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GLOBAL ASYMPTOTIC STABILITY IN A PERIODIC INTEGRODIFFERENTIAL SYSTEM

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A set of easily verifiable sufficient conditions are derived for the existence of a globally stable periodic solution in a system of nonlinear Volterra integrodifferential equations with periodic coefficients.

1. Introduction. The purpose of this article is to derive a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (componentwise) periodic solution of the integrodifferential system

$$\frac{dx_i(t)}{dt} = x_i(t)\left\{b_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-u)x_j(u)du\right\}, \quad i = 1, 2, \cdots, n;\ t > t_0;\ t_0 \in (-\infty, \infty) \tag{1.1}$$

where b_i, a_{ij} $(i, j = 1, 2, \cdots, n)$ are continuous, positive periodic functions with a common period ω and K_{ij}: $[0, \infty) \to [0, \infty)$, $(i, j = 1, 2 \cdots, n; i \neq j)$ denote delay kernel about which more will be said below. In mathematical ecology (1.1) denotes a model of the dynamics of an n-species system in which each individual competes with all others of the system for a common pool of resources and the interspecific competition involves a time delay extending over the entire past as typified by the delay kernels K_{ij} in (1.1). The assumption of periodicity of the parameters b_i, a_{ij} $(i, j = 1, 2, \cdots, n)$ is a way of incorporating the periodicity of the environment (e.g. seasonal effects of weather, food supplies, mating habits etc.). We will need the following preparation.

LEMMA 1.1. *Assume that the delay kernels* K_{ij} $(i, j = 1, 2, \cdots, n; i \neq j)$ *are piecewise (locally) continuous such that the series* $\sum_{r=0}^{\infty} K_{ij}(u + rw)$ *converges uniformly with respect to* u *on* $[0, \omega]$. *Then any* ω*-periodic solution of* (1.1) *is also an* ω*-periodic solution of*

$$\frac{dx_i(t)}{dt} = x_i(t)\left\{b_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)\int_{t-\omega}^{t} H_{ij}(t-u)x_j(u)du\right\}, \quad i = 1, 2, \cdots, n, \tag{1.2}$$

AMS Subject Classification: 34K20, 34J05, 92A15 *where*

$$H_{ij}(u) = \sum_{r=0}^{\infty} K_{ij}(u + r\omega);\ i, j = 1, 2, \cdots, n;\ i \neq j \tag{1.3}$$

and conversely any ω*-periodic solution of* (1.2)-(1.3) *is a* ω*-periodic solution of* (1.1).

PROOF. The proof follows immediately from the fact that if $(x_1, x_2, \cdots, x_n)$ is any periodic solution of period ω of (1.1) then we have

$$\begin{aligned}\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds &= \sum_{r=0}^{\infty}\int_{t-(r+1)\omega}^{t-r\omega} K_{ij}(t-s)x_j(s)ds \\ &= \sum_{r=0}^{\infty}\int_{t-\omega}^{t} K_{ij}(t-s+r\omega)x_j(s-r\omega)ds = \int_{t-\omega}^{t} H_{ij}(t-s)x_j(s)ds\ ,\end{aligned} \tag{1.4}$$

implying that the ω-periodic solution $(x_1, \cdots, x_n)$ of (1.1) is also a solution of (1.2)-(1.3). The converse is similarly proved by retracing the steps backwards and the proof is complete.

Now let $\boldsymbol{R}$ and $\boldsymbol{R}_n$ denote respectively the set of all real numbers and the real n-dimensional Euclidean space; $\boldsymbol{R}_n^+$ will denote the nonnegative cone of $\boldsymbol{R}_n$ under a componentwise ordering. Define the constants $b_i^l, b_i^u, a_{ij}^l, a_{ij}^u$ $(i, j = 1, 2, \cdots, n)$ by the following:

$$\begin{aligned}&\inf_{t\in\boldsymbol{R}} b_i(t) = \min_{t\in[0,\omega]} b_i(t) = b_i^l \\ &\inf_{t\in\boldsymbol{R}} a_{ij}(t) = \min_{t\in[0,\omega]} a_{ij}(t) = a_{ij}^l \\ &\sup_{t\in\boldsymbol{R}} b_i(t) = \max_{t\in[0,\omega]} b_i(t) = b_i^u \\ &\sup_{t\in\boldsymbol{R}} a_{ij}(t) = \max_{t\in[0,\omega]} a_{ij}(t) = a_{ij}^u \qquad i, j = 1, 2, \cdots, n\ .\end{aligned}$$

