Tôhoku Math. Journ.
37 (1985), 533-540.

ASYMPTOTIC ESTIMATES FOR MODULI OF EXTREMAL RINGS

GLEN D. ANDERSON AND MAVINA K. VAMANAMURTHY

(Received November 12, 1984)

Abstract. For $n \geqq 2$ and $0 < a < 1$ let $R_n(a)$ denote the extremal ring domain consisting of the unit ball in n-space minus the closed slit $[-a, a]$ along the x_1-axis. Significant lower and upper limits as n tends to ∞ are obtained for the expressions

$$\operatorname{mod} R_n(a) - n + \frac{1}{2}\log n$$

and

$$n^{1/2-n} \operatorname{mod} R_n(a)^n ,$$

where mod denotes the conformal modulus.

1. Introduction. In this paper we find asymptotic lower and upper limits as n tends to ∞ for the modulus of certain extremal rings in n-space.

For $n \geqq 2$ and $0 < a < 1$ we let $R = R_n(a)$ denote the ring in $\boldsymbol{R}^n$ consisting of the open unit ball B^n minus the closed slit $[-a, a]$ along the x_1-axis. The *conformal capacity* of R is defined to be

$$\operatorname{cap} R = \inf_u \int_R |\nabla u|^n d\omega ,$$

where $u \in C^1(R)$, $u = 0$ on the slit $[-a, a]$, and $u = 1$ on the boundary sphere S^{n-1}. The *modulus* of R is defined by

$$\operatorname{mod} R = (\sigma_{n-1}/\operatorname{cap} R)^{1/(n-1)} , \qquad \sigma_{n-1} = m_{n-1}(S^{n-1}) .$$

The rings $R_n(a)$ are *extremal* in the following sense: If R is any ring in $\boldsymbol{R}^n$ consisting of the unit ball minus a continuum whose diameter is at least $2a$, then $\operatorname{mod} R \leqq \operatorname{mod} R_n(a)$ (cf. [An1]). This extremal property of the rings $R_n(a)$ makes them useful in the study of the distortion properties of quasiconformal mappings in n-space (cf. [G], [AVV]), and we therefore wish to obtain all possible information about these rings.

The asymptotic behavior of $R_n(a)$ has been studied as a tends to 0 and to 1 and as n tends to ∞. In particular, it has been shown [An2, Theorem 2, p. 7] that for each a, $0 < a < 1$,

AMS (MOS) subject classification (1980). Primary 30C60.

$$A_n(K'/\pi K)^{1/(n-1)} \leqq \operatorname{mod} R_n(a) \leqq A_n\left(\log \frac{1+a}{1-a}\right)^{1/(1-n)}, \tag{1}$$

where K and K' are the complete elliptic integrals of the first kind defined by

$$\begin{aligned} K = K(k) &= \int_0^1 [(1-t^2)(1-k^2t^2)]^{-1/2}dt\,, \\ K' = K(k')\,, &\qquad k' = (1-k^2)^{1/2} \end{aligned} \tag{2}$$

with $k = a^2$ and where $A_n = I_n^{1/(n-1)} J_n$ with

$$I_n = \int_0^{\pi/2} \sin^{n-2} t dt\,, \qquad J_n = \int_0^{\pi/2} (\sin t)^{(2-n)/(n-1)} dt\,. \tag{3}$$

Also for fixed a, $0 < a < 1$, it is known [An2, Theorem 5, p. 18] that

$$\lim_{n\to\infty} (1/n) \operatorname{mod} R_n(a) = 1\,.$$

It is the purpose of this paper to make more precise the dependence of $R_n(a)$ upon the dimension n. Specifically, we shall prove the following theorems.

THEOREM 1. *For $n \geqq 3$ and $0 < a < 1$ let $R_n(a)$ denote the ring in $\boldsymbol{R}^n$ consisting of the unit ball B^n minus the slit $[-a, a]$ along the x_1-axis. Then*

$$\begin{aligned} &\limsup_{n\to\infty} \left\{\operatorname{mod} R_n(a) - n + \frac{1}{2}\log n\right\} \\ &\qquad \leqq -1 + \frac{1}{2}\log(2\pi) - \log\log\frac{1+a}{1-a} \end{aligned} \tag{4}$$

and

$$\liminf_{n\to\infty} \left\{\operatorname{mod} R_n(a) - n + \frac{1}{2}\log n\right\} \geqq -1 + \frac{1}{2}\log(2\pi) + \log(K'/\pi K) \tag{5}$$

where K and K' are the elliptic integrals in (2) with $k = a^2$.

