SUBMANIFOLDS OF THE GRASSMANN MANIFOLD WITH VANISHING CHERN FORMS

Dedicated to the memory of Professor Yozo Matsushima

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Introduction. We shall adopt the notation of Griffiths [2] throughout this paper. A holomorphic mapping between manifolds induces a holomorphic foliation on the domain manifold. In particular, for a complex manifold M^n immersed in C^N, we have

$$\Gamma(n, N): M^n \dashrightarrow G(n, N) ,$$

where $\Gamma = \Gamma(n, N)$ is the generalized Gauss map which is a holomorphic mapping and Γ_* has constant rank off a set of measure zero. Thus, $\ker(\Gamma_*)$ is an integrable subbundle almost everywhere. In fact, if E denotes the universal bundle of $G(n, N)$, then $\Gamma^*(E) = TM$, and the Chern forms on E pullback to the Chern forms on TM. Therefore, we are interested in studying the Chern forms on the universal bundle along submanifolds of $G(n, N)$.

In this paper we will give conditions in terms of the Chern forms on the universal bundle of the Grassmannian which imply that a submanifold of sufficiently high dimension has a parallel subbundle in the universal bundle.

Let $c_j(\Omega_E|_M)$ denote the j-th Chern form on the universal bundle of $G(n, N)$ restricted to M. Let $c_1(\Omega_q|_M)$ denote the first Chern form on CP^q. We shall prove the following:

THEOREM 2.8. *Assume that M^r is a complex submanifold of $G(n, N)$, with $r \geqq 2$. If $c_k(\Omega_E|_M) \neq 0$, and $c_{k+1}(\Omega_E|_M) = 0$, for*

$$k \leqq (r + N - n - 2)/(N - n) ,$$

then there exists a constant $(n - k)$-subbundle of $E|_M$.

The following Corollary is a direct result of the Theorem.

COROLLARY 2.9. *Assume that M^r is a submanifold of $G(n, N)$, with $r \geqq 2$, and $c_2(\Omega_E|_M) = 0$. Then M is contained in $G(1, N - n + 1) \subset G(n, N)$.*

We will also investigate some special cases: For example, the case given by requiring the k-th Chern form to be positive definite.

THEOREM 4.1. *Assume that* M^{k+1} *is a complex submanifold of* $G(n, n+2)$. *Suppose that* $(-1)^k c_k(\Omega_E|_M)$ *is positive definite on* M, *and* $c_{k+1}(\Omega_E|_M) = 0$. *Then* M *is a complex submanifold of* $G(k, k+2) \subset G(n, n+2)$.

THEOREM 5.1. *Assume that* $M \subset G(n, N)$ *as a complex manifold, and* $2 \leqq \dim(M)$. *Then* $c_2(\Omega_E|_M) = c_1(\Omega_q|_M)^2$ *if and only if* $M \subset CP^n \subset G(n, N)$, *where* c_1 *is the Chern form on* CP^q, $(q = \binom{N}{n} - 1)$.

We remark that $M \subset G(1, N-n+1) \subset G(n, N)$ means that $G(1, N-n+1)$ is the subset of $G(n, N)$ given by all complex n-planes which contain a fixed plane of dimension $(n-1)$. In general, when we write $M \subset G(k, N-n+k) \subset G(n, N)$ we shall mean that $G(k, N-n+k)$ is the subset of $G(n, N)$ given by the set of all n-planes in C^N which contain a fixed k-plane in C^N. This convention will be assumed throughout the paper.

1. Some Preliminaries. We will assume the following convention on indices throughout the paper:

$$1 \leqq A, B \leqq N; 1 \leqq i, j \leqq n; n+1 \leqq \mu, \nu \leqq N;$$
$$1 \leqq \alpha, \beta \leqq k; k+1 \leqq \gamma, \delta \leqq n.$$

We let $PU(C^N)$ denote the principal unitary bundle over C^N. We have the following structure equations

$$(1.1) \qquad dz = \omega^A e_A, \qquad d\omega^A = \omega^B \wedge \omega_B^A,$$
$$d\omega_A^B = \omega_A^C \wedge \omega_C^B, \qquad -\omega_A^B = \bar{\omega}_B^A$$

where the ω_A^B are the connection forms on the principal bundle, and the ω^A are the cannonical forms. For the projection map,

$$\pi: PU_o(C^N) \to G(n, N) = U(N, C)/U(n, C) \times U(N-n, C),$$

given by $\pi: (o, e_1, \cdots, e_N) \to P(e_1 \wedge \cdots \wedge e_n)$, where P is defined by $P: C^k - \{0\} \to CP^{k-1}$, such that $P(z) = \{cz | c \in C\}$. The 1-forms ω_i^μ are constant along fibres of the map π, and they are of type $(1, 0)$, so they push down to a coframe for $T_+^* G(n, N)$, where T_+ is the $(1, 0)$ tangent space. In fact, $T_+ G(n, N) = \text{Hom}(C^n, C^{N-n})$.

We would like to study the following bundle over $G(n, N)$:

$$E(n, N) = \{(z, T) \mid T \in G(n, N), \text{ and } z \in T\},$$

where T is an n-dimensional complex linear subspace of C^N. We shall call $E(n, N)$ the universal bundle of the Grassmann manifold. Every Hermitian holomorphic vector bundle has a corresponding complex Hermitian connection with the following properties,

(i) $D = D' + D''$,

(ii) $D'' = \bar\partial$ on functions

(iii) $Dh(v_1, v_2) = h(Dv_1, v_2) + h(v_1, Dv_2)$,

where $h(\ ,\ )$ denotes the Hermitian metric.

The 1-forms ω_i^j are defined on $E(n, N)$ and we have,

$$d\omega_i^j = \omega_i^k \wedge \omega_k^j + \Omega_i^j$$

and

$$\Omega_i^j = \omega_i^\mu \wedge \omega_\mu^j = -\omega_i^\mu \wedge \bar\omega_j^\mu$$

by (1.1).

