Tôhoku Math. Journ.
37 (1985), 553-560.

MINIMAL SURFACES IN A SPHERE WITH GAUSSIAN CURVATURE NOT LESS THAN 1/6

TAKASHI OGATA

(Received December 11, 1984)

1. Introduction. Let $S^n(r)$ be an n-dimensional sphere in the $(n+1)$-dimensional Euclidean space $\boldsymbol{R}^{n+1}$ with radius r and $x: M \to S^n(r)$ be an isometric minimal immersion of a differentiable 2-manifold M into $S^n(r)$ $(n \geqq 3)$. If M is compact and the Gaussian curvature K of M is non-negative, not identically zero, then the genus of M is zero, by the Gauss-Bonnet theorem. In [3], Borůvka has constructed a series of isometric minimal immersions $\psi_k: S^2((k(k+1)/2)^{1/2}) \to S^{2k}(1)$ by making use of spherical harmonic polynomials of degree k. ψ_2 is the Veronese surface with $K = 1/3$ in $S^4(1)$ and ψ_3 is called the generalized Veronese surface with $K = 1/6$ in $S^6(1)$. Later, in [4], Calabi has proved that, any isometric full minimal immersion of $S^2(K^{-1/2})$ into $S^n(1)$ is congruent to some ψ_k and so there exists an integer k such that $K = 2/k(k+1)$ and $n = 2k$.

On the other hand, Lawson [13] and Benko et al. [2] have proved the following:

THEOREM A. *Let $x: M \to S^n(1)$ be an isometric minimal immersion of a complete, connected, oriented 2-manifold M into $S^n(1) (n \geqq 3)$. If $1/3 \leqq K \leqq 1$, then either $x(M)$ is totally geodesic and $K \equiv 1$, or the Veronese surface in $S^4(1)$ and $K \equiv 1/3$.*

In this paper, we shall prove:

THEOREM B. *Let $x: M \to S^n(1)$ be an isometric minimal immersion of a complete, connected, oriented 2-manifold M into $S^n(1)$ $(n \geqq 3)$. If $1/6 \leqq K \leqq 1$, then either* (1) *$x(M)$ is totally geodesic and $K \equiv 1$,* (2) *the generalized Veronese surface in $S^6(1)$ and $K \equiv 1/6$, or* (3) *a minimal surface in $S^4(1)$ with $1/6 \leqq K \leqq 1$.*

As a corollary to Theorem B, we can prove the following:

COROLLARY C. *If $1/6 \leqq K \leqq 1/3$, then $K \equiv 1/3$ or $1/6$, and either $x(M)$ is the Veronese surface in $S^4(1)$ in the case of $K \equiv 1/3$, or the generalized Veronese surface in $S^6(1)$ in the case of $K \equiv 1/6$.*

Recently, Kozlowski and Simon [12] proved Corollary C by studying the properties of eigenfunctions of the Laplacian for $S^2(1)$.

In Section 2, we shall explain the notion of the third fundamental form and calculate the Laplacian of the square of its length. In Section 3, we shall give proofs of Theorem B and Corollary C. The main idea of our proof is to calculate the Laplacian of some functions defined globally on a surface M. In Section 4, we shall give a one-parameter family of minimal immersions of a differentiable 2-sphere S^2 into $S^4(1)$, from which we can get infinitely many examples of the case (3) of Theorem B and show that the hypothesis on K in Theorem A is the best possible.

The author would like to express particular thanks to Professor K. Kenmotsu for his advice and encouragement during the development of this work.

