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A DIMENSION FORMULA FOR A CERTAIN SPACE OF AUTOMORPHIC FORMS OF $SU(p, 1)$, II: THE CASE OF $\Gamma(N)$ WITH $N \geqq 3$

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0. Introduction. In the previous paper [5], we derived a dimension formula for the spaces of cusp forms of $SU(p, 1)$ in a closed form in the case of neat lattices in $SU(p, 1)$. With the use of this formula, we shall give, in the present paper, more explicit expressions for such dimensions in the case of the congruence subgroups $\Gamma(N)$ with $N \geqq 3$ in terms of the arithmetic quantities.

For $SU(2, 1)$, explicit description of such dimensions was given by Cohn [3] for $\Gamma(1)$ defined for the base field $\boldsymbol{Q}(\sqrt{-1})$. There he calculated the volume of $\Gamma(1)\backslash SU(2, 1)$ and explained in detail how elliptic elements contribute to the dimension formula. On the other hand, for $SU(p, 1)$ Zeltinger has calculated the volume of $\Gamma(1)\backslash SU(p, 1)$ in [12]. Thus, in our case, in view of the result in [5] (Theorem 1.1 in this paper), we have only to describe in terms of the arithmetic quantities the contribution of unipotent elements to the dimension formula. We shall obtain such a description in this paper.

In § 1, we shall recall the definitions and the results in [5] and state the main theorem in this paper. In § 2, we explain the relation between certain quantities related to the $\Gamma(1)$-inequivalent cusps and the theory of adele groups and investigate the adelized group $SU(p, 1)_A$, following the method of Arakawa [1, § 3]. We also give another proof of the result concerning the number of $\Gamma(1)$-inequivalent cusps obtained in Zeltinger [12]. (By a similar method, one can also prove a more general result concerning $SU(p, q)$, conjectured by Zeltinger. See Corollary 2.7.) The third section is devoted to a proof of the main theorem.

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NOTATION. We denote by $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{Q}$, $\boldsymbol{Z}$ and $\boldsymbol{N}$, respectively, the field of complex numbers, the field of real numbers, the field of rational numbers, the ring of rational integers and the set consisting of all natural numbers. In the following, we mean by k an imaginary quadratic field and by $\mathfrak{O}$ the ring of integers of k. For a commutative ring S with an identity element, we denote by $GL_n(S)$ and $SL_n(S)$ the group of invertible elements in the full matrix ring $M_n(S)$ and the group of all elements in $M_n(S)$ with determinant one, respectively. By 0 and 1_n, we denote the zero matrix and the identity matrix of $M_n(S)$, respectively. We also denote by i the complex number $\sqrt{-1}$ and by $\bar{z}$ the complex conjugate of $z \in \boldsymbol{C}$. For $Z = (z_j) \in \boldsymbol{C}^n$, we denote by $|Z|$ the norm $(\sum_{j=1}^n |z_j|^2)^{1/2}$ of Z. The cardinality of a set X is denoted by $^{\sharp}X$. We denote by $\zeta(z)$ the Riemann zeta function.

1. Statement of the main results. Let G denote an algebraic group defined over $\boldsymbol{Q}$ such that one has

$$G_{\boldsymbol{Q}} = \{g \in SL_{p+1}(k);\ {}^t\bar{g}Rg = R\}\ ,$$
$$G_{\boldsymbol{R}} = \{g \in SL_{p+1}(\boldsymbol{C});\ {}^t\bar{g}Rg = R\}\ ,$$

with $p \in \boldsymbol{N}$ and

$$R = \begin{pmatrix} 1_p & 0 \\ 0 & -1 \end{pmatrix} \in GL_{p+1}(k)\ .$$

Let D be a bounded domain in $\boldsymbol{C}^p$ defined by

$$D = \{Z \in \boldsymbol{C}^p;\ |Z| < 1\}\ .$$

The group $G_{\boldsymbol{R}}$ acts on D naturally by

$$gZ = (g_{11}Z + g_{12})(g_{21}Z + g_{22})^{-1}\ ,$$

where

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in G_{\boldsymbol{R}}$$

with blocks corresponding to those of R, and $Z \in D$. Set $\mu(g, Z) = g_{21}Z + g_{22}$ with $g \in G_{\boldsymbol{R}}$ and $Z \in D$. We denote by K the subgroup of $G_{\boldsymbol{R}}$ consisting of those elements which fix the origin $0 \in D$:

$$K = \{g \in G_{\boldsymbol{R}};\ g0 = 0\}\ . \tag{1.1}$$

The group K is a maximal compact subgroup of $G_{\boldsymbol{R}}$ and we have

$$K = \left\{\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \in SL_{p+1}(\boldsymbol{C});\ u_1 \in U(p),\ u_2 \in \boldsymbol{C},\ |u_2| = 1\right\}\ .$$

Put

$$(1.2)\qquad A = \left\{ a(v) = \begin{pmatrix} 1_{p-1} & 0 & 0 \\ 0 & (\sqrt{v} + \sqrt{v}^{-1})/2 & (\sqrt{v} - \sqrt{v}^{-1})/2 \\ 0 & (\sqrt{v} - \sqrt{v}^{-1})/2 & (\sqrt{v} + \sqrt{v}^{-1})/2 \end{pmatrix} \in GL_{p+1}(\boldsymbol{C});\ v > 0 \right\},$$

$$(1.3)\qquad N = \left\{ [x, y] = \begin{pmatrix} 1_{p-1} & -x & x \\ {}^t\bar{x} & 1 - |x|^2/2 + iy & |x|^2/2 - iy \\ {}^t\bar{x} & -|x|^2/2 + iy & 1 + |x|^2/2 - iy \end{pmatrix} \in GL_{p+1}(\boldsymbol{C});\ \begin{matrix} x \in \boldsymbol{C}^{p-1} \\ y \in \boldsymbol{R} \end{matrix} \right\}.$$

Then we have the Iwasawa decomposition $g = [x, y]a(v)k \in G_{\boldsymbol{R}} = NAK$. By dg we denote the Haar measure on $G_{\boldsymbol{R}}$ normalized by

$$dg = 2^{-1}v^{-(p+1)}dxdydvdk .$$

Here dx is the standard Euclidean measure on $\boldsymbol{C}^{p-1}$ ($\cong \boldsymbol{R}^{2(p-1)}$), dy and dv stand for the Euclidean measure on $\boldsymbol{R}$ and dk is the Haar measure on K normalized by $\int_K dk = 1$. We denote by P the normalizer of N in $G_{\boldsymbol{R}}$ and by $P_{\boldsymbol{Q}}$ (resp. $\boldsymbol{N}_{\boldsymbol{Q}}$) the group $P \cap G$ (resp. $\boldsymbol{N} \cap G_{\boldsymbol{Q}}$).

