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STABILITY IN FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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The purpose of this article is to discuss some stability properties of a solution of functional differential equations with delay by using the method of Liapunov-Razumikhin type. The method of Liapunov-Razumikhin type is powerful for the analysis of functional differential equations with delay, and many results have been presented (for instance, [1, 2], [4], [7-9], [11]). However, as the example given by Seifert in [8, 9] shows, the (uniform) asymptotic stability of a solution of functional differential equations with infinite delay cannot necessarily be deduced from the familiar settings of Liapunov-Razumikhin theory. In order to investigate the (uniform) asymptotic stability of a solution of functional differential equations with infinite delar, in this paper, we shall take the following approach: First we decompose a given equation with infinite delay as a sum of an equation with finite delay and the remainders; Next we obtain some perturbation theorems for the equation with finite delay in terms of the arguments of Liapunov-Razumikhin type; Finally we discuss the stability of a solution of the original equation with infinite delay. In particular, this approach is useful for the analysis of integrodifferential equations. Indeed, we shall consider some integrodifferential equations, and obtain some results on the stability properties of a solution (Theorems 3 and 4). Our Theorem 3 improves a result in [11].

Now, we explain the notations and definitions employed throughout this paper. Let $\boldsymbol{R}^n$ be the n-dimensional real Euclidean space. Let BC be the set of all bounded and continuous functions defined on $(-\infty, 0]$ with values in $\boldsymbol{R}^n$. For any ϕ in BC we set $\|\phi\| = \sup_{\theta\leqq 0}|\phi(\theta)|$. Then $(\mathrm{BC}, \|\cdot\|)$ is a Banach space. For any $H, 0 < H < \infty$, we set $\mathrm{BC}_H = \{\phi \in \mathrm{BC}: \|\phi\| < H\}$. Furthermore, if $x(\cdot)$ is a continuous and bounded function on $(-\infty, t_1), t_1 \leqq \infty$, to $\boldsymbol{R}^n$, then for each $t < t_1$ we define the function $x_t \in \mathrm{BC}$ by $x_t(\theta) = x(t+\theta), \theta \leqq 0$.

Consider a system of functional differential equations with infinite delay

$$\dot{x}(t) = f(t, x_t)\,, \tag{1}$$

where f is a function on $I\times\mathrm{BC}$, $I := [0, \infty)$, to $\boldsymbol{R}^n$ and $f(t, 0) \equiv 0$ on I.

For any $(\sigma, \phi) \in I \times \mathrm{BC}$, a continuous function $x: (-\infty, T) \to \boldsymbol{R}^n$, $T > \sigma$, is said to be a solution of (1) through (σ, ϕ) on $[\sigma, T)$ if $x_\sigma = \phi$ and $x(t)$ satisfies (1) for $\sigma \leqq t < T$. We denote by $x(t, \sigma, \phi, f)$ a (noncontinuable) solution of (1) through (σ, ϕ). By the assumption that $f(t, 0) \equiv 0$ on I, (1) has the zero solution $x(t) \equiv 0$.

We now introduce the following assumptions on the function $f(t, \phi)$:

(i) For each T, $0 < T < \infty$, the function $f(t, x_t)$ is continuous in $(t, x) \in [0, T] \times \widetilde{\mathrm{BC}}_T$, where $\widetilde{\mathrm{BC}}_T = \{x: (-\infty, T] \to \boldsymbol{R}^n, x_T \in \mathrm{BC}\}$ is a Banach space with the uniform norm.

(ii) f takes closed bounded sets of $I \times \mathrm{BC}$ into bounded sets of $\boldsymbol{R}^n$.

We denote by Ω the set of all functions $f: I \times \mathrm{BC} \to \boldsymbol{R}^n$ satisfying (i) and (ii). Clearly, the set C which consists of all continuous functions on I to $\boldsymbol{R}^n$ is considered as a subset of Ω. Henceforth, we assume that the function $f(t, \phi)$ is in Ω. Hence, it follows from the well known results (cf. [1]) that given $(\sigma, \phi) \in I \times \mathrm{BC}$, a solution of (1) through (σ, ϕ) exists and that if $x(t)$ is a (noncontinuable) solution of (1) on $[\sigma, T)$, then $\limsup_{t \to T^-} \|x_t\| = \infty$ or $T = \infty$.