2. The third fundamental forms. Let $\bar{M}$ be an n-dimensional Riemannian manifold of constant curvature c and $x: M \to \bar{M}$ be a minimal isometric immersion of a complete, connected, orientable Riemannian 2-manifold M into $\bar{M}$. We shall use the same notations and terminologies as in Kenmotsu [10] unless otherwise stated and denote by $h_{\alpha ij}$ or $h_{\alpha ij,k}$ the components of the second fundamental form or of its covariant derivatives, respectively. We first introduce scalar fields $K_{(2)}$, $N_{(2)}$ and $f_{(2)}$ on M used in [10] by the following equations:

$$K_{(2)} = \sum (h_{\alpha 11}^2 + h_{\alpha 12}^2) \tag{2.1}$$

$$N_{(2)} = \sum h_{\alpha 11}^2 \sum h_{\alpha 12}^2 - (\sum h_{\alpha 11} h_{\alpha 12})^2 \tag{2.2}$$

$$f_{(2)} = K_{(2)}^2 - 4N_{(2)} . \tag{2.3}$$

Note that these are globally defined on M and independent of the choice of the frame fields. If $K_{(2)}$ is identically zero on M, then M is totally geodesic.

LEMMA 2.1 (Chern [6] and Kenmotsu [10]). *Let M be a compact oriented minimal surface in $S^n(1)$. If the Gaussian curvature of M is strictly positive, then $f_{(2)}$ is identically zero on M.*

Suppose that the Gaussian curvature K is bounded from below by some positive constant. Hence, if M is complete and simply connected, then M is compact. By Lemma 2.1, we have $f_{(2)} = 0$ and so $\Delta K_{(2)} = 2\Delta(\sum h_{\alpha 11}^2)$. By [10, $(4.27)_2$], we have

$$\Delta(\sum h_{\alpha 11}^2) = 4(\sum h_{\alpha 11}^2)\cdot K - 4(\sum h_{\alpha 11}^2)^2 + 2\sum (h_{\alpha 11,1}^2 + h_{\alpha 11,2}^2) . \tag{2.4}$$

At any point p of M, we denote by $T_p^{(2)}$ the subspace of $T_p(\bar{M})$ spanned by e_1, e_2, $\sum h_{\alpha 11} e_\alpha$ and $\sum h_{\alpha 12} e_\alpha$, which is called the *second osculating* *space*. We identify the *first osculating space* $T_p^{(1)}$ with the tangent space $T_p(M)$ of M. In general we have $2 \leqq \dim T_p^{(2)} \leqq 4$. We set

$$\Omega_2 = \{p \in M;\ N_{(2)} \neq 0 \quad \text{at} \quad p\}\,,$$

which is an open subset of M. Hereafter, we assume that x is not totally geodesic. By Lemma 2.1, Ω_2 is not empty and so $\dim T_p^{(2)} = 4$ for any $p \in \Omega_2$. Let $\{e_A\}$ be a system of local orthonormal frame fields such that $\{e_i;\ i = 1, 2\}$ and $\{e_\lambda;\ \lambda = 3, 4\}$ span $T_p^{(2)}$ for $p \in \Omega_2$. We then have

$$\omega_{\alpha i} = 0 \quad \text{for} \quad \alpha \geqq 5 \quad \text{on} \quad \Omega_2\,. \tag{2.5}$$

By taking the exterior derivative of (2.5), and making use of the structure equations of M, we get

$$\omega_{i3} \wedge \omega_{3\alpha} + \omega_{i4} \wedge \omega_{4\alpha} = 0 \qquad (\alpha \geqq 5)\,. \tag{2.6}$$

This allows us to introduce quantities $h_{\alpha ijk}$ $(\alpha \geqq 5)$ defined by the equation

$$h_{3ij}\omega_{3\alpha} + h_{4ij}\omega_{4\alpha} = \sum h_{\alpha ijk}\omega_k \qquad (\alpha \geqq 5)\,. \tag{2.7}$$

$\{h_{\alpha ijk}\}$ is symmetric in the Latin indices i, j and k. By (2.7) and the minimality of $x(M)$, we get

$$\sum h_{\alpha iik} = 0 \qquad (\alpha \geqq 5)\,. \tag{2.8}$$

$\sum_{\alpha \geqq 5} (\sum_{i,j,k} h_{\alpha ijk}\omega_i \otimes \omega_j \otimes \omega_k) \otimes e_\alpha$ is called the third fundamental form of the immersed manifold $x(M)$ (cf. [6]). Note that, for any $\alpha \geqq 5$, we have

$$h_{\alpha ijk} = h_{\alpha ij,k} \tag{2.9}$$

which follows easily from the definition of the covariant derivatives of $h_{\alpha ij}$.