Let L be the lattice $\mathfrak{O}^{p+1}$ in k^{p+1}. Following Shimura [11], we mean by the *congruence subgroup* $\Gamma(N)$ of $G_{\boldsymbol{Q}}$, the subgroup

$$(1.4)\qquad \Gamma(N) = \{\gamma \in G_{\boldsymbol{Q}};\ L(\gamma - 1) \subset NL\}$$

with $N \in \boldsymbol{N}$. The group $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$, and is a lattice in $G_{\boldsymbol{R}}$, that is, a discrete subgroup of $G_{\boldsymbol{R}}$ such that the volume of $\Gamma(N)\backslash G_{\boldsymbol{R}}$ with respect to the measure dg is finite. With $m \in \boldsymbol{Z}$, let $S_m(\Gamma(N))$ denote the space of holomorphic functions $F(Z)$ on D satisfying the following conditions:

(i) $F(\gamma Z) = \mu(\gamma, Z)^m F(Z)$ for $\gamma \in \Gamma(N)$, $Z \in D$.

(ii) $(1 - |Z|^2)^{m/2} F(Z)$ is bounded on D.

A function in $S_m(\Gamma(N))$ is called a $\Gamma(N)$-*cusp form of weight* m. Set, for any $h \in \Gamma(1)\backslash G_{\boldsymbol{Q}}/P_{\boldsymbol{Q}}$ and $N \in \boldsymbol{N}$,

$$\Gamma(N)_h^* = h^{-1}\Gamma(N)h \cap P , \quad \Gamma(N)_h = h^{-1}\Gamma(N)h \cap \boldsymbol{N}$$

and $w_h = [\Gamma(1)_h^* : \Gamma(1)_h]$. In [5], we obtained the following dimension formula for $S_m(\Gamma(N))$, using the Selberg trace formula.

THEOREM 1.1 ([5, Corollary 4.9]). *Suppose $m > 2p$ and $N \geqq 3$. Then the following dimension formula for $S_m(\Gamma(N))$ holds.*

$$\dim S_m(\Gamma(N)) = [\Gamma(1)\colon \Gamma(N)]\frac{(m-1)!}{\pi^p(m-p-1)!}\operatorname{vol}(\Gamma(1)\backslash G_R) + 2^{p-1}\zeta(1-p)[\Gamma(1)\colon \Gamma(N)]m_\infty(\Gamma(1), \Gamma(N)) ,$$

where

$$m_\infty(\Gamma(1), \Gamma(N)) = \sum_{h \in \Gamma(1)\backslash G_Q/P_Q} w_h^{-1}\nu_{N,h}^{-p}\operatorname{vol}(\Gamma(1)_h\backslash N) ,$$

with $\nu_{N,h} = \min\{b > 0;\ [0, b] \in \Gamma(N)_h\}$.

In particular, if the number p is odd and greater than one, then we get

$$\dim S_m(\Gamma(N)) = [\Gamma(1)\colon \Gamma(N)]\frac{(m-1)!}{\pi^p(m-p-1)!}\operatorname{vol}(\Gamma(1)\backslash G_R) .$$

REMARK 1.2. By Borel [2], the number of double cosets of $\Gamma(1)\backslash G_Q/P_Q$ is equal to the number of $\Gamma(1)$-inequivalent cusps. On the other hand, the number $m_\infty(\Gamma(1), \Gamma(N))$ depends only on $\Gamma(1)$ and $\Gamma(N)$ (cf. [5, p. 473]).

The "Euler volume" of $\Gamma(1)\backslash D$ (i.e., the volume with respect to the p-th Chern form) was calculated by Zeltinger [12].

THEOREM 1.2 (Zeltinger [12]). *The Euler volume of $\Gamma(1)\backslash D$ is given by*

$$(-1)^p(p+1)2^{-p}a(p, d)c(p, d)\prod_{\substack{r=1\\ r\equiv 0(2)}}^{p} L(-r, \chi)\prod_{\substack{r=1\\ r\equiv 1(2)}}^{p}\zeta(-r) ,$$

where

(1.5)

$$a(p, d) = \begin{cases} 1 & (p \equiv 0\ (2)) \\ D^{(p+1)/2}\prod_{p\mid D} 1 + \left(\dfrac{-1}{p}\right)^{(p-1)/2} p^{-(p+1)/2} \begin{cases} 1 - 2^{-(p+1)/2} & \begin{pmatrix} p \equiv 1\ (2) \\ -d \equiv 2,\ 3\ (4)\end{pmatrix} \\ 1 & \begin{pmatrix} p \equiv 1\ (2) \\ -d \equiv 1\ (4)\end{pmatrix} \end{cases} \end{cases} ,$$

$$c(p, d) = \begin{cases} 2 & (p \equiv 1\ (2)) \\ 1 & (p \equiv 0\ (2)) \end{cases} \quad (d \neq 1, 3)$$