For a scalar C^1-function $V(t, x)$: $\boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}$, the derivative of V along the solution of (1) is defined by

$$\dot{V}_{(1)}(t, \phi) := \frac{\partial V}{\partial t}(t, \phi(0)) + \langle \operatorname{grad} V(t, \phi(0)), f(t, \phi) \rangle$$

for $t \in I$ and $\phi \in \mathrm{BC}$, where $\langle\ ,\ \rangle$ denotes the inner product in $\boldsymbol{R}^n$. Clearly, we have $(d/dt) V(t, x(t)) = \dot{V}_{(1)}(t, x_t)$ for $\sigma \leqq t < T$ if $x(t)$ is a solution of (1) on $[\sigma, T)$.

In what follows we need the following definitions (cf. [3, Chapter 1]).

Definition 1. The zero solution of (1) is totally stable (for brevity, TS), if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $(\sigma, \phi, g) \in I \times \mathrm{BC}_\delta \times \Omega$ and $|g(t, \phi)| < \delta(\varepsilon)$ for all $(t, \phi) \in [\sigma, \infty) \times \overline{\mathrm{BC}_\varepsilon}$, then $|x(t, \sigma, \phi, f+g)| < \varepsilon$ for all $t \geqq \sigma$.

Definition 2. The zero solution of (1) is totally asymptotically stable (for brevity, TAS), if it is TS and if there exist $\delta_0 > 0$ and $\gamma_0 > 0$ with the property that for any $\varepsilon > 0$ there exist $\gamma(\varepsilon) > 0$ and $T(\varepsilon) > 0$ such that if $(\sigma, \phi, g) \in I \times \mathrm{BC}_{\delta_0} \times \Omega$ and $|g(t, \phi)| < \gamma(\varepsilon)$ for all $(t, \phi) \in [\sigma, \infty) \times \overline{\mathrm{BC}_{\gamma_0}}$, then $|x(t, \sigma, \phi, f + g)| < \varepsilon$ for all $t \geqq \sigma + T(\varepsilon)$.

Now, we have the following lemma.

Lemma. *The zero solution of* (1) *is* TS *if and only if for any* $\varepsilon > 0$ *there exists a* $\bar{\delta} = \bar{\delta}(\varepsilon) > 0$ *such that if* $(\sigma, \phi, p) \in I \times \mathrm{BC}_{\bar{\delta}} \times C$ *and* $\sup_{t \geqq \sigma} |p(t)| <$

$\bar{\delta}(\varepsilon)$, then $|x(t, \sigma, \phi, f+p)| < \varepsilon$ for all $t \geqq \sigma$. Moreover, the zero solution of (1) *is* TAS *if and only if it is* TS *and there exists a $\bar{\delta}_0 > 0$ with the property that for any $\varepsilon > 0$ there exist $\bar{\gamma}(\varepsilon) > 0$ and $\bar{T}(\varepsilon) > 0$ such that if $(\sigma, \phi, p) \in I \times \mathrm{BC}_{\bar{\delta}_0} \times C$ and $\sup_{t \geqq \sigma}|p(t)| < \bar{\gamma}(\varepsilon)$, then $|x(t, \sigma, \phi, f+p) < \varepsilon$ for all $t \geqq \sigma + \bar{T}(\varepsilon)$.*

We can prove the lemma by employing almost the same arguments as in [3, Chapter 1, Section 8] (also, refer to [5, Lemma 1]). For instance, if we let $\delta(\cdot) = \bar{\delta}(\cdot)$, $\delta_0 = \min(\bar{\delta}(1), \bar{\delta}_0)$, $\gamma_0 = 1$, $\gamma(\cdot) = \min(\bar{\gamma}(\cdot), \bar{\delta}(1))$ and $T(\cdot) = \bar{T}(\cdot)$, then $\delta(\cdot)$ (resp. $\delta(\cdot)$, δ_0, γ_0, $\gamma(\cdot)$, $T(\cdot)$) satisfies the condition in Definition 1 (resp. Definition 2). We omit the details.

Now, let K be the set of all continuous and strictly monotone increasing functions $u(s)$ on I with $u(0) = 0$. In what follows, we shall investigate the total stability and total asymptotic stability of the zero solution of (1) by using the method of Liapunov-Razumikhin type.