We define the covariant derivatives $h_{\alpha ijk,l}$ of $h_{\alpha ijk}$ by

$$\begin{aligned} Dh_{\alpha ijk} &= h_{\alpha ijk,l}\omega_l \\ &= dh_{\alpha ijk} + \sum h_{\alpha sjk}\omega_{si} + \sum h_{\alpha isk}\omega_{sj} + \sum h_{\alpha ijs}\omega_{sk} + \sum h_{\beta ijk}\omega_{\beta\alpha}\,. \end{aligned} \tag{2.10}$$

Then we have $\sum h_{\alpha iik,l} = 0$ by (2.8) and (2.10). By Lemma 2.1, the normal vectors $\sum h_{\alpha 11}e_\alpha$ and $\sum h_{\alpha 12}e_\alpha$ are perpendicular to each other and of the same non-zero length at any p in Ω_2. So, normalizing these vectors, we adopt them as a part of a basis of $T_p^{(2)}$ for $p \in \Omega_2$. With respect to these new frames, we have, on Ω_2,

$$h_{311} = h_{412},\ h_{312} = h_{411} = 0 \quad \text{and} \quad h_{\alpha ij} = 0 \qquad (\alpha \geqq 5)\,, \tag{2.11}$$

$$dh_{311} = h_{311,1}\omega_1 + h_{311,2}\omega_2\,, \tag{2.12}$$

(2.13) $$h_{311}(-\omega_{34}+2\omega_{12})=h_{311,2}\omega_1-h_{311,1}\omega_2\ ,$$

(2.14) $$h_{411,1}=-h_{311,2},\ h_{411,2}=h_{311,1}\ .$$

Also, (2.7), (2.8) and (2.11) imply, for $\alpha\geqq 5$,

$(2.15)_1$ $$h_{311}\omega_{3\alpha}=h_{\alpha111}\omega_1+h_{\alpha112}\omega_2$$

$(2.15)_2$ $$h_{311}\omega_{4\alpha}=h_{\alpha112}\omega_1-h_{\alpha111}\omega_2\ .$$

Taking the exterior derivative of (2.15), we have

(2.16) $$h_{\alpha111,2}=h_{\alpha112,1}\quad\text{and}\quad h_{\alpha111,1}+h_{\alpha112,2}=0\ .$$

We introduce three scalar fields $K_{(3)}$, $N_{(3)}$ and $f_{(3)}$ on Ω_2, which are defined by

(2.17) $$K_{(3)}=\sum(h_{\alpha111}^2+h_{\alpha112}^2)\ ,$$

(2.18) $$N_{(3)}=(\sum h_{\alpha111}^2)(\sum h_{\alpha112}^2)-(\sum h_{\alpha111}h_{\alpha112})^2\ ,$$

(2.19) $$f_{(3)}=K_{(3)}^2-4N_{(3)}\ .$$

Note that $f_{(3)}$ is globally defined on Ω_2 and the notions of these scalar fields can be extended to the higher order fundamental tensors if $K_{(3)}\neq 0$ (cf. [10]). As for the geometrical meaning of $K_{(i)}$ and $N_{(i)}$, $i=2,3$, we have the following:

LEMMA 2.2 (Ōtsuki [14]). (a) *If $K_{(2)}\neq 0$, $N_{(2)}\neq 0$ and $h_{\alpha11,1}=h_{\alpha11,2}=0$ ($\alpha\geqq 5$) on M, then there is a 4-dimensional totally geodesic submanifold of $\bar{M}$ such that M is contained in the submanifold.*

(b) *If $K_{(2)}K_{(3)}\neq 0$, $N_{(2)}N_{(3)}\neq 0$ and $h_{\alpha111,1}=h_{\alpha111,2}=0(\alpha\geqq 7)$ on M, then there is a 6-dimensional totally geodesic submanifold of $\bar{M}$ such that M is contained in the submanifold.*