(1.6)

$$c(p, 1) = \begin{cases} 1 & (p \equiv 0\ (2)) \\ 2 & (p \equiv 1\ (4)) \\ 4 & (p \equiv 3\ (4)) , \end{cases} \qquad c(p, 3) = \begin{cases} 1 & (p \equiv 0, 4\ (6)) \\ 2 & (p \equiv 1, 3\ (6)) \\ 3 & (p \equiv 2\ (6)) \\ 6 & (p \equiv 5\ (6)) , \end{cases}$$

and

$$(1.7)\qquad \chi(\boldsymbol{p}) = \begin{cases} 1 & \textit{if } \boldsymbol{p} \textit{ splits in } k/\boldsymbol{Q} \\ -1 & \textit{if } \boldsymbol{p} \textit{ neither splits in nor is ramified for } k/\boldsymbol{Q} \\ 0 & \textit{if } \boldsymbol{p} \textit{ is ramified for } k/\boldsymbol{Q}. \end{cases}$$

It is easy to see that the Euler volume of $\Gamma(1)\backslash D$ is equal to $(-1)^p((p+1)!/\pi^p)\mathrm{vol}(\Gamma\backslash G_R)$. Thus, to derive a more explicit formula for the dimension of the space $S_m(\Gamma(N))$, we need to calculate the following two quantities:

(I) $\nu_{N,h}^{-p}\,\mathrm{vol}(\Gamma(1)_h\backslash N)\qquad (h \in \Gamma(1)\backslash G_Q/P_Q)$,

(II) $\sum_{h\in\Gamma(1)\backslash G_Q/P_Q} w_h^{-1}$

for an even integer p. We shall calculate these quantities explicitly in § 3 and show that the quantities (I) do not depend on h. Our final result is the following:

THEOREM 1.3. *Suppose $m > 2p$ and $N \geqq 3$. Then, the following dimension formula for $S_m(\Gamma(N))$ holds.*

$$\begin{aligned} \dim S_m(\Gamma(N)) = {} & [\Gamma(1)\colon \Gamma(N)]2^{-p}(m-1)!\,((m-p-1)!)^{-1}(p!)^{-1} \\ & \times a(p,d)c(p,d)\prod_{\substack{r=1\\ r\equiv 0(2)}}^{p} L(-r,\chi)\prod_{\substack{r=1\\ r\equiv 1(2)}}^{p}\zeta(-r) \\ & + [\Gamma(1)\colon\Gamma(N)]2^t\delta^p N^{-p} d^{(p-1)/2}\zeta(1-p)e(p,d)\,, \end{aligned}$$

where

$$e(p,d) = 2^{-t-p(p-1)/2}\pi^{-p(p-1)/2}\prod_{r=0}^{p-2} r!\,|D|^{p(p-1)/4}L(1,\chi)\prod_{r=1}^{(p-2)/2} L(2r+1,\chi)\zeta(2r)\,,$$

D is the discriminant of k, δ is 1 or 2 according as $-d \equiv 1$ (4) or $-d \equiv 2, 3$ (4), t is the number of primes which divide the discriminant of k, and $a(p,d)$, $c(p,d)$ and $\chi(\boldsymbol{p})$ are given by (1.5), (1.6) *and* (1.7), *respectively*.

REMARK 1.4. The quantity $e(p,d)$ in Theorem 1.3 is the inverse of the volume of the stabilizer $\mathfrak{U}_{U(p-1)}$ in the adelized group $U(p-1)_A$ of the lattice $\mathfrak{O}^{p-1}$, with respect to the measure $dU(p-1)_A$ normalized suitably:

$$e(p,d) = \left(\int_{\mathfrak{U}_{U(p-1)}} dU(p-1)_A\right)^{-1},$$

(cf. § 3).

2. The structure of $\Gamma(1)\backslash G_Q/P_Q$. In this section, we study the structure of $\Gamma(1)\backslash G_Q/P_Q$. We keep the notations introduced in § 1.

Let L be any $\mathfrak{O}$-lattice in k^{p+1}. We mean by the *norm* of L the $\boldsymbol{Z}$-ideal generated by the elements ${}^t\bar{x}Rx$ with $x \in L$. If L is maximal among the $\mathfrak{O}$-lattices with the same norm, it is called a *maximal $\mathfrak{O}$-lattice*.

Clearly $\mathfrak{O}^{p+1}$ is a maximal $\mathfrak{O}$-lattice. For any prime number $\boldsymbol{p}$, set $L_p = L\otimes_Z \boldsymbol{Z}_p$. The adelized group G_A of G acts naturally on the lattice L:

$$Lg = \bigcap_p (L_p g_p \cap k^{p+1}) \qquad (g = (g_p) \in G_A) .$$

By $\mathfrak{U}$ we denote the stabilizer in G_A of the lattice L:

$$\mathfrak{U} = \{g \in G_A;\ Lg = L\} .$$

In the following, we assume that $L = \mathfrak{O}^{p+1}$. By the definition of $\Gamma(1)$, we see immediately that $\Gamma(1) = G_Q \cap \mathfrak{U}$.

LEMMA 2.1. *Let the map* $\Phi\colon \Gamma(1)\backslash G_Q/P_Q \to \mathfrak{U}\backslash G_A/P_Q$ *be defined by* $\Phi(\Gamma(1)gP_Q) = \mathfrak{U}gP_Q$ *for* $g \in G_Q$. *Then* Φ *is bijective.*

PROOF. By Shimura [10, Theorem 5.19], the number of double cosets in $\mathfrak{U}\backslash G_A/G_Q$ is one. Hence the surjectivity of Φ follows. On the other hand, the injectivity is an easy consequence of the relation $\Gamma(1) = \mathfrak{U} \cap G_Q$.
q.e.d.

For any prime number $\boldsymbol{p}$, we denote by G_p the group of $\boldsymbol{Q}_p$-rational points of G and by $\mathfrak{U}_p$ the group $G_p \cap GL_{p+1}(\mathfrak{O}_p)$ with $\mathfrak{O}_p = \mathfrak{O}\otimes_Z \boldsymbol{Z}_p$. Let P_A be the adelized group of P_Q. We shall deduce the following lemma from a Iwasawa decomposition of G_p (cf. Satake [8]).