We will study the system (1.1) under the following assumptions on the coefficients of (1.1):

(i) the delay kernels are normalized and are such that

$$\int_0^{\infty} K_{ij}(s)ds = 1;\ \int_0^{\infty} sK_{ij}(s)ds < \infty\ , \qquad i, j = 1, 2, \cdots, n;\ i \neq j\ , \tag{1.5}$$

(1.6) (ii) $\qquad b_i^l > 0$ and $a_{ii}^l > 0\ ;\qquad i = 1, 2, \cdots, n\ ,$

(1.7) (iii) $\qquad b_i^l > \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}^u(b_j^u/a_{jj}^l)\ ;\qquad i = 1, 2, \cdots, n\ .$

Since solutions of (1.1) corresponding to initial conditions of the form

$$\begin{aligned}&x_i(s) = \varphi_i(s) \geqq 0;\ \sup \varphi_i(s) < \infty;\ \varphi_i(0) > 0 \\ &\varphi_i \text{ is piecewise (locally) continuous on } (-\infty, 0]\end{aligned} \tag{1.8}$$

remain nonnegative, it will follow that

$$(1.9)\qquad \frac{dx_i}{dt} \leqq x_i\{b_i^u - a_{ii}^l x_i\};\ t > 0\ , \qquad i = 1, 2, \cdots, n\ ,$$

as a consequence of which we will have

$$(1.10)\qquad 0 < x_i(0) \leqq b_i^u/a_{ii}^l = x_i^u$$
$$\Rightarrow x_i(t) \leqq x_i^u \quad \text{for} \quad t > 0\ , \qquad i = 1, 2, \cdots, n\ .$$

Now (1.1) and (1.10) together lead to

$$(1.11)\qquad \frac{dx_i}{dt} \geqq x_i\left\{b_i^l - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}^u(b_j^u/a_{jj}^l) - a_{ii}^u x_i\right\}\ , \qquad t > 0;\ i = 1, 2, \cdots, n\ .$$

If $0 < x_i(0) \leqq x_i^u$ (1.6), (1.7) and (1.11) lead to

$$x_i(0) \geqq \left\{b_i^l - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}^u(b_j^u/a_{jj}^l)\right\}\Big/a_{ii}^u = x_i^l$$
$$(1.12)\qquad \Rightarrow x_i(t) \geqq x_i^l \quad \text{for} \quad t \geqq 0\ , \qquad i = 1, 2, \cdots, n\ .$$

From the foregoing preparation we have the following:

LEMMA 1.2. *Let*

$$x(t, t_0, \tilde{\varphi}) = \{x_1(t, t_0, \tilde{\varphi}), \cdots, x_n(t, t_0, \tilde{\varphi})\}$$

be a solution of (1.2)-(1.3) *with the initial conditions*

$$x_i(t_0, t_0, \tilde{\varphi}) = \varphi_i(s),\ s \in [t_0 - \omega, t_0],\ t_0 \in \boldsymbol{R}\ ,$$
$$\tilde{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_n)\ .$$

If

$$(1.13)\qquad 0 < x_* = \max_{1\leqq i\leqq n} x_i^l \leqq \varphi_i(s) \leqq x^* = \min_{1\leqq i\leqq n} x_i^u\ , \quad s \in [t_0 - \omega, t_0]\ ,$$
$$i = 1, 2, \cdots, n;\ t_0 \in \boldsymbol{R}$$

then we have

$$(1.14)\qquad x_* \leqq x_1(t, t_0, \tilde{\varphi}) \leqq x^* \quad \textit{for} \quad t \geqq t_0;\ t_0 \in \boldsymbol{R}\ , \qquad i = 1, 2, \cdots, n\ .$$