Let $Z[A, B]$ denote the set of integers between A and B inclusive. Let $\varepsilon\binom{(i)}{(j)}$ denote the sign of the permutation $\binom{i1,\ \cdots,\ ik}{j1,\ \cdots,\ jk}$, where $(j) = (j1, \cdots, jk)$ and $(i) = (i1, \cdots, ik)$ are k-tuples of distinct elements taken from the set $Z[1, n]$. We define Chern forms in the following way

$$\mathrm{Det}(tI + (\sqrt{-1}/2\pi)\Omega_E) = \sum_{k=0}^{n} t^{n-k} c_k(\Omega_E) ,$$

or

$$c_k(\Omega_E) = (\sqrt{-1}/2\pi)^k/k! \sum \varepsilon\binom{(j)}{(i)} \Omega_{i1}^{j1} \wedge \cdots \wedge \Omega_{ik}^{jk} ,$$

$$= (1/k!)(\sqrt{-1}/2\pi)^k (-1)^k \sum \varepsilon\binom{(i)}{(j)} \omega_{i1}^{\mu1} \wedge \bar\omega_{j1}^{\mu1} \Big/ \qquad \wedge \bar\omega_{jk}^{\mu k}$$

where the summation is over the set of all k-tuples $(i1, \cdots, ik)$, with distinct elements taken from the set $Z[1, n]$ and over all possible permutations (j) of each k-tuple. Now, let $r = k(k + 1)/2$. Then

$$c_k(\Omega_E) = (-1)^r (\sqrt{-1}/2\pi)^k/k! \sum \varepsilon\binom{(i)}{(j)} \omega_{i1}^{\mu1} \wedge \cdots \wedge \omega_{ik}^{\mu k} \wedge \bar\omega_{j1}^{\mu1} \wedge \cdots \wedge \bar\omega_{jk}^{\mu k} .$$

We drop the summation over the permutations assuming that $1 \leq i1 < \cdots < ik \leq n$, and define

$$(\omega_{i1}^{\nu1} \wedge \cdots \wedge \omega_{ik}^{\nu k})^\sigma = \omega_{i1}^{\mu1} \wedge \cdots \wedge \omega_{ik}^{\mu k}$$

where σ denotes the permutation such that $\nu\sigma(i) = \mu i$. We also replace the permutation (j) with a permutation $\sigma2$ to obtain

$$c_k(\Omega_E) = (-1)^r (\sqrt{-1}/2\pi)^k/k! \sum (\omega_{i1}^{\mu1} \wedge \cdots \wedge \omega_{ik}^{\mu k})^{\sigma 1} \wedge (\bar{\omega}_{i1}^{\mu1} \wedge \cdots \wedge \bar{\omega}_{ik}^{\mu k})^{\sigma 2} ,$$

where the summation is over the set of all distinct ordered k-tuples with elements in the set $Z[1, n]$, and over the set of all permutations $\sigma 1$, and $\sigma 2$.

Now we define,

$$(1.2) \qquad\qquad A_{i1,\cdots,ik}^{\mu1,\cdots,\mu k} = \sum_{\sigma \in S_k} (\omega_{i1}^{\mu1} \wedge \cdots \wedge \omega_{ik}^{\mu k})^{\sigma} .$$

Then

$$A_{i1,\cdots,ik}^{\mu1,\cdots,\mu k} \wedge \bar{A}_{i1,\cdots,ik}^{\mu1,\cdots,\mu k} = \sum_{\sigma 1} \sum_{\sigma 2} (\omega_{i1}^{\mu1} \wedge \cdots \wedge \omega_{ik}^{\mu k})^{\sigma 1} \wedge (\bar{\omega}_{i1}^{\mu1} \wedge \cdots \wedge \bar{\omega}_{ik}^{\mu k})^{\sigma 2} .$$

Finally, we have,

$$(1.3) \qquad c_k(\Omega_E) = (-1)^{k(k+1)/2} (\sqrt{-1}/2\pi)^k/k! \sum A_{i1,\cdots,ik}^{\mu1,\cdots,\mu k} \wedge \bar{A}_{i1,\cdots,ik}^{\mu1,\cdots,\mu k} .$$

It is now clear that the Chern form has a special property, which we define in the following way:

DEFINITION 1.4. A (k, k)-form α is said to be strongly positive when it satisfies the following relation:

$$\alpha = (-1)^{k(k-1)/2} (\sqrt{-1}/2\pi)^k \sum_x \beta_x \wedge \bar{\beta}_x ,$$

where β_x is of type $(k, 0)$.

It is straightforward to show that a strongly positive form is a real valued form which is positive semi-definite. We immediately have the following from (1.3):

PROPOSITION 1.5. *The form* $(-1)^k c_k(\Omega_E)$ *is strongly positive.*

On the other hand, we define

$$F(n, N) = \{(z, T) \,|\, T \in G(n, N), \quad \text{and} \quad z \in T^\perp\} .$$

2. Chern Forms on $G(n, N)$. Recall that ω_i^μ give a coframe for $T_+^* G(n, N)$, and we know that

$$(2.1) \qquad d\omega_i^\mu = \omega_i^j \wedge \omega_j^\mu = (\delta_\nu^\mu \omega_i^j - \delta_i^j \omega_\nu^\mu) \wedge \omega_j^\nu .$$

Thus,

$$(2.2) \qquad\qquad \omega_{i,\nu}^{j,\mu} = (\omega_i^j \delta_\nu^\mu - \delta_i^j \omega_\nu^\mu)$$

is the connection form on $TG(n, N)$ as can be seen from (1.1). Similarly,

$$d\omega_{i,\nu}^{j,\mu} = \omega_{i,\nu}^{k,\nu'} \wedge \omega_{k,\nu'}^{j,\mu} + \Omega_{i,\nu}^{j,\mu} .$$

Consequently,

$$(2.3) \qquad\qquad \Omega_{i,\nu}^{j,\mu} = \delta_\nu^\mu (\omega_i^{\nu'} \wedge \omega_{\nu'}^j) - \delta_i^j (\omega_\nu^k \wedge \omega_k^\mu) .$$

Therefore the Chern forms on $G(n, N)$ are determined by the Chern forms on $E(n, N)$ and $F(n, N)$. We now have:

LEMMA 2.4. *If* $f: M^n \to G(n, N)$ *is a holomorphic mapping with* $\mathrm{rank}(f_*) = r$, *then the matrix of 1-forms* (ω_i^μ) *has exactly* r *linearly independent forms.*

PROOF. This follows since ω_i^μ give a coframe for $T_+^* G(n, N)$.