LEMMA 2.2. $\mathfrak{U}\backslash G_A/P_Q = (\mathfrak{U} \cap P_A)\backslash P_A/P_Q$: *namely, as a complete set of representatives of* $\mathfrak{U}\backslash G_A/P_Q$, *one can take a complete set of representatives of* $(\mathfrak{U} \cap P_A)\backslash P_A/P_Q$.

PROOF. We note that G_p is isomorphic to $SL_{p+1}(\boldsymbol{Q}_p)$ in the case where $\boldsymbol{p}$ splits in $k/\boldsymbol{Q}$, and that G_p is isomorphic to $SU_{p+1}(k\otimes\boldsymbol{Q}_p)$ otherwise. Then, since L is a maximal $\mathfrak{O}$-lattice, we see easily that, for any $g = (g_p) \in G_A$, there are $u_p \in U_p$ and $h_p \in P_{Q_p}$ such that $g_p = u_p h_p$, in view of Satake [8, Chapter III § 8.4 and § 9.2]. Take the element $h = (h_p) \in P_A$ with $h_\infty = g_\infty$, as a representative of $\mathfrak{U}gP_Q$. Now our assertion follows easily.
q.e.d.

Let H $(= U(p-1))$ be an algebraic group defined over $\boldsymbol{Q}$ such that

$$H_Q = \{g \in GL_{p-1}(k);\ {}^t\bar{g}g = 1_{p-1}\} .$$

Let L_H denote the $\mathfrak{O}$-lattice $\mathfrak{O}^{p-1}$ in k^{p-1}, H_A the adelized group of H and $\mathfrak{U}_H$ the stabilizer in H_A of the lattice L_H:

$$\mathfrak{U}_H = \{g \in H_A;\ L_H g = L_H\} .$$

By Lemmas 2.1 and 2.2, it is possible to choose an element h of P_A as a representative of each double coset in $\Gamma(1)\backslash G_Q/P_Q$. As is seen easily, moreover, one may assume that the element h is of the following form:

$$h=\begin{pmatrix}\alpha & 0 & 0\\ 0 & (\nu+\bar{\nu}^{-1})/2 & (\nu-\bar{\nu}^{-1})/2\\ 0 & (\nu-\bar{\nu}^{-1})/2 & (\nu+\bar{\nu}^{-1})/2\end{pmatrix}$$

with some $\alpha\in H_A$ and some $\nu\in k_A^\times$. Here $k_A^\times$ is the adelized group of $k^\times$, where $k^\times$ is regarded as an algebraic group defined over $\boldsymbol{Q}$.

Now we shall recall some facts from class field theory. Let C be the ideal class group of k. It is well-known that C is isomorphic to $k^\times\backslash k_A^\times/\mathfrak{U}_0$, where $\mathfrak{U}_0$ is the stabilizer in $k_A^\times$ of $\mathfrak{O}$. Let $\mathfrak{a}$ be any ideal of k and c the ideal class containing $\mathfrak{a}$. By $\bar{\mathfrak{a}}$ we denote the complex conjugate of $\mathfrak{a}$ and by $\bar{c}$ the ideal class containing $\bar{\mathfrak{a}}$. Then the set $A_k=\{c=\bar{c};\, c\in C\}$ is a subgroup of C, whose element is said to be an *ambig class* of k. It is known that the number of ambig classes of k is 2^{t-1}. Here t is the number of primes which divide the discriminant of k.

The following lemma will be used in the proof of Proposition 2.4.

LEMMA 2.3. (i) *The following maps φ_1, φ_2 are surjective*:

$$\varphi_1\colon H_Q\ni\alpha\mapsto\det\alpha\in k^1=\{\kappa\in k;\ N(\kappa)=1\}\ ,$$
$$\varphi_2\colon \mathfrak{U}_H\ni\alpha\mapsto\det\alpha\in\mathfrak{U}_0^1=\{u\in\mathfrak{U}_0;\ N(u)=1\}\ ,$$

with $N(\nu)=\nu\bar{\nu}$ *for* $\nu\in k_A^\times$.

(ii) *Let*

$$\begin{pmatrix}\alpha & 0 & 0\\ 0 & (\nu+\bar{\nu}^{-1})/2 & (\nu-\bar{\nu}^{-1})/2\\ 0 & (\nu-\bar{\nu}^{-1})/2 & (\nu+\bar{\nu}^{-1})/2\end{pmatrix}$$

be an element of P_A, and $\mathfrak{D}$ be an element of some ambig class of k. Then, there exists an element $\alpha_\mathfrak{D}$ of H_A such that $\alpha_\mathfrak{D}\in\mathfrak{U}_H\alpha H_Q$ and

$$\begin{pmatrix}\alpha_\mathfrak{D} & 0 & 0\\ 0 & (\nu\mathfrak{D}+(\overline{\nu\mathfrak{D}})^{-1})/2 & (\nu\mathfrak{D}-(\overline{\nu\mathfrak{D}})^{-1})/2\\ 0 & (\nu\mathfrak{D}-(\overline{\nu\mathfrak{D}})^{-1})/2 & (\nu\mathfrak{D}+(\overline{\nu\mathfrak{D}})^{-1})/2\end{pmatrix}\in P_A\ .$$

PROOF. For any $\kappa\in k^1$ (resp. $u\in\mathfrak{U}_0^1$), put

$$\alpha=\begin{pmatrix}\kappa & 0\\ 0 & 1_{p-2}\end{pmatrix}\quad\left(\text{resp. }\ \alpha=\begin{pmatrix}u & 0\\ 0 & 1_{p-2}\end{pmatrix}\right).$$

Then, the first part of the lemma follows from $\alpha\in H_Q$ (resp. $\alpha\in\mathfrak{U}_H$).