2. Existence of a periodic solution. Our strategy for proving the existence of a periodic solution of (1.2) is as follows; we show that a class of solutions of (1.2) converge as $t \to \infty$ to an asymptotically almost periodic function and then show that such an asymptotically almost periodic function is itself a periodic solution of (1.2). For convenience we note the following definitions:

DEFINITION 2.1. (Halanay [5], p. 343). Let $\tilde{\varphi}, \tilde{\psi}$: $[t_0 - \omega, t_0] \to \boldsymbol{R}_n$ for $t_0 \in \boldsymbol{R}$ and let $\tilde{\varphi}, \tilde{\psi}$ be continuous on $[t_0 - \omega, t_0]$. If $\tilde{\varphi} = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$

then a solution

$$x(t, t_0, \tilde{\varphi}) = \{x_1(t, t_0, \tilde{\varphi}), \cdots, x_n(t, t_0, \tilde{\varphi})\}, \quad t > t_0$$

of (1.2) with

$$x_i(t_0, t_0, \tilde{\varphi}) = \tilde{\varphi}_i(s), \quad s \in [t_0 - \omega, t_0], \quad t_0 \in \boldsymbol{R}, \qquad i = 1, 2, \cdots, n \tag{2.1}$$

is said to be uniformly stable if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\max_{s \in [t_0-\omega, t_0]} \sum_{i=1}^{n} |\varphi_i(s) - \psi_i(s)| < \delta$$

$$\Rightarrow \sum_{i=1}^{n} |x_i(t, t_0, \tilde{\varphi}) - y_i(t, t_0, \tilde{\psi})| < \varepsilon, \quad t \geqq t_0, \tag{2.2}$$

where $y(t, t_0, \tilde{\psi}) = \{y_1(t, t_0, \tilde{\psi}), y_2(t, t_0, \tilde{\psi}), \cdots, y_n(t, t_0, \tilde{\psi})\}$ $(t \geqq t_0)$ is a solution of (1.2) with

$$y_i(t_0, t_0, \tilde{\psi}) = \psi_i(s), \quad s \in [t_0 - \omega, t_0], \qquad i = 1, 2, \cdots, n.$$

DEFINITION 2.2. A function $p = (p_1, p_2, \cdots, p_n): \boldsymbol{R} \to \boldsymbol{R}_n$ is said to be almost periodic if for every $\varepsilon > 0$ there exists a $l = l(\varepsilon) > 0$ such that within any interval $(a, a + l(\varepsilon))$ of length l there is a number β for which

$$\sum_{i=1}^{n} |p_i(t + \beta) - p_i(t)| < \varepsilon \quad \text{for} \quad t \in \boldsymbol{R}.$$

A function $p: \boldsymbol{R} \to \boldsymbol{R}_n$ is said to be asymptotically almost periodic if it is a sum of an almost periodic function $f(t)$ and a continuous function $g(t)$ defined on $\boldsymbol{R}$ such that (Yoshizawa [6])

$$p(t) = f(t) + g(t), t \in \boldsymbol{R} \quad \text{and} \quad g(t) \to 0 \text{ as } t \to \infty.$$

The following result will be used in the proof of our existence theorem below.

LEMMA 2.1. (Halanay [5], p. 486, Th. 4.37). *Every bounded and uniformly stable solution of a system of the form* (1.2) *converges asymptotically* (*as* $t \to \infty$) *to an almost periodic function.*

Our main result on the existence of a periodic solution of (1.2) is the following:

THEOREM 2.1. *Assume that* (1.5)-(1.7) *hold. Furthermore, suppose that there exists a positive constant* m *such that*

$$\min_{t \in [0,\omega]} a_{jj}(t) > \sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\max_{t \in [0,\omega]} a_{ij}(t) \right) + m, \qquad j = 1, 2, \cdots, n. \tag{2.3}$$