THEOREM 2. *For $n \geqq 3$ and $0 < a < 1$ let $R_n(a)$ denote the ring in Theorem 1. Then*

$$\limsup_{n\to\infty} n^{1/2-n} \operatorname{mod} R_n(a)^n \leqq (\sqrt{2\pi}/e)\left(\log\frac{1+a}{1-a}\right)^{-1} \tag{6}$$

and

$$\liminf_{n\to\infty} n^{1/2-n} \operatorname{mod} R_n(a)^n \geqq (\sqrt{2\pi}/e)(K'/\pi K)\,, \tag{7}$$

where K and K' are the elliptic integrals in (2) with $k = a^2$.

We shall accomplish the proofs of these theorems by studying the asymptotic behavior of the constant A_n, appearing in (1), as n tends to ∞.

We shall follow mostly standard notation, consistent with [AV].

2. Proof of Theorem 1. The proof of Theorem 1 will depend upon a knowledge of the behavior of the constant A_n in (1) as a function of n. Since $A_n = I_n^{1/(n-1)} J_n$, where I_n and J_n are the integrals in (3), we begin by studying these.

LEMMA 1. *For* $n \geqq 3$ let $I_n = \int_0^{\pi/2} \sin^{n-2} t dt$. *Then*

$$(\pi/(2n-2))^{1/2} < I_n < (\pi/(2n-4))^{1/2} .$$

PROOF. This is Lemma 1 of [AV]. □

REMARK. In the sequel we shall frequently need to use the facts that

$$\log c < n(c^{1/n} - 1) < c - 1 \tag{8}$$

and

$$c^{1/n} - 1 < (\log c)/(n - \log c) \tag{9}$$

for $c > 1$ and $n \geqq 2$.

LEMMA 2. *For $n \geqq 3$ let* $J_n = \int_0^{\pi/2} (\sin t)^{(2-n)/(n-1)} dt$. *Then*

$$n - 1 + \log 2 - (n-1)^{-1} < J_n < n - 1 + \log 2 - \left(\frac{\pi}{2}\log 2 - 1\right)(n-1)^{-1} .$$

In particular, $n - 1 < J_n < n$ for $n \geqq 3$, and $J_n - n$ increases to $-1 + \log 2$ as n tends to ∞.

PROOF. By an elementary estimate and by (8) with $c = \csc t$ we have

$$\begin{aligned} J_n - n + 1 - \log 2 &= \int_0^{\pi/2} (1 - \cos t)((\sin t)^{(2-n)/(n-1)} - \csc t) dt \\ &= \int_0^{\pi/2} (1 - \cos t)(\sin t)^{(2-n)/(n-1)}(1 - (\sin t)^{1/(1-n)}) dt \\ &< \int_0^{\pi/2} (1 - \cos t)(1 - (\sin t)^{1/(1-n)}) dt \\ &< (n-1)^{-1} \int_0^{\pi/2} (1 - \cos t) \log \sin t dt = \left(1 - \frac{\pi}{2}\log 2\right)(n-1)^{-1} . \end{aligned}$$

Thus the upper bound is established.

Again using elementary estimates and (8) we obtain

$$J_n - n + 1 - \log 2 = \int_0^{\pi/2}(1 - \cos t)(\sin t)^{(2-n)/(n-1)}(1 - (\sin t)^{1/(1-n)})dt$$
$$> (n-1)^{-1}\int_0^{\pi/2}(1 - \cos t)(\csc t)(1 - \csc t)dt$$
$$> -(n-1)^{-1}\int_0^{\pi/2}(1 - \cos t)\csc^2 t dt = -(n-1)^{-1} ,$$

and the lower bound follows.