By (2.17), (2.18) and (2.19), we have

(2.20) $$f_{(3)}=\Big(\sum_{\alpha\geqq 5}(h_{\alpha111}^2-h_{\alpha112}^2)\Big)^2+4\Big(\sum_{\alpha\geqq 5}h_{\alpha111}h_{\alpha112}\Big)^2\ .$$

LEMMA 2.3 (Chern [6] and Kenmotsu [10, p. 300, Proposition]). *Let M be a compact, oriented, connected minimal surface in $S^n(1)$. Suppose that M is not totally geodesic and the Gaussian curvature of M is strictly positive. Then $f_{(3)}$ is identically zero on Ω_2.*

3. Proofs of Theorem B and Corollary C. We assume that M is not totally geodesic. By virtue of the curvature condition and Lemma 2.3, we have $\sum h_{\alpha111}^2=\sum h_{\alpha112}^2$ and $\sum h_{\alpha111}h_{\alpha112}=0$ on Ω_2.

LEMMA 3.1. $\Delta(\sum h_{\alpha111}^2)=6(\sum h_{\alpha111}^2)K-(4/h_{311}^2)(\sum h_{\alpha111}^2)^2+2\sum(h_{\alpha111,1}^2+h_{\alpha111,2}^2)$.

PROOF. Since the proof of this lemma in [10] is incorrect, we give it here for completeness. We first get

$$d(\sum h_{\alpha 111}^2) = 2\sum (h_{\alpha 111}h_{\alpha 111,1}\omega_1 + h_{\alpha 111}h_{\alpha 111,2}\omega_2)\ ,$$
$$\Delta(\sum h_{\alpha 111}^2)\omega_1 \wedge \omega_2 = 2d\{\sum (h_{\alpha 111}h_{\alpha 111,1}\omega_2 - h_{\alpha 111}h_{\alpha 111,2}\omega_1)\}\ .$$

On the other hand, by (2.16), we have,

$$2\sum (h_{\alpha 111}h_{\alpha 111,1}\omega_2 - h_{\alpha 111}h_{\alpha 111,2}\omega_1) = -2\sum h_{\alpha 111}(dh_{\alpha 112} + \sum h_{\beta 112}\omega_{\beta\alpha} + 3h_{\alpha 111}\omega_{12})\ .$$

Hence, by direct calculation, we have

$$\Delta(\sum h_{\alpha 111}^2)\omega_1 \wedge \omega_2 = 6K\sum h_{\alpha 111}^2\omega_1 \wedge \omega_2 - 2\sum h_{\alpha 111}h_{\beta 112}\omega_{\beta 3} \wedge \omega_{3\alpha} - 2\sum h_{\alpha 111}h_{\beta 112}\omega_{\beta 4} \wedge \omega_{4\alpha} + 2(\sum (h_{\alpha 111,1}^2 + h_{\alpha 111,2}^2)\omega_1 \wedge \omega_2\ .$$

Substituting (2.15) into the above equation, we have Lemma 3.1.

Making use of (2.4), (2, 11) and Lemma 2.3, we have

$$\Delta h_{311}^2 = 4h_{311}^2\cdot K - 4h_{311}^4 + 4(h_{311,1}^2 + h_{311,2}^2) + 4\sum h_{\alpha 111}^2\ . \tag{3.1}$$

$(h_{311}^2\sum h_{\alpha 111}^2)$ is a smooth function on Ω_2 by (2.1) and (2.11). We can compute the Laplacian of this function by using Lemma 3.1 and (3.1):

$$\begin{aligned}\Delta(h_{311}^2\sum h_{\alpha 111}^2) = {} & 10h_{311}^2K(\sum h_{\alpha 111}^2) - 4h_{311}^4(\sum h_{\alpha 111}^2) + 2h_{311}^2\sum (h_{\alpha 111,1}^2 \\ & + h_{\alpha 111,2}^2) + 4\sum h_{\alpha 111}^2(h_{311,1}^2 + h_{311,2}^2) \\ & + 8(h_{311}h_{311,1}\sum h_{\alpha 111}h_{\alpha 111,1} + h_{311}h_{311,2}\sum h_{\alpha 111}h_{\alpha 111,2})\ .\end{aligned} \tag{3.2}$$