Then (1.2) *has a periodic solution of period* ω *say* $x^*(t) = \{x_1(t), \cdots, x_n(t)\}$

such that

$$x_* \leqq x_i(t) \leqq x^* , \qquad i = 1, 2, \cdots, n;\ t \in [0, \omega] . \tag{2.4}$$

PROOF. Let

$$x(t, t_0, \tilde{\varphi}) = \{x_1(t, t_0, \tilde{\varphi}), \cdots, x_n(t, t_0, \tilde{\varphi})\} ,$$
$$y(t, t_0, \tilde{\psi}) = \{y_1(t, t_0, \tilde{\psi}), \cdots, y_n(t, t_0, \tilde{\psi})\}$$

be two solutions of (1.2) corresponding to continuous initial conditions $\tilde{\varphi}$ and $\tilde{\psi}$ such that

$$\begin{aligned} &x_* \leqq \varphi_i(s) \leqq x^* , \quad s \in [t_0 - \omega, t_0] \\ &x_* \leqq \psi_i(s) \leqq x^* , \quad s \in [t_0 - \omega, t_0] , \qquad i = 1, 2, \cdots, n;\ t_0 \in \boldsymbol{R} . \end{aligned} \tag{2.5}$$

Consider a Lyapunov-functional $v(t) = V(t, x, y)$ defined by

$$\begin{aligned} v(t) = V(t, x, y) = \sum_{i=1}^{n} \Big(&|\log x_i(t, t_0, \tilde{\varphi}) - \log y_i(t, t_0, \tilde{\psi})| \\ &+ \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_{t-\omega}^{t} \Big\{ a_{ij}(s) \Big(\int_s^t H_{ij}(s + \omega - u) |x_j(u, t_0, \tilde{\varphi}) - y_j(u, t_0, \psi)| du \Big) \Big\} ds , \\ &\qquad t \geqq t_0 . \end{aligned} \tag{2.6}$$

Since

$$\begin{aligned} &x_* \leqq x_i(t, t_0, \tilde{\varphi}) \leqq x^* \\ &x_* \leqq y_i(t, t_0, \tilde{\psi}) \leqq x^* \qquad i = 1, 2, \cdots, n;\ t \geqq t_0 \end{aligned}$$

we have (by the elementary mean value theorem)

$$|\log x_i(t, t_0, \tilde{\varphi}) - \log y_i(t, t_0, \tilde{\psi})| \leqq |x_i(t, t_0, \tilde{\varphi}) - y_i(t, t_0, \tilde{\psi})|/x^* \tag{2.7}$$

and hence

$$v(t_0) \leqq [\alpha(t_0)] \max_{s \in [t_0 - \omega, t_0]} \sum_{i=1}^{n} |x_i(s, t_0, \tilde{\varphi}) - y_i(s, t_0, \tilde{\psi})| , \tag{2.8}$$

where

$$[\alpha(t_0)] = \frac{1}{x_*} + \max_{1 \leqq i \leqq n} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ji}^{u} \Big(\int_{t_0-\omega}^{t_0} \Big\{ \int_{t_0-\omega}^{t_0} H_{ji}(s + \omega - u) du \Big\} ds \Big) . \tag{2.9}$$

We have from

$$\begin{aligned} \int_{t_0-\omega}^{t_0} \Big\{ \int_s^{t_0} H_{ji}(s + \omega - u) du \Big\} ds &\leqq \int_{t_0-\omega}^{t_0} du \Big\{ \int_{t_0-\omega}^{t_0} H_{ji}(s + \omega - u) ds \Big\} \\ &\leqq \int_{t_0-\omega}^{t_0} du \Big\{ \int_{t_0-\omega}^{t_0} H_{ji}(s + \omega - u) ds \Big\} \leqq \int_{t_0-\omega}^{t_0} du \Big\{ \int_{t_0-u}^{t_0+\omega-u} H_{ji}(\eta) d\eta \Big\} \\ &\leqq \omega \int_0^{2\omega} H_{ji}(\eta) d\eta \end{aligned} \tag{2.10}$$