THEOREM 1. *Let $V(t, x)$ be a scalar C^1-function with the properties that;*

(i) *there exist $u(s)$ and $v(s)$ in K such that*

$$u(|x|) \leqq V(t, x) \leqq v(|x|),\ t \in \boldsymbol{R},\ x \in \boldsymbol{R}^n\ ;$$

(ii) *there exist positive constants k and r such that*

$$|V(t, x) - V(t, y)| \leqq k|x-y|,\ t \in \boldsymbol{R},\ |x| \leqq r,\ |y| \leqq r\ ;$$

(iii) *there exists a $w(s)$ in K such that*

$$\dot{V}_{(1)}(t, \phi) \leqq -w(|\phi(0)|)$$

for all $(t, \phi) \in I \times \mathrm{BC}$ satisfying $V(t, \phi(0)) = \sup_{\theta \leqq 0} V(t+\theta, \phi(\theta))$. Then the zero solution of (1) *is* TS, *and $\delta(\cdot)$ in Definition* 1 *can be chosen so that it depends on only the functions u, v, w and the constants k, r.*

PROOF. Let an $\varepsilon \in (0, r)$ be given, and select a positive constant $\eta = \eta(\varepsilon)$ so that $v(\eta) < u(\varepsilon)$. Set $c = w(\eta)$, and define $\delta = \delta(\varepsilon)$ by $\delta(\varepsilon) := \min(\eta, c/k)$. By Lemma it suffices to show that if $(\sigma, \phi, p) \in I \times \mathrm{BC}_\delta \times C$ and $\sup_{t \leqq \sigma}|p(t)| < \delta(\varepsilon)$, then $|x(t)| < \varepsilon$ for all $t \geqq \sigma$, where $x(t) = x(t, \sigma, \phi, f+p)$. Suppose that this is not the case. Then there is a T, $T > \sigma$, such that $|x(T)| = \varepsilon$ and $|x(t)| < \varepsilon$ for all $t < T$. Set $V(t) = V(t, x(t))$. Then $\sup_{\theta \leqq 0} V(\sigma+\theta) < V(T)$, since $\sup_{\theta \leqq 0} V(\sigma+\theta) \leqq \sup_{\theta \leqq 0} v(|\phi(\theta)|) < v(\delta) \leqq v(\eta) < u(\varepsilon) \leqq V(T)$ by (i). Hence, there is a T_0, $\sigma < T_0 \leqq T$, such that $V(T_0) = \sup_{u \leqq T} V(u) =: M$. Now, we have $\eta \leqq |x(T_0)| \leqq \varepsilon$, since $|x(T_0)| \leqq |x(T)| = \varepsilon$ and $v(\eta) < u(\varepsilon) \leqq V(T) \leqq V(T_0) \leqq v(|x(T_0)|)$ by (i). Moreover, $V(T_0) = M \geqq V(u)$ for all $u \leqq T_0$. Thus, we obtain $\dot{V}_{(1)}(T_0, x_{T_0}) \leqq -w(|x(T_0)|) \leqq$

$-c$ by (iii), and consequently $\dot{V}(T_0) \leqq \dot{V}_{(1)}(T_0, x_{T_0}) + k|p(T_0)| < -c + k\delta(\varepsilon) < 0$ by (ii). Therefore, there is a T_1, $T_1 < T_0$, such that $V(T_1) > V(T_0) = M$, which is a contradiction q.e.d.

EXAMPLE. Suppose that there exist positive constants ν and h, $\nu h < 1$, such that

$$|\phi(0) - \psi(0) + h(F(t, \phi) - F(t, \psi))| \leqq (1 - \nu h)\|\phi - \psi\|$$

for all (t, ϕ) and (t, ψ) in $I \times \mathrm{BC}$. If $F \in \Omega$ and $u(t)$ is a solution of $\dot{x}(t) = F(t, x_t)$ on I, then $u(t)$ is TS, that is, the zero solution of $\dot{z}(t) = \bar{F}(t, z_t)$ is TS, where $\bar{F}(t, \phi) := F(t, u_t + \phi) - F(t, u_t)$ (cf. [10]). Indeed, by the fact that $F \in \Omega$, we easily see that $\bar{F} \in \Omega$. Furthermore, as can be easily checked, the function $V(t, z) = |z|^2$ satisfies all of the conditions in Theorem 1. Hence, the zero solution of $\dot{z}(t) = \bar{F}(t, z_t)$ is TS.

