alpha)}$.

PROOF. Let k be the greatest integer not exceeding α and let z_0 be in D. By (1) and simple calculations, there exist a positive constant C and a neighborhood V of z_0 in D such that

$$\left| \frac{\partial^{p+q} R_{m,n}^{(\alpha)}}{\partial x^p \partial y^q}(z) \right| \leqq C$$

on V for $0 \leqq p + q \leqq k$ and all m, n. This implies that the functions in $A^{(\alpha)}$ have k continuous derivatives on D and the functional

$$f \mapsto \frac{\partial^{p+q} f}{\partial x^p \partial y^q}(z_0)$$

on $A^{(\alpha)}$ is continuous. Let $I_1 = \{f \in A^{(\alpha)}; f(z_0) = 0\}$ and $I_2 = \{f \in A^{(\alpha)}; f(z_0) = (\partial f/\partial x)(z_0) = 0\}$. Then I_1 and I_2 are distinct closed ideals for $\alpha \geqq 1$. This proves the theorem.

3. Sets of interpolation with respect to $A^{(\alpha)}$. A closed set E in $\bar{D}$ will be called a *set of interpolation* with respect to $A^{(\alpha)}$, if every continuous function on E is the restriction of a function in $A^{(\alpha)}$ to E. Vinogradov [9], Kahane [4; Ch. XI §4] and [3] suggest the following observations.

A finite subset of $\bar{D}$ is evidently a set of interpolation with respect to $A^{(\alpha)}$. Let $\boldsymbol{T}$ be the circle group $\boldsymbol{R}/2\pi\boldsymbol{Z}$ and $A(\boldsymbol{T})$ be the algebra of absolutely convergent Fourier series $f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$, $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. A closed set E in $\boldsymbol{T}$ is called a Helson set, if every continuous function on E is the restriction of a function in $A(\boldsymbol{T})$ to E (cf. [4; Ch. IV]). The image of a Helson set by the map $t \mapsto e^{it}$ will be called a Helson set on the boundary ∂D. For $f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$ in $A(\boldsymbol{T})$, put $f(z) = \sum_{n=0}^{\infty} a_n R_{n,0}^{(\alpha)}(z) + \sum_{n=1}^{\infty} a_{-n} R_{0,n}^{(\alpha)}(z)$. Then $f(z)$ belongs to $A^{(\alpha)}$. Thus a Helson set on the boundary ∂D is a set of interpolation with respect to $A^{(\alpha)}$. Also, the union of a finite set in $\bar{D}$ and a Helson set on the boundary ∂D is a set of interpolation with respect to $A^{(\alpha)}$. We will consider the converse.

THEOREM 3. *Suppose that $\alpha > 0$. Then every set of interpolation with respect to $A^{(\alpha)}$ is the union of a finite set in the open unit disk D and a Helson set on the boundary ∂D.*

LEMMA 3. *Let $\alpha, \beta \geqq 0$ and $0 < \theta < \pi$. Then*

$$|R_n^{(\alpha,\beta)}(\cos\theta)| \leqq C_\alpha n^{-\alpha}\{\sin^{-\alpha}(\theta/2)\}\{\cos^{-\beta}(\theta/2)\} ,$$

where C_α is a positive constant depending only on α.

PROOF. Let $F(w)$ be the generating function for Jacobi polynomials of the form $F(w) = 2^{\alpha+\beta}\Phi(w)\Psi(w)/Q(w)$, where $Q(w) = (1 - 2w\cos\theta + w^2)^{1/2}$, $\Phi(w) = \{1 - w + Q(w)\}^{-\alpha}$ and $\Psi(w) = \{1 + w + Q(w)\}^{-\beta}$ with the branches of $Q(w)$, $\Phi(w)$ and $\Psi(w)$ being chosen positive for $w = 0$. Then, for $0 < \theta < \pi$, Jacobi polynomials are given by the formula

$$\binom{n+\alpha}{n} R_n^{(\alpha,\beta)}(\cos\theta) = \frac{1}{2\pi i}\int F(w) w^{-n-1} dw ,$$