that

$$v(t_0) \leqq \varepsilon$$

for arbitrary $\varepsilon > 0$ whenever

$$\max_{s \in [t_0-\omega, t_0]} \sum_{i=1}^{n} |x_i(s, t_0, \tilde{\varphi}) - y_i(s, t_0, \tilde{\psi})| \leqq \delta_1(\varepsilon) , \tag{2.11}$$

where

$$\delta_1(\varepsilon) = \varepsilon\left[\max_{1 \leqq i \leqq n} \sum_{j=1}^{n} a_{ji}^{u} \omega \int_0^{2\omega} H_{ji}(\eta) d\eta + (1/x_*)\right]^{-1} . \tag{2.12}$$

Calculating the right derivative D^+v of v and simplifying,

$$\begin{aligned} D^+v(t) &\leqq -m \sum_{i=1}^{n} |x_i(t, t_0, \tilde{\varphi}) - y_i(t, t_0, \tilde{\psi})| , \qquad (t \geqq t_0;\ t, t_0 \in \boldsymbol{R}) \\ &\leqq 0 \end{aligned} \tag{2.13}$$

showing that

$$v(t) \leqq (t_0) \quad \text{for} \quad t \geqq t_0 , \tag{2.14}$$

which implies that $v(t)$ is nonincreasing for $t \geqq t_0$; furthermore we have from

$$\begin{aligned} v(t) &\geqq \sum_{i=1}^{n} |\log x_i(t, t_0, \tilde{\varphi}) - \log y_i(t, t_0, \tilde{\psi})| \\ &\geqq \left(\sum_{i=1}^{n} |x_i(t, t_0, \tilde{\varphi}) - y_i(t, t_0, \tilde{\psi})|\right)\Big/ x^* \end{aligned} \tag{2.15}$$

that

$$\sum_{i=1}^{n} |x_i(t, t_0, \tilde{\varphi}) - y_i(t, t_0, \tilde{\psi})| \leqq x^* v(t_0) < \varepsilon \tag{2.16}$$

whenever

$$\max_{s \in [t_0-\omega, t_0]} \sum_{i=1}^{n} |x_i(s, t_0, \tilde{\varphi}) - y_i(s, t_0, \tilde{\psi})| \leqq \delta(\varepsilon) , \tag{2.17}$$

where

$$0 < \delta(\varepsilon) < \delta_1(\varepsilon)/x^* . \tag{2.18}$$

It follows from (2.16)-(2.17) that all solutions of (1.2) having components of initial values in the interval (x_*, x^*) are uniformly stable. Now by Lemma 2.1 such solutions converge as $t \to \infty$ to almost periodic functions, that is, there exists an almost periodic function $p = (p_1, p_2, \cdots, p_n)$ such that

$$x_i(t, t_0, \tilde{\varphi}) - p_i(t) \to 0 \quad \text{as} \quad t \to \infty , \qquad i = 1, 2, \cdots, n , \tag{2.19}$$

and

$$(2.20) \qquad x_* \leqq p_i(t) \leqq x^* \quad \text{for} \quad t \geqq t_0,\ t_0 \in \boldsymbol{R}\ , \qquad i = 1, 2, \cdots, n\ .$$

Our task is now to show that $p = (p_1, p_2, \cdots, p_n)$ is itself a solution of (1.2). Since $t_0 \in \boldsymbol{R}$ is arbitrary, we can consider p to be defined on $\boldsymbol{R}$.