It should be noted that without any additional condition, the asymptotic stability of the zero solution of (1) cannot necessarily be deduced from the assumptions of Theorem 1. Indeed, if $f(t, \phi) = -2\phi(0) + \phi(-t)$, then the zero solution of (1) is not asymptotically stable, whereas the function $V(t, x) = |x|^2$ satisfies all of the conditions in Theorem 1 (cf. [8], [9]). In Theorem 2 below, we shall provide such an additional condition.

THEOREM 2. *Let* $V(t, x)$ *be a scalar* C^1*-function satisfying the conditions* (i) *and* (ii) *in Theorem* 1 *and the condition;*

(iii)′ *there exist a positive constant* h, *a* $w(s)$ *in* K *and a continuous function* $\rho(s)$, $\rho(s) > s$ *for* $s > 0$, *such that*

$$\dot{V}_{(1)}(t, \phi) \leqq -w(|\phi(0)|)$$

for all $(t, \phi) \in I \times \mathrm{BC}$ *satisfying* $\rho(V(t, \phi(0))) \geqq \sup_{-h \leqq \theta \leqq 0} V(t + \theta, \phi(\theta))$.

Then the zero solution of (1) *is* TAS, *and* δ_0, γ_0, $\delta(\cdot)$, $\gamma(\cdot)$ *and* $T(\cdot)$ *in Definition* 2 *can be chosen so that* δ_0, γ_0, $\delta(\cdot)$ *and* $\gamma(\cdot)$ *depend only on the functions* u, v, w, ρ *and the constants* k, r, *while* $T(\cdot)$ *depends also on the constant* h.

PROOF. Since (iii)′ implies (iii) in Theorem 1, the zero solution of (1) is TS with $\delta(\cdot)$ which depends only on the functions u, v, w and the constants k, r. Therefore, we can choose a positive constant δ_0 so that if $(\sigma, \phi, p) \in I \times \mathrm{BC}_{\delta_0} \times C$ and $\sup_{t \geqq \sigma}|p(t)| < \delta_0$, then $|x(t, \sigma, \phi, f + p)| < r$ for all $t \geqq \sigma$. Now, let an $\varepsilon \in (0, \delta_0)$ be given, and let η and c be the numbers given in the proof of Theorem 1. Set $a = a(\varepsilon) := \inf\{\rho(s) - s : u(\varepsilon) \leqq s \leqq v(r)\}$. Clearly, $a > 0$. Let $N = N(\varepsilon)$ be the first positive integer such that $u(\varepsilon) + Na \geqq v(r)$, and set $\gamma(\varepsilon) = \min(\delta_0, c/(2k))$ and $T(\varepsilon) = 2Nv(r)/c + (N-1)h$. By the proof of Lemma, it suffices to prove that if $(\sigma, \phi, p) \in$

$I \times BC_{\delta_0} \times C$ and $\sup_{t \geqq \sigma}|p(t)| < \gamma(\varepsilon)$, then $|x(t, \sigma, \phi, f+p)| \leqq \varepsilon$ for all $t \geqq \sigma + T(\varepsilon)$. Set $x(t) = x(t, \sigma, \phi, f+p)$ and $V(t) = V(t, x(t, \sigma, \phi, f+p))$. First we show that

$$(2) \qquad V(t_1) \leqq u(\varepsilon) + (N-1)a \quad \text{for a} \quad t_1 \in [\sigma, \sigma + 2v(r)/c] .$$

Suppose that $V(t) > u(\varepsilon) + (N-1)a$ for all $t \in [\sigma, \sigma + 2v(r)/c]$. Then we have $\rho(V(t)) \geqq V(t) + a > u(\varepsilon) + Na \geqq v(r) \geqq \sup_{-h \leqq \theta \leqq 0} V(t+\theta)$ for all $t \in [\sigma, \sigma + 2v(r)/c]$, since $|x(t)| \leqq r$. Moreover, since $|x(t)| \geqq \eta$ on $[\sigma, \sigma + 2v(r)/c]$, it follows from (iii)$'$ that $(d/dt)V(t) \leqq k|p(t)| - w(|x(t)|) < k\gamma(\varepsilon) - c \leqq -c/2$ for all $t \in [\sigma, \sigma + 2v(r)/c]$. Hence $V(\sigma + 2v(r)/c) < V(\sigma) + (-c/2) \cdot 2v(r)/c \leqq v(r) - v(r) = 0$. This is a contradiction. Thus, the assertion (2) holds.