Next, noting that $\mathfrak{D}$ is an element of some ambig class, write

$$\bar{\mathfrak{D}}\mathfrak{D}^{-1}=ua\quad\text{for some}\quad a\in k^\times\quad\text{and some}\quad u\in\mathfrak{U}_0\ .$$

Then, since $N(u)N(a)=1$, we have $u\in\mathfrak{U}_0^1$ and $a\in k^1$. Thus, by part (i), there exist $\delta \in \mathfrak{U}_H$ and $\beta \in H_Q$ such that $u = \det \delta$ and $a = \det \beta$. Put $\alpha_D = \delta\alpha\beta$. Now the proof of the assertion (ii) is immediate. q.e.d.

PROPOSITION 2.4. *Suppose p is an even positive integer. Let h be the number of double cosets of $\mathfrak{U}_H\backslash H_A/H_Q$. Then, there are matrices*

$$h_{ij} = \begin{pmatrix} \alpha_{ij} & 0 & 0 \\ 0 & (\nu_{ij} + (\overline{\nu_{ij}})^{-1})/2 & (\nu_{ij} - (\overline{\nu_{ij}})^{-1})/2 \\ 0 & (\nu_{ij} - (\overline{\nu_{ij}})^{-1})/2 & (\nu_{ij} + (\overline{\nu_{ij}})^{-1})/2 \end{pmatrix} \in P_A$$

with $\alpha_{ij} \in H_A$, $\nu_{ij} \in k_A^\times$; $1 \leqq i \leqq h$, $1 \leqq j \leqq 2^{t-1}$ having the following properties:

(i) *The set $\Lambda = \{h_{ij};\ 1 \leqq i \leqq h,\ 1 \leqq j \leqq 2^{t-1}\}$ is a complete set of representatives of double cosets of $(\mathfrak{U} \cap P_A)\backslash P_A/P_Q$.*

(ii) *For a fixed j_0; the set $\{\alpha_{ij_0};\ 1 \leqq i \leqq h\}$ is a complete set of representatives of $\mathfrak{U}_H\backslash H_A/H_Q$.*

(iii) *For a fixed pair i_0, j_0, the set $\{\nu_{i_0j}(\nu_{i_0j_0})^{-1};\ 1 \leqq j \leqq 2^{t-1}\}$ is a complete set of repesentatives of ambig classes of k.*

PROOF. Let $\{\alpha_i;\ 1 \leqq i \leqq h\}$ (resp. $\{\mathfrak{D}_j;\ 1 \leqq j \leqq 2^{t-1}\}$ be a complete set of representatives of $\mathfrak{U}_H\backslash H_A/H_Q$ (resp. of ambig classes of k). Now we fix i, j. It is easy to see that there is $h_i \in P_A$ of the form

$$h_i = \begin{pmatrix} \alpha_i & 0 & 0 \\ 0 & (\nu_i + \overline{\nu_i}^{-1})/2 & (\nu_i - \overline{\nu_i}^{-1})/2 \\ 0 & (\nu_i - \overline{\nu_i}^{-1})/2 & (\nu_i + \overline{\nu_i}^{-1})/2 \end{pmatrix}$$

for some $\nu_i \in k_A^\times$. Set $\nu_{ij} = \nu_i\mathfrak{D}_j$. Then, in view of Lemma 2.3 (ii), there exists $h_{ij} \in P_A$ such that

$$h_{ij} = \begin{pmatrix} \alpha_{ij} & 0 & 0 \\ 0 & (\nu_{ij} + (\overline{\nu_{ij}})^{-1})/2 & (\nu_{ij} - (\overline{\nu_{ij}})^{-1})/2 \\ 0 & (\nu_{ij} - (\overline{\nu_{ij}})^{-1})/2 & (\nu_{ij} + (\overline{\nu_{ij}})^{-1})/2 \end{pmatrix}$$

for some $\alpha_{ij} \in \mathfrak{U}_H\alpha_iH_Q$. We show that $\Lambda = \{h_{ij};\ 1 \leqq i \leqq h,\ 1 \leqq j \leqq 2^{t-1}\}$ is a complete set of representatives of $(\mathfrak{U} \cap P_A)\backslash P_A/P_Q$. To do so, it suffices to prove that, for any $h \in P_A$, there exists $h_{ij} \in \Lambda$ satisfying the condition $h \in (\mathfrak{U} \cap P_A)h_{ij}P_Q$.

Let

$$h = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & (\nu + \bar{\nu}^{-1})/2 & (\nu - \bar{\nu}^{-1})/2 \\ 0 & (\nu - \bar{\nu}^{-1})/2 & (\nu + \bar{\nu}^{-1})/2 \end{pmatrix}$$

be any element of P_A, with $\alpha \in H_A$, $\nu \in k_A^\times$. Assume that $\alpha \in \mathfrak{U}_H\alpha_iH_Q$.

Since $\bar{\nu}^{-1}\nu \in \mathfrak{U}_0(\overline{\nu_i}^{-1}\nu_i)k^\times$, we have $\nu = u\nu_{ij}\kappa$ for some j, $u \in \mathfrak{U}_0$ and $\kappa \in k^\times$. Write $\alpha = \delta\alpha_{ij}\beta$ with $\delta \in \mathfrak{U}_H$, $\beta \in H_Q$. Since p is even, one can take $\delta \in \mathfrak{U}_H$ and $\beta \in H_Q$ so that $\det(\delta)uu^{-1} = \det(\beta)\kappa\bar{\kappa}^{-1} = 1$. Thus we obtain

$$h = \begin{pmatrix} \delta & 0 & 0 \\ 0 & (u+\bar{u}^{-1})/2 & (u-\bar{u}^{-1})/2 \\ 0 & (u-\bar{u}^{-1})/2 & (u+\bar{u}^{-1})/2 \end{pmatrix} h_{ij} \begin{pmatrix} \beta & 0 & 0 \\ 0 & (\kappa+\bar{\kappa}^{-1})/2 & (\kappa-\bar{\kappa}^{-1})/2 \\ 0 & (\kappa-\bar{\kappa}^{-1})/2 & (\kappa+\bar{\kappa}^{-1})/2 \end{pmatrix}$$

$$\in (\mathfrak{U} \cap P_A)h_{ij}P_Q. \qquad \text{q.e.d.}$$

The above proposition gives us another proof of the following corollary, which is due to Zeltinger [12].