where the path of integration is a small closed curve around the origin in the positive direction. Thus

$$\binom{n+\alpha}{n} R_n^{(\alpha,\beta)}(\cos\theta) = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\left\{\int_{-(\theta-\varepsilon)}^{\theta-\varepsilon} + \int_{\theta+\varepsilon}^{2\pi-(\theta+\varepsilon)}\right\} 2^{\alpha+\beta}\frac{\Phi(w)\Psi(w)}{Q(w)} e^{-int} dt$$

for $0<\theta<\pi$, where $w=e^{-it}$. From this and the inequality $\binom{n+\alpha}{n}^{-1} \leqq C_\alpha n^{-\alpha}$ with a constant C_α depending only on α, it suffices to show that $|\Phi(w)| \leqq \{2\sin(\theta/2)\}^{-\alpha}$ and $|\Psi(w)| \leqq \{2\cos(\theta/2)\}^{-\beta}$, which follow from the inequalities;

$$|1-w+Q(w)| \geqq 2\sin(\theta/2) \tag{6}$$

$$|1+w+Q(w)| \geqq 2\cos(\theta/2) \tag{7}$$

for $w=e^{-it}$, $t\in(-\theta,\theta)\cup(\theta,2\pi-\theta)$. Write

$$\begin{aligned}1-w&+(1-2w\cos\theta+w^2)^{1/2}\\ &=e^{-it/2}(e^{it/2}-e^{-it/2})+[e^{-it}\{(e^{it}+e^{-it})-2\cos\theta\}]^{1/2}\\ &=e^{-it/2}2i\sin(t/2)+e^{-it/2}(2\cos t-2\cos\theta)^{1/2}\end{aligned}$$

for $t\in(-\theta,\theta)$. Then a branch of $(2\cos t-2\cos\theta)^{1/2}$ should be chosen positive for $t=0$. Thus we have

$$|1-w+Q(w)|=[\{2\sin(t/2)\}^2+2\cos t-2\cos\theta]^{1/2}=2\sin(\theta/2)$$

for $t\in(-\theta,\theta)$. Also, write

$$\begin{aligned}1-w&+(1-2w\cos\theta+w^2)^{1/2}\\ &=e^{-it/2}2i\sin(\theta/2)+e^{-it/2}i(2\cos\theta-2\cos t)^{1/2}\end{aligned}$$

for $t\in(\theta,2\pi-\theta)$. Then the branch of $(2\cos\theta-2\cos t)^{1/2}$ should be positive, since the branch of $(1-2w\cos\theta+w^2)^{1/2}$ is positive for $w=-1$. This shows that

$$|1-w+Q(w)|=2\sin(t/2)+(2\cos\theta-2\cos t)^{1/2}>2\sin\,(\theta/2)$$

for $t\in(\theta,2\pi-\theta)$. Thus we have (6). Similarly, we have (7) by the identities;

$$1+w+Q(w)=e^{-it/2}2\cos(t/2)+e^{-it/2}(2\cos t-2\cos\theta)^{1/2}$$

for $t\in(-\theta,\theta)$, where the branch of $(2\cos t-2\cos\theta)^{1/2}$ is chosen positive, and

$$1+w+Q(w)=e^{-it/2}2\cos(t/2)+e^{-it/2}i(2\cos\theta-2\cos t)^{1/2}$$

for $t\in(\theta,2\pi-\theta)$, where the branch of $(2\cos\theta-2\cos t)^{1/2}$ is chosen positive. q.e.d.

PROOF OF THEOREM 3. Let E be a set of interpolation with respect to $A^{(\alpha)}$. Any closed subset E is also a set of interpolation with respect to $A^{(\alpha)}$ and the restriction of a function in $A^{(\alpha)}$ to ∂D can be regarded as a function in $A(\boldsymbol{T})$. Thus $E\cap\partial D$ is a Helson set on the boundary ∂D.