Next, we show that

$$(3) \qquad V(t) \leqq u(\varepsilon) + (N-1)a \quad \text{for all} \quad t \geqq \sigma + 2v(r)/c .$$

Indeed, if we suppose that $V(t_2) > u(\varepsilon) + (N-1)a$ for some $t_2 \in [\sigma + 2v(r)/c, \infty)$, then it follows from (2) that there is a $t_3 \in [t_1, t_2)$ satisfying $V(t_3) = u(\varepsilon) + (N-1)a$ and $\dot{V}(t_3) \geqq 0$. Note that $\rho(V(t_3)) \geqq V(t_3 + \theta)$ for all $\theta \in [-h, 0]$ and $|x(t_3)| \geqq \eta$. Then, by (iii)$'$ we have $\dot{V}(t_3) \leqq k|p(t_3)| - w(|x(t_3)|) \leqq -c/2 < 0$, which contradicts $\dot{V}(t_3) \geqq 0$. Thus, the assertion (3) holds.

Finally, we show that

$$(4) \qquad |x(t)| \leqq \varepsilon \quad \text{for all} \quad t \geqq \sigma + T(\varepsilon) .$$

If $N = 1$, then (3) implies $V(t) \leqq u(\varepsilon)$ for all $t \geqq \sigma + T(\varepsilon)$, and hence the assertion (4) holds. Suppose $N \geqq 2$. By repeating the same arguments as in the proof of the assertion (2), we can show that $V(t_4) \leqq u(\varepsilon) + (N-2)a$ for a $t_4 \in [\sigma + 2v(r)/c + h, \sigma + 4v(r)/c + h]$ by (3). Hence, we obtain $V(t) \leqq u(\varepsilon) + (N-2)a$ for all $t \geqq \sigma + 4v(r)/c + h$ by the same type of reasoning as in (3). Repeat this procedure. Then $V(t) \leqq u(\varepsilon) + (N-j)a$ for all $t \geqq \sigma + 2jv(r)/c + (j-1)h$, $j = 1, 2, \cdots, N$. Consequently, we have $V(t) \leqq u(\varepsilon)$ for all $t \geqq \sigma + T(\varepsilon)$, and hence the assertion (4) holds. q.e.d.

We consider a system of integrodifferential equations

$$(5) \qquad \dot{x}(t) = Ax(t) + F(x_t) + \int_{-\infty}^{0} g(t, s, x(t+s))ds .$$

Impose the following hypotheses on (5):

(H1) A is an $n \times n$ real constant matrix, all the eigenvalues of which have negative real parts;

(H2) $F\colon C([-h_1, 0]) \to \boldsymbol{R}^n$, $0 \leqq h_1 < \infty$, is a continuous function, $C([-h_1, 0])$ the space of continuous functions on $[-h_1, 0]$ to $\boldsymbol{R}^n$ with the usual supremum norm $\|\cdot\|_C$, and $|F(\phi)| \leqq b\|\phi\|_C$ for all $\phi \in C([-h_1, 0])$ and some constant b;

(H3) $g(t, s, x)\colon I\times(-\infty, 0]\times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is continuous and satisfies

$$|g(t, s, x)| \leqq m(s)|x| \quad \text{for all} \quad (t, s, x) \in I\times(-\infty, 0]\times \boldsymbol{R}^n ,$$

where $\int_{-\infty}^0 m(s)ds < \infty$.

From (H1), (H2) and (H3), we see that the function $f\colon I\times \mathrm{BC} \to \boldsymbol{R}^n$ defined by $f(t, \phi) = A\phi(0) + F(\phi) + \int_{-\infty}^0 g(t, s, \phi(s))ds$ is in Ω. As an application of Theorem 2, we shall investigate the stability property of the zero solution of (5). Now, from the well known result in matrix theory there exists a unique positive definite symmetric real matrix B such that $BA + A^T B = -E$; here E denotes the identity matrix and A^T the transpose of A. Let Λ and λ be positive numbers such that Λ^2 and λ^2 are the greatest and least eigenvalues of B, respectively. Clearly, we have $\lambda^2|x|^2 \leqq \langle Bx, x\rangle \leqq \Lambda^2|x|^2$ for all x in $\boldsymbol{R}^n$.