LEMMA 2.5. *Let* U *be a unitary transformation on the unitary frame* $\{e_1, \cdots, e_N\}$ *in* C^N, *such that*

$$U = \left[\begin{array}{c|c} U_1 & O \\ \hline O & U_2 \end{array} \right]$$

where $U_1 = U_i^j$ *is a unitary transformation on* $\mathrm{span}\{e_1, \cdots, e_n\}$ *and* $U_2 = U_\nu^\mu$ *is a unitary transformation on* $\mathrm{span}\{e_{n+1}, \cdots, e_N\}$. *Then*

$$\omega_i^{\mu *} = U_i^j \omega_j^\nu \bar{U}_\mu^\nu .$$

PROOF. We have

$$\omega_A^{B*} = d U_A^B \bar{U}_C^B + U_A^B \omega_B^C \bar{U}_D^C ,$$

but $U_i^\mu = U_\mu^i = 0$, and so we obtain the lemma.

We now have:

THEOREM 2.6. *Assume* $M^n \subset C^N$ *and* $\mathrm{rank}(\Gamma_*) = r$. *Then* M *is holomorphically foliated by totally geodesic submanifolds which are totally geodesic submanifolds of* C^N.

PROOF. A proof of this fact can be given, if we observe that the kernel of the Gauss map is exactly the relative nullity space, but we shall give here an alternate proof which makes use of a lemma which we will need later.

We first observe that $\ker(\Gamma_*)$ is a holomorphic subbundle and that for a suitable choice of unitary frames $\ker(\Gamma_*) = \mathrm{span}\{e_{r+1}, \cdots, e_n\}$. Moreover, $D_{e_i} e_\gamma - D_{e_\gamma} e_i = [e_i, e_\gamma] \in TM$, thus $0 = \omega_i^\mu(e_\gamma) = \omega_\gamma^\mu(e_i)$, where $\gamma \in Z[r + 1, n]$ and $i \in Z[1, n]$, since $\Gamma_*(e_\gamma) = (\omega_i^\mu(e_\gamma)) = 0$. Finally we can write

$$\Gamma_* = \left| \begin{array}{c} \omega_\alpha^\mu \\ 0 \end{array} \right|$$

where $\alpha \in Z[1, r]$, and $\mu \in Z[n + 1, N]$. We will need the following lemma to complete the proof.

LEMMA 2.7. *Assume* Γ_* *has rank* r. *Then* $\omega_\gamma^\alpha(e_\delta) = 0$, *for each* $\gamma, \delta \in$

$Z[r + 1, n]$, and for each $\alpha \in Z[1, r]$.

PROOF. Notice that if $\omega_\alpha^{n+1} = \omega_\alpha^{n+2} = \cdots = \omega_\alpha^N = 0$, then $e_\alpha \in \ker(\Gamma_*)$, for $\alpha \in Z[1, r]$. But this is not possible, since $\ker(\Gamma_*) = \mathrm{span}\{e_{r+1}, \cdots, e_n\}$.

Now let us assume that $\omega_1^{n+1}, \cdots, \omega_{s1}^{n+1}$ make up a maximal linearly independent set for the 1-forms in the $(n + 1)$-st column. However, ω_{s1+1}^{n+1} can be written as a linear combination of the others. In fact, after a suitable choice of unitary frames for $\mathrm{span}\{e_1, \cdots, e_r\}$ we can assume that ω_{s1+1}^{n+1} is a multiple of ω_1^{n+1}. In other words,

$$\omega_{s1+1}^{n+1} = c(x)\omega_1^{n+1} .$$

Therefore we consider the following unitary transformation:

$$(2.7a) \qquad\qquad e_1^* = (e_1 + \bar{c}e_{s1+1})/(1 + c\bar{c})^{1/2} ,$$
$$e_{s1+1}^* = (-ce_1 + e_{s1+1})/(1 + c\bar{c})^{1/2} ,$$

so that $\omega_1^{n+1*} = (1 + c\bar{c})^{1/2}\omega_1^{n+1}$, and $\omega_{s1+1}^{n+1*} = 0$. If we repeat this process on the forms $\omega_{s1+2}^{n+1}, \cdots, \omega_r^{n+1}$ we will obtain $\omega_{s1+1}^{n+1} = \cdots = \omega_r^{n+1} = 0$. On the other hand, $\omega_{r+1}^{n+1} = 0$, so that

$$d\omega_{r+1}^{n+1} = \omega_{r+1}^A \wedge \omega_A^{n+1} = 0 \quad \text{(sum over } A\text{)}$$
$$= \omega_{r+1}^\alpha \wedge \omega_\alpha^{n+1} = 0 \quad \text{(sum over } \alpha \in Z[1, r]) .$$

We conclude Cartan's lemma that $\omega_{r+1}^1, \cdots, \omega_{r+1}^{s1}$ can be written as a linear combination of the set $\{\omega_1^{n+1}, \cdots, \omega_{s1}^{n+1}\}$, and consequently

$$\omega_{r+1}^1(e_\gamma) = \cdots = \omega_{r+1}^{s1}(e_\gamma) = 0 .$$

Moreover,

$$\omega_r^1(e_\delta) = \cdots = \omega_r^{s1}(e_\delta) = 0 ,$$

for each $\gamma, \delta \in Z[r + 1, n]$. Now from the set $\{\omega_{s1+1}^{n+2}, \cdots, \omega_r^{n+2}\}$ we can choose a maximal linearly independent set, say $\{\omega_{s1+1}^{n+2}, \cdots, \omega_{s2}^{n+2}\}$. If we repeat the above process we will have $\omega_r^1(e_\delta) = \cdots = \omega_r^{s2}(e_\delta) = 0$. Finally, observe that we can continue this process until ω_r^μ is among a maximal linearly independent set, since otherwise $\omega_r^{n+1} = \cdots = \omega_r^N = 0$, which implies that $e_r \in \ker(\Gamma_*)$. This completes the proof of the lemma.