COROLLARY 2.5. *Suppose that p is an even integer. Let h be the number of double cosets in $\mathfrak{U}_H\backslash H_A/H_Q$ and let t be the number of primes which divide the discriminant of k. Then the number of double cosets of $\Gamma(1)\backslash G_Q/P_Q$ is equal to $2^{t-1}h$.*

REMARK 2.6. Suppose that $|p-q|$ is an even integer. Then, by a similar method, one can deduce the following corollary for $SU(p, q)$, which was conjectured by Zeltinger [12].

COROLLARY 2.7. *Let $\boldsymbol{G}$ $(= SU(p, q))$ be an algebraic group defined over $\boldsymbol{Q}$ such that*

$$\boldsymbol{G}_Q = \left\{ g \in SL_{p+q}(k);\ {}^t\bar{g}\begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} g = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} \right\},$$

and let $\boldsymbol{P}$ be a minimal parabolic subgroup of $\boldsymbol{G}$ defined over $\boldsymbol{Q}$. Set $\Gamma = SL_{p+q}(\mathfrak{O}) \cap \boldsymbol{G}_Q$. We suppose that $|p-q|$ is an even integer. Then the number of double cosets of $\Gamma\backslash\boldsymbol{G}_Q/\boldsymbol{P}_Q$ is equal to $2^{t-1}c(k)^{\min(p,q)-1}h_{|p-q|}(k)$. Here we denote by $h_{|p-q|}(k)$ the number of double cosets of $\mathfrak{U}_H\backslash\boldsymbol{H}_A/\boldsymbol{H}_Q$ with $\boldsymbol{H} = U(|p-q|)$, by $c(k)$ the class number of k and by t the number of primes which divide the discriminant of k.

3. Proof of the main theorem. The notations in this section will be as in § 1 and § 2. In this section, we shall calculate the quantities (I), (II) in § 1 to derive Theorem 1.3. First we calculate the quantity (I).

PROPOSITION 3.1. *Let h be an element in P_A of the form*

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & (\nu+\bar{\nu}^{-1})/2 & (\nu-\bar{\nu}^{-1})/2 \\ 0 & (\nu-\bar{\nu}^{-1})/2 & (\nu+\bar{\nu}^{-1})/2 \end{pmatrix}.$$

Then, for a positive integer N, we have

$$\mathrm{vol}(\Gamma(1)_h\backslash \boldsymbol{N})\nu_{N,h}^{-p}=\begin{cases}2^{1-p}N^{-p}d^{(p-1)/2} & \text{for} \quad -d\equiv 1\ (4)\\ 2N^{-p}d^{(p-1)/2} & \text{for} \quad -d\equiv 2,3\ (4)\,.\end{cases}$$

PROOF. First we recall the relation $\mathfrak{O}=\boldsymbol{Z}+\boldsymbol{Z}\rho$, where

$$\rho=\begin{cases}(1+\sqrt{-d})/2 & \text{for} \quad -d\equiv 1\ (4)\\ \sqrt{-d} & \text{for} \quad -d\equiv 2,3\ (4)\,.\end{cases}$$

Let $N(\nu)$ denote the norm $\nu\bar{\nu}$ of $\nu\in k_A^\times$. By (1.3), it is easy to check that

$$h[x,y]h^{-1}=\begin{pmatrix}1_{p-1} & -\alpha x\bar{\nu} & \alpha x\bar{\nu}\\ {}^t\bar{x}\nu\alpha^{-1} & 1-|x|^2N(\nu)/2+\sqrt{-d}yN(\nu) & |x|^2N(\nu)/2-\sqrt{-d}yN(\nu)\\ {}^t\bar{x}\nu\alpha^{-1} & -|x|^2N(\nu)/2+\sqrt{-d}yN(\nu) & 1+|x|^2N(\nu)/2-\sqrt{-d}yN(\nu)\end{pmatrix}$$

with $[x,y]\in N_{\boldsymbol{Q}}$. Hence $[x,y]\in N_{\boldsymbol{Q}}$ is an element of $\Gamma(1)_h$ if and only if $[x,y]$ satisfies the following conditions: (i) ${}^tx\in L_H\bar{\alpha}\bar{\nu}^{-1}$, (ii) $|x|^2/2-\sqrt{-d}y\in\mathfrak{O}N(\nu)^{-1}$. Let $\boldsymbol{Q}_A^\times$ be the idele group of $\boldsymbol{Q}$, and $\mathfrak{U}_{\boldsymbol{Q}}$ the stabilizer in $\boldsymbol{Q}_A^\times$ of $\boldsymbol{Z}$. Then we have

$$N(\nu)=ul \quad \text{for some} \quad u\in\mathfrak{U}_{\boldsymbol{Q}} \quad \text{and} \quad l\in\boldsymbol{Q}^\times\,.$$

By a similar argument, we obtain

$$\{[0,y]\in h^{-1}\Gamma(N)h\cap\boldsymbol{N}\}=\{[0,y]\in N_{\boldsymbol{Q}};\ y\in\boldsymbol{Z}N(\nu)^{-1}N\}\,.$$

In particular, the relation $\nu_{N,h}=l^{-1}N$ holds.