Taking the exterior derivative of $\sum h_{\alpha 111}^2 = \sum h_{\alpha 112}^2$ and $\sum h_{\alpha 111}h_{\alpha 112} = 0$, we have

$$\begin{aligned}&\sum (h_{\alpha 111}h_{\alpha 111,2} + h_{\alpha 112}h_{\alpha 111,1}) = 0 \quad \text{and} \\ &\sum (h_{\alpha 111}h_{\alpha 111,1} - h_{\alpha 112}h_{\alpha 111,2}) = 0\ .\end{aligned} \tag{3.3}$$

Hence, by the Gauss equation, (3.3) and Lemma 2.3, (3.2) implies

$$\begin{aligned}\Delta(h_{311}^2\sum h_{\alpha 111}^2) = {} & 2h_{311}^2\sum h_{\alpha 111}^2(5 - 12h_{311}^2) + 2\sum \{(h_{311}h_{\alpha 111,1} + h_{311,1}h_{\alpha 111} \\ & - h_{311,2}h_{\alpha 112})^2 + (h_{311}h_{\alpha 111,2} + h_{311,1}h_{\alpha 112} + h_{311,2}h_{\alpha 111})^2\}\ .\end{aligned} \tag{3.4}$$

Since $K \geqq 1/6$ implies $h_{311}^2 \leqq 5/12$ on M, we have $\Delta(h_{311}^2\sum h_{\alpha 111}^2) \geqq 0$ on Ω_2. Note that $M - \Omega_2$ is atmost finite (cf. [6] or [10, p. 300, Proposition]). Hence, Ω_2 is parabolic or compact and the maximum principle holds good. If there exists a point p of $M - \Omega_2$ such that $\limsup_{x\to p}(h_{311}^2\sum h_{\alpha 111}^2)|_x = +\infty$, then we have $\limsup_{x\to p}\sum h_{\alpha 111}^2|_x = +\infty$. So, by (3.1), it follows that $\limsup_{x\to p}(\Delta h_{311}^2) \geqq \limsup_{x\to p} 4\sum h_{\alpha 111}^2 = +\infty$, because of $\lim_{x\to p} h_{311}^2 = 0$. This contradicts the boundedness of Δh_{311}^2 on the compact manifold M. Hence, $h_{311}^2\sum h_{\alpha 111}^2$ is an upper bounded, subharmonic function on the parabolic surface Ω_2, hence is constant on Ω_2. Thus, we have $(5 - 12h_{311}^2) \equiv 0$ or $\sum h_{\alpha 111}^2 \equiv 0$. If $(5 - 12h_{3111}^2) \equiv 0$, we have $M = \Omega_2$, $K \equiv 1/6$ and $\sum h_{\alpha 111}^2 = 5/48$ by (3.1). Moreover, by Lemma 3.1, we have $\sum(h_{\alpha 111,1}^2 + h_{\alpha 111,2}^2) = 0$, which shows $n = 6$ by Lemma 2.3 (b). On the other hand, if $\sum h_{\alpha 111}^2 = 0$, we have $n = 4$ by Lemma 2.3 (a), which completes the proof of Theorem A.

Next, we shall give the proof of Corollary C. By the assumption $1/6 \leqq K \leqq 1/3$, the case (1) in Theorem B does not happen. Hence, in proving Corollary C, it is sufficient to show that a minimal surface in $S^4(1)$ satisfying $1/6 \leqq K \leqq 1/3$ is the Veronese surface. From (3.1), we have

$$\Delta h_{311}^2 = 4h_{311}^2(1 - 3h_{311}^2) + 4(h_{311,1}^2 + h_{311,2}^2) \,. \tag{3.5}$$

On the other hand, $1/6 \leqq K \leqq 1/3$ implies $1/3 \leqq h_{311}^2 \leqq 5/12$ by the Gauss equation. From (3.5), we have $\Delta \log(1/h_{3111}^2) = 4(3h_{311}^2 - 1) \geqq 0$. Since $1/h_{311}^2$ is a positive scalar function on M, we have $h_{311}^2 = 1/3$, which gives $K = 1/3$.