Using the lemma we see that $\omega_r^\alpha(e_\delta) = \omega_r^\mu(e_\delta) = 0$, for $\gamma, \delta \in Z[r + 1, n]$ and $\alpha \in Z[1, r]$. Therefore, along the integrable submanifolds of $\ker(\Gamma_*)$ the second fundamental form is zero. This completes the proof of the theorem.

THEOREM 2.8. Assume that M^r is a complex submanifold of $G(n, N)$, with $r \geq 2$. If $c_k(\Omega_E|_M) \neq 0$, and $c_{k+1}(\Omega_E|_M) = 0$, for

$$k \leq (r + N - n - 2)/(N - n) ,$$

then there exists a constant $(n - k)$-subbundle of $E|_M$.

COROLLARY 2.9. *If M^r is a submanifold of $G(n, N)$, with $r \geq 2$, and $c_2(\Omega_E|_M) = 0$, then M is contained in $G(1, N - n + 1) \subset G(n, N)$.*

PROOF. Because of the complexity of the proof of Theorem 2.8 we will first give a proof of Corollary 2.9 and then sketch the proof of Theorem 2.8. We begin by observing that the matrix of forms (ω_i^μ) has at least two linearly independent forms. We assume without loss of generality that ω_1^{n+1} is a nonvanishing form in some neighborhood. Now using the transformation of (2.7a), we can assume that $\omega_2^{n+1} = \cdots = \omega_n^{n+1} = 0$, since otherwise $c_2 \neq 0$. Now observe that

$$A_{1,2}^{n+1, \mu} = 0 \, , \quad \text{for each} \quad \mu \in Z[n + 2, N] \, .$$

In fact $\omega_\gamma^\mu \wedge \omega_1^{n+1} = 0$ for each $\gamma \in Z[2, n]$ and each $\mu \in Z[n + 1, N]$. Therefore we can assume without loss of generality that ω_1^{n+2} is nonvanishing and linearly independent of ω_1^{n+1}. By the same reasoning as above $\omega_2^{n+2} = \cdots = \omega_n^{n+2} = 0$. Moreover, $\omega_\gamma^\mu = 0$, for each $\gamma \in Z[2, n]$, and $\mu \in Z[n + 1, N]$. Now we have

(a) $0 = d\omega_\gamma^{n+1} = \omega_\gamma^1 \wedge \omega_1^{n+1}$,

(b) $0 = d\omega_\gamma^{n+2} = \omega_\gamma^1 \wedge \omega_1^{n+2}$.

By Cartan's Lemma $\omega_\gamma^1 = 0$, and so $de_\gamma = \omega_\gamma^\delta e_\delta$ and the universal bundle contains a constant $(n - 1)$-dimensional subbundle which implies that $M^n \subset G(1, N - n + 1) \subset G(n, N)$.

PROOF OF THEOREM 2.8. We say that $\{i1, \cdots, ik\}$ and $\{\mu1, \cdots, \mu k\}$ satisfy condition $(*)k$ whenever,

$$1 \leq i1 < \cdots < ik \leq n$$

and

$$n + 1 \leq \mu1 \leq \cdots \leq \mu k \leq N \, .$$

Now, since $c_k(\Omega_E|_M) \neq 0$, there exist the sets $\{i1, \cdots, ik\}$ and $\{\mu1, \cdots, \mu k\}$ satisfying condition $(*)k$, such that $A_{i1, \cdots, ik}^{\mu1, \cdots, \mu k} \neq 0$, where A is defined by (1.2). On the other hand $c_{k+1}(\Omega_E|_M) = 0$ at each point of M, so that for any set of indices satisfying $(*)k + 1$, we have

$$A_{i1, \cdots, i(k+1)}^{\mu1, \cdots, \mu(k+1)} = 0 \, ,$$

and in particular

$$A_{i1, \cdots, i(k+1)}^{\mu1, \cdots, \mu1} = 0 \, .$$

In other words, $\omega_{i1}^{\mu1} \wedge \cdots \wedge \omega_{i(k+1)}^{\mu1} = 0$, for each $\mu1 \in Z[n + 1, N]$. Therefore, the matrix (ω_i^μ) can have at most k linearly independent forms

in any one column.   Now, since $\dim M = r$, the matrix contains exactly r linearly independent forms.   If each column contains no more than $(k-1)$ linearly independent forms, then there would be at most $(N-n) \times (k-1)$ linearly independent 1-forms.   However,

$$(N-n)(k-1) \leqq r-2$$

implies that there are at least two columns containg k linearly independent 1-forms from the set of r linearly independent 1-forms.   We assume without loss of generality that $\omega_1^{n+1}, \cdots, \omega_k^{n+1}$ are linearly independent. Then ω_{k+1}^{n+1} can be written as a linear combination of $\omega_1^{n+1}, \cdots, \omega_k^{n+1}$.   But by (2.7a) we can choose unitary frames so that $\omega_{k+1}^{n+1} = \cdots = \omega_n^{n+1} = 0$. And if we assume that the $(n+2)-nd$ column contains k linearly independent forms, then $c_{k+1}(\Omega_E|_M) = 0$ implies

$$A_{1,\cdots,k+1}^{n+1,\cdots,n+1,n+2} = 0$$

or,

$$\omega_{k+1}^{n+2} \wedge \omega_1^{n+1} \wedge \cdots \wedge \omega_k^{n+1} = 0 \ ,$$

since $\omega_{k+1}^{n+1} = 0$.   In fact, since ω_r^{n+2} is a linear combination of the set $\omega_1^{n+1}, \cdots, \omega_k^{n+1}$.   Therefore, we may assume that $\omega_1^{n+2}, \cdots, \omega_k^{n+2}$ are linearly independent and ω_{k+1}^{n+2} must be a linear combination of the set $\omega_1^{n+2}, \cdots, \omega_k^{n+2}$. We conclude that $\omega_{k+1}^{n+2} = \cdots = \omega_n^{n+2} = 0$.   By the same argument we have just given we obtain that $\omega_r^\mu = 0$, for $\gamma \in Z[k+1, n]$.   Now we write