Thus we have

$$\begin{aligned}\nu_{N,h}^{-p}\,\mathrm{vol}(\Gamma(1)_h\backslash\boldsymbol{N}) &=(l^{-1}N)^{-p}\delta\int_{L_H\bar{\alpha}\bar{\nu}^{-1}\backslash \boldsymbol{C}^{p-1}}dx\int_{\boldsymbol{Z}l^{-1}\backslash\boldsymbol{R}}dy\\ &=\delta N^{-p}\int_{L_H\backslash\boldsymbol{C}^{p-1}}dx\int_{\boldsymbol{Z}\backslash\boldsymbol{R}}dy\,,\end{aligned}$$

where δ is 1 or 2 according as $-d\equiv 1$ (4) or $-d\equiv 2,3$ (4). Here we have used the relation

$$|\det(\bar{\alpha}\bar{\nu}^{-1})|^2=N(\nu)^{1-p}\,.$$

Hence, noting the relation $\mathfrak{O}=\boldsymbol{Z}+\boldsymbol{Z}\rho$, we obtain Proposition 3.1. q.e.d.

Next we shall calculate the quantity (II) in § 1. Let

$$h=\begin{pmatrix}\alpha & 0 & 0\\ 0 & (\nu+\bar{\nu}^{-1})/2 & (\nu-\bar{\nu}^{-1})/2\\ 0 & (\nu-\bar{\nu}^{-1})/2 & (\nu+\bar{\nu}^{-1})/2\end{pmatrix}\in P_A$$

with $\alpha\in H_A$, $\nu\in k_A^\times$. For $\alpha\in H_A$, put

$$w^*(\alpha) = {}^\sharp\{a \in H_Q;\ L_H\alpha a\alpha^{-1} = L_H\} = {}^\sharp(\alpha^{-1}\mathfrak{U}_H\alpha \cap H_Q)\ .$$

Then the following holds.

LEMMA 3.2. *Suppose that p is an even positive integer. Then the equality $w_h = w^*(\alpha)$ holds.*

PROOF. As can be seen easily, we have

$$\begin{aligned} w_h = {}^\sharp\Bigg\{ & \begin{pmatrix} a & 0 & 0 \\ 0 & (\lambda+\bar\lambda^{-1})/2 & (\lambda-\bar\lambda^{-1})/2 \\ 0 & (\lambda-\bar\lambda^{-1})/2 & (\lambda+\bar\lambda^{-1})/2 \end{pmatrix} \in P_Q; \\ & Lh\begin{pmatrix} a & 0 & 0 \\ 0 & (\lambda+\bar\lambda^{-1})/2 & (\lambda-\bar\lambda^{-1})/2 \\ 0 & (\lambda-\bar\lambda^{-1})/2 & (\lambda+\bar\lambda^{-1})/2 \end{pmatrix}h^{-1} = L;\ \begin{matrix} a \in H_Q \\ \lambda \in k^\times \end{matrix} \Bigg\} \\ = {}^\sharp\Bigg\{ & \begin{pmatrix} a & 0 & 0 \\ 0 & (\lambda+\bar\lambda^{-1})/2 & (\lambda-\bar\lambda^{-1})/2 \\ 0 & (\lambda-\bar\lambda^{-1})/2 & (\lambda+\bar\lambda^{-1})/2 \end{pmatrix} \in P_Q;\ L_H\alpha a\alpha^{-1} = L_H,\ a \in H_Q,\ \lambda \in \mathfrak{O}^\times \Bigg\}\ . \end{aligned}$$

Now we suppose that k is neither $\boldsymbol{Q}(\sqrt{-1})$ nor $\boldsymbol{Q}(\sqrt{-3})$. Then, noting that $\mathfrak{O}^\times = \pm 1$ and $\det(-a) = -\det(a)$, we get

$$\begin{aligned} w_h &= 2{}^\sharp\Bigg\{\begin{pmatrix} a & 0 & 0 \\ 0 & (\lambda+\bar\lambda^{-1})/2 & (\lambda-\bar\lambda^{-1})/2 \\ 0 & (\lambda-\bar\lambda^{-1})/2 & (\lambda+\bar\lambda^{-1})/2 \end{pmatrix} \in P_Q;\ L_H\alpha a\alpha^{-1} = L_H,\ \begin{matrix} a \in H_Q \\ \det(a) = 1 \end{matrix}\Bigg\} \\ &= {}^\sharp\{a \in H_Q;\ L_H\alpha a\alpha^{-1} = L_H,\ \det(a) = \pm 1\} \\ &= w^*(\alpha)\ . \end{aligned}$$

When k is $\boldsymbol{Q}(\sqrt{-1})$ or $\boldsymbol{Q}(\sqrt{-3})$, the equality $w_h = w^*(\alpha)$ can be proved similarly. q.e.d.

Let ω be a left invariant highest differential form on H defined over $\boldsymbol{Q}$, and ω_p the Haar measure on H_{Q_p} induced by ω for each prime p. Let χ be the Dirichlet character associated to $k/\boldsymbol{Q}$, i.e.,

$$\chi(p) = \begin{cases} 1 & \text{if } p \text{ splits in } k/\boldsymbol{Q} \\ -1 & \text{if } p \text{ neither splits in nor is ramified for } k/\boldsymbol{Q} \\ 0 & \text{if } p \text{ is ramified for } k/\boldsymbol{Q}. \end{cases} \tag{3.1}$$

Now following Ono [7], we define a measure dH_A as follows:

$$dH_A = \rho_H^{-1}\omega_\infty \prod_{p:\text{prime}} (1 - p^{-1}\chi(p))^{-1}\omega_p\ ,$$

where $\rho_H = L(1, \chi)$.