Next we will show that $E \cap D$ is finite. Suppose that the assersion does not hold. Then there exist a sequence $\{z_j\}_{j=1}^{\infty}$ in E such that $0 < |z_j| < 1$ for $j = 1, 2, 3, \cdots$ and $z_i \neq z_j$ for $i \neq j$, and a point z_0 in $\bar{D}$ such that $\{z_j\}$ converges to z_0. Let $A^{(\alpha)}(E)$ be the quotient algebra $A^{(\alpha)}/I(E)$ with quotient norm $\|\cdot\|_{A^{(\alpha)}(E)}$ and $C(E)$ be the Banach algebra of continuous functions on E with uniform norm $\|\cdot\|_{C(E)}$. Since E is a set of interpolation with respect to $A^{(\alpha)}$, we have $A^{(\alpha)}(E) = C(E)$, and the norms in $A^{(\alpha)}(E)$ and in $C(E)$ are equivalent. Let g_k be a function in $C(E)$ such that $g_k(z_{2j}) = 1$ and $g_k(z_{2j-1}) = 0$ for $j = 1, 2, 3, \cdots, k$, $g_k(z_j) = 0$ for $j = 2k + 1, 2k + 2, \cdots$ and $\|g_k\|_{C(E)} = 1$. By the norm equivalence we can choose a function $f_k = \sum a_{m,n}(k) R_{m,n}^{(\alpha)}$ in $A^{(\alpha)}$ for every $k = 1, 2, 3, \cdots$ so that $f_k = g_k$ on E and $\|f_k\| \leqq C$, where C is a constant not depending on k. Let c_0 be the space of double sequences $\{c_{m,n}\}_{m,n=0}^{\infty}$ vanishing at infinity. Since $A^{(\alpha)}$ is isometric to l^1, $A^{(\alpha)}$ is identified with the dual of c_0. This implies that there exists a subsequence $\{f_{k(p)}\}_{p=1}^{\infty}$ of $\{f_k\}_{k=1}^{\infty}$ which converges to a function $f = \sum a_{m,n} R_{m,n}^{(\alpha)}$ in the weak $*$ topology $\sigma(A^{(\alpha)}, c_0)$. Let z be in D and put $z = e^{i\phi}\cos(\theta/2)$. By Lemma 3, we have

$$(8) \qquad |R_{m,n}^{(\alpha)}(z)| \leqq \begin{cases} C_\alpha n^{-\alpha} \sin^{-\alpha}(\theta/2) & \text{for} \quad m \geqq n > 0\,, \\ C_\alpha m^{-\alpha} \sin^{-\alpha}(\theta/2) & \text{for} \quad n > m > 0\,. \end{cases}$$

Since $|R_n^{(\alpha,\beta)}(\cos\theta)| \leqq \binom{n+\beta}{n} \Big/ \binom{n+\alpha}{n}$ for $\beta \geqq \alpha$ (see, [8; (7.32.2)]) and $\binom{n+\beta}{n} \Big/ \binom{n+\alpha}{n} \leqq C_{\alpha,n}\beta^n$ with a constant $C_{\alpha,n}$ not depending on β, we have

$$(9) \qquad |R_{m,n}^{(\alpha)}(z)| \leqq \begin{cases} C_{\alpha,n}(m-n)^n \{\cos(\theta/2)\}^{m-n} & \text{for} \quad m - n \geqq \alpha\,, \\ C_{\alpha,m}(n-m)^m \{\cos(\theta/2)\}^{n-m} & \text{for} \quad n - m \geqq \alpha\,. \end{cases}$$

Thus, if $\alpha > 0$, then the complex sequence $\{R_{m,n}^{(\alpha)}(z)\}_{m,n=0}^{\infty}$ belongs to c_0 for every z in D by (8) and (9). By the definition of the weak $*$ topology, we have that $f_{k(p)}(z)$ coverges to $f(z)$ as $p \to \infty$ for every z in D. In particular, we have that $f(z_{2j}) = 1$ and $f(z_{2j-1}) = 0$ for $j = 1, 2, 3, \cdots$, which contradicts the continuity of f in $\bar{D}$. The proof is complete.

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