(a) $0 = d\omega_r^{n+1} = \omega_r^A \wedge \omega_A^{n+1}$,

(b) $0 = d\omega_r^{n+2} = \omega_r^A \wedge \omega_A^{n+2}$.   (sum over A)

However, by Cartan's lemma ω_r^α is a linear combination of $\omega_1^{n+1}, \cdots, \omega_k^{n+1}$, as a consequence of equation (a), but ω_r^α is also a linear combination of $\omega_1^{n+2}, \cdots, \omega_k^{n+2}$, as a consequence of (b).

We conclude that ω_r^α is identically zero for each $\alpha \in Z[1, k]$, and $\gamma \in Z[k+1, n]$.   We have shown that

$$de_r = \omega_r^\delta e_\delta \ .$$

In other words, the span of $\{e_{k+1}, \cdots, e_n\}$ is constant along M.   This completes our proof.

THEOREM 2.10.   *Assume that M^n is a submanifold of C^N, and $\mathrm{rank}(\Gamma_*) = r$.   Furthermore, assume that $c_k(\Omega_E|_M) \neq 0$, and $c_{k+1}(\Omega_E|_M) = 0$.   Then $r \leqq (k-1)(N-n) + 1$.*

PROOF.   If r does not satisfy the inequality, we shall have a constant subbundle of $E_{\Gamma(M)}$, which pulls back to TM as a subbundle of $\ker(\Gamma_*)$, which is parallel for the connection on C^N.   In fact, $\omega_\alpha^\mu(e_r) = \omega_r^\mu(e_\alpha) = 0$,

for each $\alpha \in Z[1, k]$, and each $\gamma \in Z[k + 1, n]$. Therefore, $e_r \in \ker(\Gamma_*)$. However, the subbundle has dimension $(n - k)$, and $\ker(\Gamma_*)$ has dimension $(n - r)$. Therefore, $n - k \leq n - r$, implies that

$$0 \leq k - r; \quad \text{or}, \quad (k - 1)(N - n) + 1 < r \leq k \leq (k - 1)(N - n) + 1 \,.$$

But this is a contradiction. Consequently r must satisfy the inequality which proves the theorem.

We end this section with a direct application of Theorem 2.8, and Corollary 2.9.

COROLLARY 2.11. *Assume M^n is a submanifold of C^N. If $c_1 \neq 0$, and $c_2 = 0$, then $\mathrm{rank}(\Gamma_*) = 1$, and if M is a complete manifold, then M is cylindrical.*

PROOF. To see that M is cylindrical we use Theorem 7 from Abe [1].

We remark that results similar to Theorem 2.8 and Corollary 2.9 hold for the bundle $F(n, N)$.

3. A Preliminary Result. We begin with:

PROPOSITION 3.1. *Let $\{X_1, \cdots, X_{m+k}\}$ be a frame on a vector bundle over a manifold M. Let $\{\theta^1, \cdots, \theta^{m+k}\}$ denote the corresponding coframe. Let $B \in GL(k, C)$, $I \in GL(n, C)$ and assume that $X_i^* = B_i^j X_j$, for each $i \in Z[1, k]$, and $X_\delta^* = X_\delta$ for each $\delta \in Z[k+1, k + m]$. Then*

$$\theta^{i*} = (B^{-1})_j^i \theta^j$$

for each $i \in Z[1, k]$ and $(\theta^{k+1})^* = \theta^{k+1}$, $\cdots$, $(\theta^{k+m})^* = \theta^{k+m}$.

The proof is left to the reader.

PROPOSITION 3.2. *Assume that M is a complex submanifold of $G(n, N)$, such that*

$$A_{1,\cdots,k}^{\mu 1,\cdots,\mu k}(X_1, \cdots, X_k) \neq 0$$

for some $X_1 \wedge \cdots \wedge X_k \neq 0$. Then there exists a change of frames on $E|_M$ such that

$$(A_{1,\cdots,k}^{n+1,\cdots,n+1})^* \neq 0 \,.$$

PROOF. Consider the unitary transformation

$$(e_{n+1})^* = t_1 e_{n+1} + \cdots + t_s e_{n+s} \,,$$

with the other elements of the frame given by the Graham-Schmidt process. We have assumed above that $\{n + 1, \cdots, n + s\}$ includes all the indices given by $\{\mu 1, \cdots, \mu k\}$. In addition, we assume that $|t_1|^2 + \cdots +$

$|t_s|^2 = 1$.   Then

$$(\omega_i^{n+1})^* = \bar{t}_1\omega_i^{n+1} + \cdots + \bar{t}_s\omega_i^{n+s} .$$

Thus we have

$$(A_{1,\cdots,k}^{n+1,\cdots,n+1})^* = \bar{t}_1^k A_{1,\cdots,k}^{n+1,\cdots,n+1} + \cdots + \bar{t}_1^{r(1)} \cdots \bar{t}_s^{r(s)} A_{1,\cdots,k}^{n+1,\cdots,n+s}$$
$$+ \cdots + \bar{t}_s^k A_{1,\cdots,k}^{n+s,\cdots,n+s} ,$$

where $n + j$ occurs in the superscript $r(j)$ times.   Now, observe that the right hand side is nonzero for some choice of $t_1, \cdots, t_s$.