LEMMA 3.3. *We have*

$$\sum_{\alpha\in\mathfrak{U}_H\backslash H_A/H_Q} w^*(\alpha)^{-1} = \tau(H)\Big(\int_{\mathfrak{U}_H} dH_A\Big)^{-1}$$

Here $\tau(H)$ means the Tamagawa number of H:

$$\tau(H) = \int_{H_A/H_Q} dH_A .$$

PROOF. Write

$$H_A = \coprod_{\alpha\in\mathfrak{U}_H\backslash H_A/H_Q} \mathfrak{U}_H \alpha H_Q .$$

Then, it follows immediately that

$$H_A/H_Q = \coprod_{\alpha\in\mathfrak{U}_H\backslash H_A/H_Q} \mathfrak{U}_H\alpha/(\alpha^{-1}\mathfrak{U}_H\alpha\cap H_Q) .$$

Thus, noting that $^\sharp(\alpha^{-1}\mathfrak{U}_H\alpha\cap H_Q) < \infty$, we see that

$$\tau(H) = \int_{H_A/H_Q} dH_A = \sum_{\alpha} \{^\sharp(\alpha^{-1}\mathfrak{U}_H\alpha\cap H_Q)\}^{-1}\int_{\mathfrak{U}_H\alpha} dH_A = \sum_{\alpha} w^*(\alpha)^{-1}\int_{\mathfrak{U}_H} dH_A .$$

q.e.d.

Now one deduces that $\tau(H) = 2$, using the fact that the Tamagawa number of $SU(p-1)$ (resp. $U(1)$) is one (resp. two) (cf. Ono [7]). Therefore, by virtue of Proposition 2.4, Lemma 3.2 and Lemma 3.3, the following holds.

PROPOSITION 3.4. *Let p be an even positive integer. Then we obtain*

$$\sum_{h\in\Gamma(1)\backslash G_Q/P_Q} w_h^{-1} = 2^t\Big(\int_{\mathfrak{U}_H} dH_A\Big)^{-1} ,$$

where t is the number of primes which divide the discriminant of k.

On the other hand, by the methods employed in the calculation of the volume $\mathrm{vol}(\Gamma(1)\backslash G_R)$ in Zeltinger [12, Chapter II], one can obtain the following:

LEMMA 3.5. *Suppose p is an even positive integer. Then it follows that*

$$\omega_{\mathbf{p}}(H_{Z_{\mathbf{p}}}) = f_{\mathbf{p}}(p, d)(1-\chi(\mathbf{p})\mathbf{p}^{-1})\prod_{r=1}^{(p-2)/2}(1-\chi(\mathbf{p})\mathbf{p}^{-(2r+1)})(1-\mathbf{p}^{-2r})$$

for any prime $\mathbf{p}\neq\infty$ and that

$$\omega_\infty(H_R) = \delta^{p-1}2^{p(p-1)/2}\pi^{p(p-1)/2}\Big(\prod_{r=0}^{p-2} r!\Big)^{-1}|D|^{-p(p-1)/4} .$$

Here $f_{\mathbf{p}}(p, d)$ stands for

$$f_p(p, d) = \begin{cases} 1 & (p \nmid D) \\ 2 & (p \neq 2,\ p \mid D) \\ 2^{2-p} & (p = 2,\ 2 \mid D)\ , \end{cases}$$

D is the discriminant of k, δ is 1 *or* 2 *according as* $-d \equiv 1\ (4)$ *or* $-d \equiv 2, 3\ (4)$ *and* $\chi(p)$ *is given by* (3.1).

THE SKETCH OF PROOF. For $N \in \boldsymbol{N}$, set

$$H_{Z_p}(N) = \{g \in H_{Z_p};\ g \equiv 1_{p-1}(\mathrm{mod}\ \boldsymbol{p}^N)\}$$

and

$$U(p-1;\ \mathfrak{O}/\boldsymbol{p}^N\mathfrak{O}) = \{g \in M_{p-1}(\mathfrak{O}/\boldsymbol{p}^N\mathfrak{O});\ {}^t\bar{g}g = 1_{p-1}\}\ .$$

Then, by the argument in [12, II, § 3.5 and § 3.6], we have

$$\begin{aligned} \omega_p(H_{Z_p}) &= p^{-N\dim H}[H_{Z_p}\colon H_{Z_p}(N)] \\ &= p^{-N(p-1)^2}\delta^{1-p}\,{}^{\#}U(p-1;\ \mathfrak{O}/\boldsymbol{p}^N\mathfrak{O}) \end{aligned}$$

for $N \geqq 3$. The numbers ${}^{\#}U(p-1;\ \mathfrak{O}/\boldsymbol{p}^N\mathfrak{O})$ are calculated in [12, II, § 1.7, § 2.7 and § 2.11].

Next, to calculate the quantity $\omega_\infty(H_R)$, let $\mathrm{vol}_{\tilde{h}}(U(p-1))$ be the volume of $U(p-1)$ with respect to the $U(p-1)$- invariant metric $\tilde{h}$ such that $\tilde{h}_{1_p}(X, Y) = -\mathrm{Tr}(XY)/2$ for $X, Y \in \mathfrak{U}(p-1)$. Here $\mathfrak{U}(p-1)$ is the Lie algebra corresponding to $U(p-1)$. It is straightforward to see that

$$\omega_\infty(H_R) = \delta^{p-1}2^{(p-1)^2/2}|D|^{-p(p-1)/4}\,\mathrm{vol}_{\tilde{h}}(U(p-1))\ ,$$

(cf. [12, II, § 4.5]). On the other hand, it follows from [12, II, § 4.7] that

$$\mathrm{vol}_{\tilde{h}}(U(p-1)) = 2^{(p-1)/2}\pi^{p(p-1)/2}\Big(\prod_{r=0}^{p-2} r!\Big)^{-1}\ ,$$

which implies the second assertion of the lemma. q.e.d.

COROLLARY 3.6. *Retain the notations and the assumption of Lemma* 3.5. *Then one has*

$$\Big(\int_{\mathfrak{U}_H} dH_A\Big)^{-1} = 2^{-t-p(p-1)/2}\pi^{-p(p-1)/2}\prod_{r=0}^{p-2} r!\,|D|^{p(p-1)/4}L(1, \chi)\prod_{r=1}^{(p-2)/2} L(2r+1, \chi)\zeta(2r)\ .$$

Combining Proposition 3.4 with Corollary 3.6, we obtain the quantity (II) in § 1, and finally complete the proof of the main theorem.

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