We can write (2.19) in the form

$$(2.21) \qquad x_i(t, t_0, \tilde{\varphi}) = p_i(t) + q_i(t)\ , \qquad i = 1, 2, \cdots, n;\ t \geqq t_0 \in \boldsymbol{R}$$

for some q_i continuous for $t \geqq t_0 \in \boldsymbol{R}$ such that $q_i(t) \to 0$ as $t \to \infty$, $i = 1, 2, \cdots, n$. By means of arguments similar to those in the proof of Theorem 16.1 on p. 182 of Yoshizawa [6] one can show that the almost periodic limit p is itself a solution of (1.2). To show that $p(t) \equiv p(t+\omega)$ on $\boldsymbol{R}$, we replace x and y in the Lyapunov functional V by $p(t)$ and $p(t+\omega)$ respectively. As a consequence of Theorems 1.7 and 4.1 of Corduneanu [1] it will follow that $v(t) = V(t, p(t), p(t+\omega))$ is itself almost periodic in $t \in \boldsymbol{R}$. We have already seen that v is nonincreasing in t (see (2.13)) and hence the convergence of $v(t)$ as $t \to \infty$ to a limit say $v(\infty) \geqq 0$ follows, i.e.

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} V(t, p(t), p(t+\omega)) = v(\infty)\ .$$

By the almost periodicity of v in t, it will follow that for any $\varepsilon > 0$ and for any integer m exists a $\sigma_m \in (m, m + l(\varepsilon))$ such that

$$(2.22) \qquad 0 \leqq v(t) - v(t+\sigma_m) < \varepsilon \quad \text{for} \quad t \in \boldsymbol{R}\ .$$

Considering the limit in (2.22) as $m \to \infty$ and $\varepsilon \to 0$ we have

$$(2.23) \qquad v(t) \equiv v(\infty) \quad \text{on} \quad \boldsymbol{R}\ .$$

We have from (2.6) and (2.13) that

$$D^+v(t) \leqq -\sum_{i=1}^{n} |p_i(t) - p_i(t+\omega)|$$

implying

$$(2.24) \qquad v(t) + \sum_{i=1}^{n} \int_0^t |p_i(s) - p_i(s+\omega)|\,ds \leqq v(0)$$

and hence

$$(2.25) \quad \sum_{i=1}^{n} |p_i(t) - p_i(t+\omega)| + x^* \sum_{i=1}^{n} \int_0^\infty |p_i(s) - p_i(s+\omega)|\,ds \leqq x^* v(0)\ .$$

The uniform continuity of $|p_i(t) - p_i(t+\omega)|$ on $\boldsymbol{R}$ and its integrability on $[0, \infty)$ as in (2.25) will imply that

$$(2.26) \qquad p_i(t) - p_i(t+\omega) \to 0 \quad \text{as} \quad t \to \infty\ .$$

A consequence of (2.26) is that $\lim_{t\to\infty} v(t) = v(\infty) = 0$; then $v(t) \equiv v(\infty) = 0$

shows that $v(t) \equiv 0$ on $\boldsymbol{R}$ and hence $p_i(t) \equiv p_i(t+\omega)$ on $\boldsymbol{R}$, $i = 1, 2, \cdots, n$ and the proof is complete.

3. Global asymptotic stability. Let $p(t) = \{p_1(t), \cdots, p_n(t)\}$ be a strictly positive (componentwise) periodic solution of (1.2)-(1.3) such that

$$x_* \leqq p_i(t) \leqq x^*;\ t \in \boldsymbol{R};\ i = 1, 2, \cdots, n\ . \tag{3.1}$$

Such a solution $p(t)$ is by Theorem 2.1 a periodic solution of (1.1) and we say $p(t)$ is globally asymptotically stable (or attractive) if any other solution $x(t) = \{x_1(t), \cdots, x_n(t)\}$ of (1.1) such that

$$x_i(s) = \varphi_i(s) \geqq 0;\ s \in (-\infty, t_0];\ \varphi_i(t_0) > 0;\ \sup_{s \leqq t_0} \varphi_i(s) < \infty \tag{3.2}$$

where φ_i is continuous on $(-\infty, t_0]$, $t_0 \in \boldsymbol{R}$ has the property

$$\lim_{t \to \infty} \sum_{i=1}^{n} |x_i(t) - p_i(t)| = 0\ . \tag{3.3}$$

It is immediate that if $p(t)$ is globally asymptotically stable then $p(t)$ is in fact unique.