THEOREM 3. *Let* (H1)-(H3) *hold, and suppose*

$$\text{(H4)} \qquad \int_{-\infty}^0 m(s)ds < \lambda/(2\Lambda^3) - b .$$

Then the zero solution of (5) *is* TAS.

PROOF. By (H4) we can choose a constant $\mu > 1$ so that

$$1 - 2\mu\Lambda^3\Big(\int_{-\infty}^0 m(s)ds + b\Big)\Big/\lambda =: l > 0 .$$

For any $h \in (h_1, \infty)$ we consider a system

$$(5)_h \qquad \dot{x}(t) = Ax(t) + F(x_t) + \int_{-h}^0 g(t, s, x(t+s))ds .$$

Set $V(x) = \langle Bx, x\rangle$. We shall show that the function $V(x)$ satisfies all of the conditions in Theorem 2 for System $(5)_h$. The conditions (i) and (ii) clearly hold. Assume that $\mu^2 V(\phi(0)) \geqq V(\phi(\theta))$ for all $\theta \in [-h, 0]$. Then $\mu^2\Lambda^2|\phi(0)|^2 \geqq \lambda^2|\phi(\theta)|^2$, and hence $|\phi(\theta)| \leqq \mu\Lambda|\phi(0)|/\lambda$ for all $\theta \in [-h, 0]$. It follows that

$$\begin{aligned}\dot{V}_{(5)_h}(t, \phi) &= \Big\langle B\Big[A\phi(0) + F(\phi) + \int_{-h}^0 g(t, s, \phi(s))ds\Big], \phi(0)\Big\rangle \\ &\quad + \Big\langle B\phi(0), A\phi(0) + F(\phi) + \int_{-h}^0 g(t, s, \phi(s))ds\Big\rangle \\ &= \langle (BA + A^T B)\phi(0), \phi(0)\rangle + 2\Big\langle B\phi(0), F(\phi) + \int_{-h}^0 g(t, s, \phi(s))ds\Big\rangle\end{aligned}$$

$$\leqq -|\phi(0)|^2 + 2\Lambda^2|\phi(0)|\Big(b + \int_{-\infty}^{0} m(s)ds\Big)\cdot\sup_{-h\leqq\theta\leqq 0}|\phi(\theta)|$$
$$\leqq -l|\phi(0)|^2 .$$

Thus, the condition (iii)′ in Theorem 2 also holds as $w(s) = ls^2$ and $\rho(s) = \mu^2 s$. Therefore, the zero solution of $(5)_h$ is TAS with δ_0, γ_0, $\delta(\cdot)$, $\gamma(\cdot)$ and $T(h, \cdot)$, where δ_0, γ_0, $\delta(\cdot)$ and $\gamma(\cdot)$ are independent of h.

Now, let $\varepsilon \in (0, \gamma_0)$ be given. Select a constant $h(\varepsilon)$, $h(\varepsilon) > h_1$, such that

$$\gamma_0\cdot\int_{-\infty}^{-h(\varepsilon)} m(s)ds < \min(\delta(\varepsilon)/2,\ \gamma(\varepsilon)/2) .$$

If $Q \in \Omega$ and $|Q(t, \phi)| < \delta(\varepsilon)/2$ for all $(t, \phi) \in [\sigma, \infty)\times\overline{\mathrm{BC}_\varepsilon}$, then

$$\left|\int_{-\infty}^{-h(\varepsilon)} g(t, s, \phi(s))ds + Q(t, \phi)\right| \leqq \varepsilon\int_{-\infty}^{-h(\varepsilon)} m(s)ds + \delta(\varepsilon)/2 < \delta(\varepsilon)$$

for all $(t, \phi) \in [\sigma, \infty]\times\overline{\mathrm{BC}_\varepsilon}$. Therefor, if $(\sigma, \phi, Q) \in I\times\mathrm{BC}_{\delta(\varepsilon)}\times\Omega$ and $|Q(t, \phi)| < \delta(\varepsilon)/2$ for all $(t, \phi) \in [\sigma, \infty)\times\overline{\mathrm{BC}_\varepsilon}$, then, from the total stability of the zero solution of $(5)_{h(\varepsilon)}$ it follows that $|x(t, \sigma, \phi)| < \varepsilon$ for all $t \geqq \sigma$, where $x(t, \sigma, \phi)$ denotes a solution of