4. Examples. In this section, we shall construct minimal surfaces in $S^4(1)$ with $K \geqq 1/6$, which give examples of the case (3) in Theorem B. Let z be an isothermal coordinate on S^2. We define a one-parameter family $\{x_t;\ t \in (0, \infty)\}$ of immersions of S^2 into $S^4(1)$.

$$x_t = \frac{1}{(t + 3|z|^2 + 3t^2|z|^4 + t|z|^6)} \begin{pmatrix} (3t)^{1/2}(z^2 + \bar{z}^2)(|z|^2 + t) \\ -i(3t)^{1/2}(z^2 - \bar{z}^2)(|z|^2 + t) \\ -i(3t)^{1/2}(z - \bar{z})(t|z|^4 - 1) \\ (3t)^{1/2}(z + \bar{z})(t|z|^4 - 1) \\ -t + 3|z|^2 + 3t^2|z|^4 - t|z|^6 \end{pmatrix} . \tag{4.1}$$

Then, for each $t \in (0, \infty)$, we have, by direct calculation,

$$ds_t^2 = \frac{12t(1 + 4t^2|z|^2 + 6t|z|^4 + 4|z|^6 + t^2|z|^8)}{(t + 3|z|^2 + 3t^2|z|^4 + t|z|^6)^2} dz \otimes d\bar{z} \tag{4.2}$$

$$K_{(t)} = 1 - \frac{2(t + 3|z|^2 + 3t^2|z|^4 + t|z|^6)^4}{3t(1 + 4t^2|z|^2 + 6t|z|^4 + 4|z|^6 + t^2|z|^8)^3} \tag{4.3}$$

$$\Delta_{(t)} x_t = -2x_t \,, \tag{4.4}$$

where ds_t^2 is the Riemannian metric of S^2 induced by x_t and $K_{(t)}$, (resp. $\Delta_{(t)}$) is the Gaussian curvature, (resp. the Laplacian), with respect to ds_t^2. From (4.4), we conclude that each immersion x_t is minimal. We see easily that x_1 is the Veronese surface and, for each $t > 0$, x_t is not totally geodesic because of $K_{(t)} \neq 1$.

PROPOSITION 4.1. *The example* (4.1) *corresponds to the one-parameter* *family* ξ_t *of directrix curves in* $\boldsymbol{CP}^4$ *given by Chern* (*cf.* [7] *or* [8]) *in homogeneous coordinate*:

$$\xi_t = \begin{pmatrix} 1 + tz^4 \\ i(1 - tz^4) \\ 2i(tz + z^3) \\ 2(-tz + z^3) \\ -2(3t)^{1/2}z^2 \end{pmatrix}. \tag{4.5}$$

PROOF. In $\boldsymbol{C}^5$ the symmetric product of two vectors $\boldsymbol{a} = (a_i)$, $\boldsymbol{b} = (b_i)$ is given by $(\boldsymbol{a}, \boldsymbol{b}) = \sum a_i b_i$. Following Barbosa [1], we compute $G_2 = \bar{\partial}^2 x_t - \{(\bar{\partial}^2 x_t, \partial x_t)/(\bar{\partial} x_t, \partial x_t)\}\bar{\partial} x_t$. We then have $\xi_t = G_2/(G_2, \bar{G}_2)$, which proves Proposition 4.1 by [1, Theorem (3.30)].

REMARK. Corresponding to Tjaden's example in [12], we have the following one-parameter family $\tilde{\xi}_t$ of directrix curves in $\boldsymbol{CP}^4$:

$$\tilde{\xi}_t = \begin{pmatrix} e^t + e^{-t}z^4 \\ i(e^t - e^{-t}z^4) \\ 2(z - z^3) \\ -2i(z + z^3) \\ 2\cdot 3^{1/2}z^2 \end{pmatrix}. \tag{4.6}$$

It is easily verified that ξ_t is isometric to some $\tilde{\xi}_{t'}$. Thus our example is the same as Tjaden's one by [1, Proposition (5.2) and Theorem (5.15)].