THEOREM 3.1. *Assume that the conditions of Theorem* 2.1 *hold. Then any periodic solution* $p(t)$ *of* (1.1) *with strictly positive components is globally asymptotically stable.*

PROOF. Let $x(t) = \{x_1(t), \cdots, x_n(t)\}$ be any solution of (1.1) and (3.2) and let $p(t) = \{p_1(t), \cdots, p_n(t)\}$ be a periodic solution of (1.1) with strictly positive components. Consider a Lyapunov functional $v(t) = V(t, x, p)$ defined by

$$\begin{aligned} v(t) = V(t, x, p) = \sum_{i=1}^{n} \Big(&|\log x_i(t) - \log p_i(t)| \\ &+ \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_0^\infty K_{ij}(s) \Big\{ \int_{t-s}^{t} a_{ij}(s+u) |x_j(u) - p_j(u)| du \Big\} ds \Big)\ , \qquad t > t_0 \end{aligned} \tag{3.4}$$

for any $t_0 \in \boldsymbol{R}$. Since both x and p are bounded and bounded away from zero (componentwise) for $t > t_0$,

$$\begin{aligned} v(t_0) \leqq \sum_{i=1}^{n} \Big\{ &|\log x_i(t_0) - \log p_i(t_0)| \\ &+ \sum_{\substack{j=1 \\ j \neq i}}^{n} (a_{ij}^u) \Big(\sup_{u \leqq t_0} |x_j(u) - p_j(u)| \Big) \Big\} < \infty \quad \text{for} \quad t_0 \in \boldsymbol{R}\ . \end{aligned} \tag{3.5}$$

Also we have

$$v(t) \geqq \sum_{i=1}^{n} |\log x_i(t) - \log p_i(t)|;\ s > t_0\ . \tag{3.6}$$

A direct calculation of the right derivative D^+v of $v(t)$ together with a simplification leads to

$$(3.7) \qquad D^+v(t) \leqq -m\sum_{j=1}^{n}|x_j(t)-p_j(t)|<0 \quad \text{if} \quad \sum_{i=1}^{n}|x_i(t)-p_j(t)|>0;\ t>t_0\,.$$

We claim that (3.7) implies (3.3). Suppose (3.3) is not valid; then there exists a sequence say $\{t_s\}$, $(s=0,1,2,\cdots)$ such that $\{t_s\}\to\infty$ as $s\to\infty$, $t_0<t_1<t_2<\cdots$ and

$$\sum_{j=1}^{n}|x_j(t_s)-p_j(t_s)|>\varepsilon;\ \text{for some positive number } \varepsilon,\ s=0,1,2,\cdots,$$

i.e.,

$$(3.8) \qquad D^+v(t_s)<-m\varepsilon;\ s=0,1,2,\cdots\,.$$

Since x_i and p_i are bounded for $t>t_0$ with bounded derivatives (from the integrodifferential equations satisfied by them), it will follow that v is uniformly continuous on $[t_0,\infty)$. If we now choose ε sufficiently small then we will have

$$(3.9) \qquad D^+v(u)<-m(\varepsilon/2) \quad \text{for} \quad u\in(t_s-\varepsilon,t_s);\ s=0,1,2,\cdots$$

and hence

$$v(t_s)-v(t_s-\varepsilon)\leqq\int_{t_s-\varepsilon}^{t_s}D^+v(u)du\leqq-m(\varepsilon^2/2)$$

implying that

$$\begin{aligned} v(t_s) &\leqq v(t_s-\varepsilon)-m(\varepsilon^2/2)\leqq v(t_{s-1})-m(\varepsilon^2/2)\leqq v(t_{s-2})-m2(\varepsilon^2/2)\\ &\leqq v(t_0)-ms(\varepsilon^2/2)\to-\infty \quad \text{as} \quad s\to\infty \end{aligned}$$

which contradicts the nonnegativity of $v(t)$. Thus our assertion (3.3) is valid and the proof is complete.

We conclude with a remark that the assumption of periodicity of the environment and the sufficient conditions (1.6), (1.7) and (2.3) have all some ecologically meaningful interpretations the details of which can be found in [2], [3], [4].

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