$$\dot{x}(t) = Ax(t) + F(x_t) + \int_{-h(\varepsilon)}^{0} g(t, s, x(t+s))ds + \int_{-\infty}^{-h(\varepsilon)} g(t, s, x(t+s))ds$$
$$+ Q(t, x_t)$$

through (σ, ϕ). Thus, the zero solution of (5) is TS. Similarly, if $(\sigma, \phi, Q) \in I\times\mathrm{BC}_{\delta_0}\times\Omega$ and $|Q(t, \phi)| < \gamma(\varepsilon)/2$ for all $(t, \phi) \in [\sigma, \infty)\times\overline{\mathrm{BC}_{\gamma_0}}$, then we obtain $|x(t, \sigma, \phi)| < \varepsilon$ for all $t \geqq \sigma + T(h(\varepsilon), \varepsilon)$. Hence the zero solution of (5) is TAS. q.e.d.

REMARK 1. When $n = 1$ and $F(\phi) = b\phi(-h_1)$, Wang [11] proved that the zero solution of (5) is uniformly asymptotically stable under all of the assumptions in Theorem 3 and the additional assumption; $\int_{-\infty}^{0} m(s)e^{-\gamma s}ds < \infty$ for a constant $\gamma > 0$ (also, refer to [4]). Thus, our Theorem 3 gives an improvement of the one in [11].

Next, we consider a system of Volterra integrodifferential equations

$$(6)\qquad \dot{x}(t) = Ax(t) + F(x_t) + \int_0^t g(t, s, x(s))ds , \qquad t \geqq 0 ,$$

and assume the hypotheses (H1), (H2) and

(H3′) $g(t, s, x)$: $I\times I\times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is continuous and satisfies $|g(t, s, x)| \leqq m(t, s)|x|$ for all $(t, s, x) \in I\times I\times\boldsymbol{R}^n$.

Clearly, from (H1), (H2) and (H3′) it follows that the function

$f: I\times \mathrm{BC}\to \boldsymbol{R}^n$ defined by $f(t, \phi) = A\phi(0) + F(\phi) + \int_0^t g(t, s, \phi(s-t))ds$ is in Ω. Now, we prove:

THEOREM 4. *Let* (H1), (H2) *and* (H3′) *hold, and suppose*

$$\text{(H4}')\qquad \sup_{t\geqq 0}\int_0^t m(t, s)ds < \lambda/(2\Lambda^3) - b, \quad \textit{and}$$

$$\text{(H5)}\qquad \sup_{t\geqq 0}\int_0^t m(t+\tau, s)ds \to 0 \quad \textit{as} \quad \tau\to\infty .$$

Then the zero solution of (6) *is* TAS.

PROOF. By (H4′) we can choose a constant $\mu > 1$ so that

$$1 - 2\mu\Lambda^3\Big(\sup_{t\geqq 0}\int_0^t m(t, s)ds + b\Big)\Big/\lambda =: l > 0 .$$

For any $h\in(h_1, \infty)$ we consider a system

$$(6)_h \qquad \dot{x}(t) = Ax(t) + F(x_t) + \int_{t-h}^t g(t, s, x(s))ds , \qquad t\geqq h .$$

Then, by almost the same argument as in the proof of Theorem 3 we can prove that the zero solution of $(6)_h$ is TAS with δ_0, γ_0, $\delta(\cdot)$, $\gamma(\cdot)$ and $T(h, \cdot)$, where δ_0, γ_0, $\delta(\cdot)$ and $\gamma(\cdot)$ are independent of h. Furthermore, by Theorem 1 we can also see that the zero solution of (6) is TS with $\delta(\cdot)$. We may assume $\delta_0 < \gamma_0$. Now, let an $\varepsilon > 0$ be given. Select a constant $h(\varepsilon)$, $h(\varepsilon) > h_1$, so that

$$\gamma_0\cdot\sup_{t\geqq 0}\int_0^t m(t+h(\varepsilon), s)ds < \gamma(\varepsilon)/2 ,$$

which is possible by (H5). Suppose that $(\sigma, \phi, Q)\in[h(\varepsilon), \infty)\times \mathrm{BC}_{\delta_0}\times\Omega$ and $|Q(t, \phi)| < \gamma(\varepsilon)/2$ for all $(t, \phi)\in[\sigma, \infty)\times\overline{\mathrm{BC}_{\gamma_0}}$. Then, for all $(t, \phi)\in[\sigma, \infty)\times\overline{\mathrm{BC}_{\gamma_0}}$ we have