4. Submanifolds with vanishing Chern forms. We are interested in the consequences of the vanishing of the Chern forms on the universal bundle. We shall show that under special circumstances the assumption that the Chern forms vanish implies that the universal bundle contains a constant subbundle.

THEOREM 4.1. *Assume that M^{k+1} is a complex submanifold of $G(n, n+2)$. Suppose that $(-1)^k c_k(\Omega_E|_M)$ is positive definite on M, and $c_{k+1}(\Omega_E|_M) = 0$. Then M is a complex submanifold of $G(k, k+2) \subset G(n, n+2)$.*

PROOF. By Proposition 3.2, we have that for any choice $X_1, \cdots, X_k \in TM$ satisfying $X_1 \wedge \cdots \wedge X_k \neq 0$, we can choose a frame $\{e_1, \cdots, e_{n+2}\}$ for $G(n, n+2) \times C^{n+2}$ which gives

$$A_{1,\cdots,k}^{n+1,\cdots,n+1}(X_1, \cdots, X_k) \neq 0 .$$

Therefore, we can assume that there are exactly k linearly independent forms in the $(n+1)$-th column. Using the transformation of (2.7a), we can assume that $\omega_{k+1}^{n+1} = \cdots = \omega_n^{n+1} = 0$. The linearly independent forms span a k-plane in the cotangent bundle, and so we assume that $X_1, \cdots, X_k$ span the associated plane in the tangent bundle. Let X_{k+1} be orthogonal to the plane generated by $X_1 \wedge \cdots \wedge X_k$, then $\omega_\delta^\mu(X_{k+1}) = 0$, for each $\delta \in Z[k+1, n]$, and each $\mu \in Z[n+1, n+2]$. Furthermore, $\omega_\alpha^{n+1}(X_{k+1}) = 0$, for each $\alpha \in Z[1, k]$.

We claim that $\omega_\delta^{n+2} = 0$, for each $\delta \in Z[k+1, n]$. To prove this, observe that $\omega_{k+1}^{n+2} = \sum_\alpha a_\alpha \omega_\alpha^{n+1}$. Otherwise,

$$A_{1,\cdots,k+1}^{n+1,\cdots,n+1,n+2}(X_1, \cdots, X_{k+1}) \neq 0 ,$$

which contradicts our assumption on $c_{k+1}(\Omega_E|_M)$. By using an appropriate transformation on $\{e_1, \cdots, e_k\}$, we can assume that $\omega_1^{n+2}(X_{k+1}) \neq 0$, and $0 = \omega_2^{n+2}(X_{k+1}) = \cdots = \omega_k^{n+2}(X_{k+1})$. Note that this transformation does not effect the span of $\omega_1^{n+1}, \cdots, \omega_k^{n+1}$. Now, choose $Y_1 \in \text{span}\{X_1, \cdots, X_k\}$, such

that $\omega_2^{n+1}(Y_1) = \cdots = \omega_k^{n+1}(Y_1) = 0$, and choose $Y_\alpha \in \text{span}\{X_1, \cdots, X_k\}$, such that $\{Y_1, \cdots, Y_k\}$ are linearly independent, and mutually orthogonal. Let $Y_{k+1} = X_{k+1}$. If $a_1 \neq 0$, then

$$A_{1,\cdots,k+1}^{n+1,\cdots,n+1,n+2,n+2}(Y_1, \cdots, Y_{k+1})$$
$$= 2\omega_1^{n+2}(Y_{k+1})\omega_2^{n+1}(Y_2) \cdots \omega_k^{n+1}(Y_k)\omega_{k+1}^{n+2}(Y_1) \neq 0 ,$$

where we have assumed for simplicity that $\omega_2^{n+1} \wedge \cdots \wedge \omega_k^{n+1}(Y_2, \cdots, Y_k) = \omega_2^{n+1}(Y_2) \cdots \omega_k^{n+1}(Y_k)$. We conclude that $a_1 = 0$. However, since $c_k(\Omega_E|_M)$ is positive definite there exist $\alpha \in Z[2, k]$, such that $\omega_\alpha^{n+2}(Y_1) \neq 0$. We assume that $\alpha = 2$. By an appropriate transformation on $\{e_2, \cdots, e_k\}$, we can assume that $\omega_2^{n+2}(Y_1) \neq 0$, and $\omega_3^{n+2}(Y_1) = \cdots = \omega_k^{n+2}(Y_1) = 0$. Now, we can repeat the above procedure to obtain $a_2 = 0$, and eventually $a_1 = a_2 = \cdots = a_k = 0$. This proves the claim, and we have $\omega_\delta^\mu = 0$, for an appropriate choice of frame.

We now apply Cartan's Lemma to
(a) $d\omega_\delta^{n+1} = \omega_\delta^\alpha \wedge \omega_\alpha^{n+1} = 0$,
(b) $d\omega_\delta^{n+2} = \omega_\delta^\alpha \wedge \omega_\alpha^{n+2} = 0$.
We observe that by (a): ω_δ^α is a linear combination of $\omega_1^{n+1}, \cdots, \omega_k^{n+1}$, and by (b): ω_δ^α is a linear combination of $\omega_1^{n+2}, \cdots, \omega_k^{n+2}$. However, $\omega_\delta^\alpha(X_{k+1}) = 0$, by (a), and consequently $\omega_\delta^\alpha = 0$. Now observe that ω_δ^α is a linear combination of $\omega_2^{n+1}, \cdots, \omega_k^{n+1}$, and therefore $\omega_\delta^\alpha(Y_1) = 0$. This implies that $\omega_\delta^2 = 0$. We repeat this analysis to obtain that

$$\omega_\delta^\alpha = 0 .$$

This gives our result.