In (4.3), we put $K_{(t)} = 1 - L_{(t)}$. Then we have

$$L_{(t)} = 2/(3t)\{t^4 + (1 - t^3)f\} \quad \text{or} \tag{4.7}$$

$$L_{(t)} = 2/(3t)\{1/(t^2 + (1 - t^3)g)\}, \tag{4.8}$$

where $f = f(t, |z|^2)$ and $g = g(t, |z|^2)$ are some positive functions of $t(>0)$ and $|z|^2$. If $1 \leqq t^3 \leqq 5/4$, then $L_{(t)} \leqq (2/3)t^3 \leqq 5/6$, by (4.7), which implies $K_{(t)} \geqq 1/6$. In the same way, if $4/5 \leqq t^3 \leqq 1$, we have $K_{(t)} \geqq 1/6$ by (4.8). Thus we have $K_{(t)} \geqq 1/6$ for each t with $4/5 \leqq t^3 \leqq 5/4$, which gives examples of the case (3) in Theorem B.

REMARK. The assumption on K is Theorem A is the best possible for the conclusion of Theorem A. Because for any $\varepsilon > 0$, we set $t^3 = 1 + (3/2)\varepsilon$ (>1). By (4.7), we have $L_{(t)} \leqq 2/3 + \varepsilon$, which implies $K_{(t)} \geqq 1/3 - \varepsilon$.

REFERENCES

[1] J. L. M. BARBOSA, On minimal immersions of S^2 into S^{2m}, Trans. Amer. Math. 210 (1975), 75-106.

[2] K. BENKO, M. KOTHE, K. D. SEMMLER AND U. SIMON, Eigenvalues of the Laplacian and curvature, Colloq. Math. 42 (1979), 19-31.

[3] O. BORŮVKA, Sur les surfaces représentées par les fonctions sphèriques de premiere espèce, J. Math. Pure et Appl. 12 (1933), 337-383.

[4] E. CALABI, Minimal immersions of surfaces in Euclidean spheres, J. Diff. Geo. 1 (1967), 111-125.

[5] S. S. CHERN, Minimal submanifolds in a Riemannian manifold, Mimeographed lecture notes, University of Kansas, 1968.

[6] S. S. CHERN, On the minimal immersions of the two-sphere in a space of constant curvature, Problems in Analysis, Princeton, 1970.

[7] S. S. CHERN, On minimal spheres in the four sphere, Studies and Essays Presented to Y. W. Chen, Taiwan, 1970, 137-150.

[8] S. S. CHERN, Holomorphic curves and minimal surfaces, Proceedings of the Carolina conference on holomorphic mappings and minimal surfaces (Mimeographed lecture notes), University of North Carolina, 1970, 1-28.

[9] S. S. CHERN, M. DO-CARMO AND S. KOBAYASHI, Minimal submanifolds of a sphere with second fundamental form of constant length, in Functional Analysis and Related Fields, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1970, 59-75.

[10] K. KENMOTSU, On compact minimal surfaces with non-negative Gaussian curvature in a space of constant curvature, I, II, Tôhoku Math. J. 25 (1973), 469-479; ibid. 27 (1975), 291-301.

[11] K. KENMOTSU, Minimal surfaces with constant curvature in 4-dimensional space forms, Proc. Amer. Math. Soc. 89 (1) (1983), 113-138.

[12] M. KOZLOWSKI AND U. SIMON, Minimal immersions of 2-manifolds into spheres, Math. Z. 186 (1984), 377-382.

[13] H. B. LAWSON, Local rigidity theorem for minimal hypersurfaces, Ann. Math. 89 (2) (1969), 187-197.

[14] T. ŌTSUKI, Minimal submanifolds with m-index 2 and generalized Veronese surfaces, J. Math. Soc. Japan 24 (1972), 89-122.

DEPARTMENT OF MATHEMATICS
FACULTY OF GENERAL EDUCATION
YAMAGATA UNIVERSITY
YAMAGATA, 990
JAPAN