$$\begin{aligned}\Big|\int_{-t}^{-h(\varepsilon)} g(t, t+s, \phi(s))ds + Q(t, \phi)\Big| &< \gamma_0\int_{-t}^{-h(\varepsilon)} m(t, t+s)ds + \gamma(\varepsilon)/2 \\ &\leqq \gamma_0\cdot\sup_{u\geqq 0}\int_0^u m(u+h(\varepsilon), s)ds + \gamma(\varepsilon)/2 < \gamma(\varepsilon) .\end{aligned}$$

Therefore, the total asymptotic stability of the zero solution of $(6)_{h(\varepsilon)}$ implies $|x(t, \sigma, \phi)| < \varepsilon$ for all $t\geqq \sigma + T(h(\varepsilon), \varepsilon)$, where $x(t, \sigma, \phi)$ denotes a solution of

$$\begin{aligned}\dot{x}(t) &= Ax(t) + F(x_t) + \int_{t-h(\varepsilon)}^t g(t, s, x(s))ds + \int_{-t}^{-h(\varepsilon)} g(t, t+s, x_t(s))ds + Q(t, x_t) \\ &= Ax(t) + F(x_t) + \int_0^t g(t, s, x(s))ds + Q(t, x_t)\end{aligned}$$

through (σ, ϕ). Set $\tilde{\delta}_0 = \delta(\delta_0)$, $\tilde{\gamma}_0 = \gamma_0$, $\tilde{\delta}(\cdot) = \delta(\cdot)$, $\tilde{\gamma}(\cdot) = \min(\delta(\delta_0), \gamma(\cdot)/2)$ and $\tilde{T}(\cdot) = T(h(\cdot), \cdot) + h(\cdot)$. Then, since $\delta_0 < \gamma_0$, we easily see that the zero solution of (6) is TAS with $\tilde{\delta}_0$, $\tilde{\gamma}_0$, $\tilde{\delta}(\cdot)$, $\tilde{\gamma}(\cdot)$ and $\tilde{T}(\cdot)$. q.e.d.

REMARK 2. Miller [6] discussed the stability properties of solutions of a system of Volterra integrodifferential equations of the form

$$\dot{x}(t) = Ax(t) + \int_0^t G(t-s)x(s)ds \, . \tag{7}$$

He showed in [6, Theorem 9 (i)] that the zero solution of (7) is uniformly asymptotically stable under the conditions;

($*$) $|G(t)|$ is integrable on $[0, \infty)$ and $\det(s - A - G^*(s)) \neq 0$ when $\operatorname{Re} s \geqq 0$,

where $G^*(s) = \int_0^\infty e^{-st}G(t)dt$;

(**) $$\int_0^\infty \left(\int_s^\infty |G(u)|du \right)^p ds < \infty \quad \text{for a constant} \quad p \in [1, 2] \, .$$

Miller's result is applicable also to the case where the matrix A does not necessarily satisfy (H1). Furthermore, the condition ($*$) is weaker than (H4′) for the function $m(t, s) := |G(t-s)|$. However, it should be noted that the condition ($**$) is stronger than (H5). Indeed, if $|G(t)| = (t+1)^{-\alpha}$ for an $\alpha \in (1, 3/2)$, then the condition ($**$) does not hold. On the one hand, whenever $|G(t)|$ is integrable on $[0, \infty)$, the function $m(t, s) = |G(t-s)|$ satisfies (H5).

REMARK 3. Though we have investigated the local stability properties, we can also obtain the global stability properties by slight modification. In particular, we can deduce that the zero solution of (6) is globally uniformly asymptotically stable under the assumptions in Theorem 4. Seifert [9] proved that the zero solution of (6) is uniformly stable and globally asymptotically stable under the assumptions in Theorem 4. Recently, Furumochi [2] has shown that the zero solution of (6) is globally uniformly asymptotically stable under the assumptions (H1), (H2), (H3′), (H4′) and

(H5′) $$\int_0^t m(t, s)ds \to 0 \quad \text{as} \quad t \to \infty \, .$$

Note that (H5′) implies (H5). Furthermore, the function $m(t, s) = e^{-(t-s)}$ does not satisfy (H5′), while it satisfies (H5).

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