In general we have:

THEOREM 4.2. *Assume that M^m is a con* $\quad G(n,$
$n + p)$ for $p \geqq 2$. If $(-1)^k c_k(\Omega_E|_M)$ is positive de $\quad _E|_M) =$
0, for $m \geqq 2k$, then $M \subset G(k, k + p) \subset G(n, n +$

PROOF. Following the proof of the previous theorem, and applying Proposition 3.2 we have

$$A_{1,\cdots,k}^{n+1,\cdots,n+1}(X_1, \cdots, X_k) \neq 0 ,$$

and $\omega_\delta^\mu(X_{k+1}) = \cdots = \omega_\delta^\mu(X_{2k}) = 0$, where $X_{k+1}, \cdots, X_{2k}$ are orthogonal to $X_1, \cdots, X_k$. Moreover we can choose frames so that

$$A_{1,\cdots,k}^{n+2,\cdots,n+2}(X_{k+1}, \cdots, X_{2k}) \neq 0 .$$

Now apply Cartan's Lemma to obtain the theorem.

COROLLARY 4.3. *Let M be a complex submanifold of $G(n, n + p)$ for*

$p \geqq 2$. *Suppose that there exists a vector subbundle V of TM such that* $\dim_c V \geqq 2k$, *and* $(-1)^k c_k(\Omega_E|_M)$ *is positive definite on V with* $c_{k+1}(\Omega_E|_M) = 0$. *Then M is a complex submanifold of* $G(k, p+k) \subset G(n, n+p)$.

PROOF. We shall apply the same procedure as in Theorem 4.2 to obtain that $\omega_\beta^\mu = 0$. Now since $(-1)^k c_k$ is positive definite on V we can apply Cartan's lemma to obtain the corollary.

5. Results in Projective Spaces. In general, projective spaces do not yield results analogous to those of Section 4. The problem arises from the fact that in the projective case we can only apply Cartan's lemma once. This fact allows us to conclude that ω_β^α is a linear combination of $\{\omega_\alpha^\mu\}$, but this is all that we can reasonably expect. The Chern forms on the tangent bundle of submanifolds of a complex projective space can be written as products of powers of the Chern forms on the universal line bundle E and the Chern forms of the vector bundle $F = E^\perp$ restricted to the manifold. From this point of view we are interested in the question: When is a submanifold of complex projective space CP^q contained as a submanifold in a complex Grassmann submanifold of CP^q.

Of course, if M is a submanifold of CP^q $\left(q = \binom{N}{n} - 1 \right)$, then the basis for C^{q+1} can be given by

$$\tilde{e}_{i1} \wedge \cdots \wedge \tilde{e}_{in}, \qquad 1 \leqq i1 < \cdots < in \leqq N$$

where $\{\tilde{e}_1, \cdots, \tilde{e}_N\}$ is a standard basis for C^N. Therefore, M is a submanifold of $G(n, N)$ if and only if each $T \in M \subset CP^q$ satisfies

$$T = X_1 \wedge \cdots \wedge X_n,$$

where $\{X_1, \cdots, X_n\}$ is a set of linearly independent vectors in C^N. In general

$$T = \sum a_{i1 \cdots in} \tilde{e}_{i1} \wedge \cdots \wedge \tilde{e}_{in},$$

and so $T \in G(n, N)$ if and only if

$$a_{i1 \cdots in} = \sum \varepsilon \binom{(i)}{(j)} a_{j1} \cdots a_{jn}$$

has a solution in terms of the variables $a_1, \cdots, a_N$. This is a nonlinear algebraic system of $\binom{N}{n}$ equations in N unknowns.

Consider $G(n, N) \subset CP^q$, and let $E(n, N)$ denote the universal bundle of $G(n, N)$. Let $E(1, q+1)$ denote the universal bundle of CP^q. Then $E(1, q+1)|_{G(n,N)} = \wedge^n E(n, N)$. The problem encountered in evaluating

the Chern forms of $E(n, N)$ in terms of the Chern forms on $E(1, q + 1)$ is that $E(n, N)$ has no natural generalization to a bundle over CP^q.

However, since $G(n, N)$ is embedded in CP^q we can restrict the universal line bundle over CP^q to $G(n, N)$. For $\{e_1, \cdots, e_N\}$ a frame for C^N, we have that $e_1 \wedge \cdots \wedge e_n$ is a frame for the universal line bundle. Differentiating this, we get

$$d(e_1 \wedge \cdots \wedge e_n) = (-1)^{i-1}\omega_i^{\mu}e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{\mu} \wedge e_{i+1} \wedge \cdots \wedge e_n .$$

From this we easily obtain the Chern form for the line bundle

$$c_1 = (-\sqrt{-1}/2\pi) \sum \omega_i^{\mu} \wedge \bar{\omega}_i^{\mu} .$$

For the remainder of this section we will assume that M is a submanifold of $G(n, N)$ (of dimension greater than one) embedded in CP^q. We note that $G(n, N) \cong G(N - n, N)$ and therefore we will also consider $G(n, N)$ as embedded in $CP^{q'}$, where $q' = \left(\begin{smallmatrix} N \\ N - n \end{smallmatrix}\right) - 1$. We will denote by Ω_q (resp. $\Omega_{q'}$) the curvature of the universal line bundle of CP^q (resp. $CP^{q'}$). We now have the following:

THEOREM 5.1. *Assume that $M \subset G(n, N)$ as a complex manifold, and* $2 \leq \dim(M)$. *Then* $c_2(\Omega_E|_M) = c_1(\Omega_q|_M)^2$ *if and only if* $M \subset G(1, N - n + 1) \subset G(n, N)$.

PROOF. We begin with the case $M \subset G(2, 4)$. We observe that one direction is obvious. Therefore, we assume that $c_2 = c_1^2$. Cancelling equal terms on either side, we obtain

(5.2) $$\Lambda = \Pi$$

where

$$\Lambda = \omega_1^{n+1} \wedge \omega_2^{n+2} \wedge \bar{\omega}_1^{n+2} \wedge \bar{\omega}_2^{n+1} + \omega_1^{n+2} \wedge \omega_2^{n+1} \wedge \bar{\omega}_1^{n+1} \wedge \bar{\omega}_2^{n+2}$$

and

$$\Pi = \omega_1^{n+1} \wedge \omega_2^{n+2} \wedge \bar{\omega}_1^{n+1} \wedge \bar{\omega}_2^{n+2} + \omega_1^{n+2} \wedge \omega_2^{n+1} \wedge \bar{\omega}_1^{n+2} \wedge \bar{\omega}_2^{n+1}$$
$$+ 2\omega_1^{n+1} \wedge \omega_1^{n+2} \wedge \bar{\omega}_1^{n+1} \wedge \bar{\omega}_1^{n+2} + 2\omega_2^{n+1} \wedge \omega_2^{n+2} \wedge \bar{\omega}_2^{n+1} \wedge \bar{\omega}_2^{n+2} .$$

We assume without loss of generality that $\omega_1^{n+1} \neq 0$. Let $\theta_1 = \omega_1^{n+1}$, and let X_1 denote the associated vector field.
(Case I) We assume that we can choose X_2 orthogonal to X_1 such that $\omega_1^{n+1}(X_2) = 0$, $\omega_2^{n+1}(X_2) \neq 1$, and $\omega_2^{n+1}(X_1) = 0$. Now we can write

$$\omega_1^{n+2} = a_1^{n+2}\theta_1 + b_1^{n+2}\theta_2 + \theta_3 ,$$

where $\theta_2(X_2) = 1$ and $\theta_2 = \omega_2^{n+1}$ and θ_3 is orthogonal to θ_1 and θ_2. Likewise

$$\omega_2^{n+2} = a_2^{n+2}\theta_1 + b_2^{n+2}\theta_2 + \theta_4 ,$$

where θ_4 is orthogonal to θ_1 and θ_2. We compute

$$\Lambda(X_1, X_2, \bar{X}_1, \bar{X}_2) = 2\,\mathrm{Re}(\bar{a}_1^{n+2}b_2^{n+2})$$
$$\Pi(X_1, X_2, \bar{X}_1, \bar{X}_2) = b_2^{n+2}\bar{b}_2^{n+2} + a_1^{n+2}\bar{a}_1^{n+2} + 2b_1^{n+2}\bar{b}_1^{n+2} + 2a_2^{n+2}\bar{a}_2^{n+2} \ .$$

Therefore (5.2) holds if and only if $\theta_3 = \theta_4 = 0$ and

$$(5.3) \qquad\qquad b_2^{n+2} = a_1^{n+2} \quad \text{and} \quad b_1^{n+2} = a_2^{n+2} = 0 \ .$$

(Case II) We assume that $\omega_1^{n+1} = \theta_1$, and $\omega_2^{n+1} = 0$. In this case

$$\Lambda(X_1, X_2, \bar{X}_1, \bar{X}_2) < \Pi(X_1, X_2, \bar{X}_1, \bar{X}_2) \ .$$

We conclude that this case is not possible.

We consider the following transformation:

$$(e_{n+1})^* = (e_{n+1} + a_1^{n+2}e_{n+2})/(1 + a_1^{n+2}\bar{a}_1^{n+2})$$
$$(e_{n+2})^* = (-\bar{a}_1^{n+2}e_{n+1} + e_{n+2})/(1 + a_1^{n+2}\bar{a}_1^{n+2})$$

and observe that $(\omega_1^{n+2})^* = (\omega_2^{n+2})^* = 0$. We apply Cartan's Lemma to the equation:

$$0 = d\omega_1^{n+2} = \omega_1^{n+1} \wedge \omega_{n+1}^{n+2} \ ,$$
$$0 = d\omega_2^{n+2} = \omega_2^{n+1} \wedge \omega_{n+1}^{n+2} \ .$$

We conclude that $de_{n+2} = \omega_{n+2}^{n+2}e_{n+2}$. Therefore, $M \subset G(1, 3) \subset G(2, 4)$. Thus, the theorem holds for $G(2, 4)$. To obtain the result in general we need only observe that

$$\Lambda(X_1, X_2, \bar{X}_1, \bar{X}_2) \leqq \Pi(X_1, X_2, \bar{X}_1, \bar{X}_2) \ ,$$

with equality holding if and only if each copy of $G(2, 4)$ in $G(n, N)$ satisfies the relation (5.3) above.

THEOREM 5.4. *If* $c_2(\Omega_E|_M) = c_1(\Omega_q|_M)^2$, *then*

$$c_k(\Omega_E|_M) = c_1(\Omega_q|_M)^k$$

PROOF. If $c_2(\Omega_E|_M) = c_1(\Omega_q|_M)^2$, then for an appropriate choice of frames we have

$$0 = \omega_i^{n+2} = \omega_i^{n+3} = \cdots = \omega_i^N \qquad (i \in \mathbf{Z}[1, n]) \ .$$

Let $r = k(k + 1)/2$. Then

$$c_k(\Omega_E|_M) = (-1)^r(\sqrt{-1}/2\pi)^k k! \sum \omega_{i1}^{n+1} \wedge \cdots \wedge \omega_{ik}^{n+1} \wedge \bar{\omega}_{i1}^{n+1} \wedge \cdots \wedge \bar{\omega}_{ik}^{n+1} \ ,$$

where $1 \leqq i1 < i2 < \cdots < ik \leqq n$. We also have

$$c_1(\Omega_q|_M)^k = (-1)^r(\sqrt{-1}/2\pi)^k k! \sum \omega_{i1}^{n+1} \wedge \cdots \wedge \omega_{ik}^{n+1} \wedge \bar{\omega}_{i1}^{n+1} \wedge \cdots \wedge \bar{\omega}_{ik}^{n+1} \ ,$$

where $1 \leqq i1 < i2 < \cdots < ik \leqq n$. This concludes the proof.

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