The fact that $J_n - n$ is increasing in n follows from the integral form of $J_n - n + 1 - \log 2$, while the limit is a consequence of the estimates we have found for J_n. □

LEMMA 3. *For* $n \geqq 3$ *let* $A_n = I_n^{1/(n-1)}J_n$, *where* I_n *and* J_n *are as in* (3). *Then*

$$\lim_n\left(A_n - n + \frac{1}{2}\log n\right) = -1 + \frac{1}{2}\log(2\pi) = -0.081\cdots .$$

In fact, for $n \geqq 3$,

$$A_n \leqq n - 1 + \log 2 - \frac{1}{2}(\pi/2)^{1/(2n-2)}\log(n-2) + (n/(2n-2))\log(\pi/2)$$

and

$$A_n \geqq (\pi/(2n-2))^{1/(2n-2)}(\log 2 - (n-1)^{-1}) - (n-2)^{-1}\log(n-1) .$$

PROOF. First,

$$A_n = I_n^{1/(n-1)}(J_n - n + 1) + (n-1)I_n^{1/(n-1)} , \tag{10}$$

where

$$\lim_n I_n^{1/(n-1)}(J_n - n + 1) = \log 2 \tag{11}$$

by Lemmas 1 and 2. Next, applying the inequality $e^{-x} > 1 - x$, $x > 0$, with $x = (2n-2)^{-1}\log(n-1)$ and using (10) along with Lemma 1, we obtain

$$A_n > I_n^{1/(n-1)}(J_n - n + 1) + (\pi/2)^{1/(2n-2)}\left(n - 1 - \frac{1}{2}\log(n-1)\right) . \tag{12}$$

Thus we have

$$A_n - n + 1 + \frac{1}{2}\log(n-1) > I_n^{1/(n-1)}(J_n - n + 1) \tag{13}$$
$$+ \left(n - 1 - \frac{1}{2}\log(n-1)\right)((\pi/2)^{1/(2n-2)} - 1) .$$

From (13) and Lemmas 1 and 2 we conclude that

$$(14)\qquad \liminf_n\left(A_n - n + \frac{1}{2}\log n\right) \geqq -1 + \frac{1}{2}\log(2\pi)\ .$$

Next, by Lemma 1 and the inequality $e^{-x} < (1+x)^{-1}$, $x > 0$, with

$$(15)\qquad x = (2n-2)^{-1}\log(n-2)$$

we may write

$$(16)\qquad (n-1)I_n^{1/(n-1)} < (n-1)(1+x)^{-1}(\pi/2)^{1/(2n-2)}\ .$$

It is easy to see that (16) implies the inequality

$$\begin{aligned}(n-1)I_n^{1/(n-1)} &- n + 1 + \frac{1}{2}\log(n-2)\\ &< (1-x)(1+x)^{-1}(n-1)((\pi/2)^{1/(2n-2)} - 1)\\ &\quad + \frac{1}{2}((\pi/2)^{1/(2n-2)} - 1)\log(n-2) + (n-1)^{-1}\log^2(n-2)\ ,\end{aligned}$$

with x as in (15). Then by employing (9) with $c = (\pi/2)^{1/2}$ and letting n tend to ∞ we have

$$(17)\qquad \limsup_n\left\{(n-1)I_n^{1/(n-1)} - n + 1 + \frac{1}{2}\log(n-2)\right\} \leqq \frac{1}{2}\log(\pi/2)\ .$$

Therefore, by (10) and (11) we have from (17),

$$(18)\qquad \limsup_n\left\{A_n - n + 1 + \frac{1}{2}\log(n-2)\right\} \leqq \frac{1}{2}\log(2\pi)\ .$$

The desired limit follows from (14) and (18).

Finally, by (1), Lemma 3, and the limit $\lim n(c^{1/n} - 1) = \log c$, $c > 1$, we have

$$\begin{aligned}\limsup_n&\left\{\operatorname{mod} R_n(a) - n + \frac{1}{2}\log n\right\}\\ &\leqq \lim\left(A_n - n + \frac{1}{2}\log n\right)\left(\log\frac{1+a}{1-a}\right)^{1/(1-n)}\\ &\quad + \lim\left(n - \frac{1}{2}\log n\right)\left(\left(\log\frac{1+a}{1-a}\right)^{1/(1-n)} - 1\right)\\ &= -1 + \frac{1}{2}\log(2\pi) - \log\log\frac{1+a}{1-a}\ .\end{aligned}$$

The lower limit in the theorem follows similarly. □

Theorem 1 has a straightforward application to the Grötzsch ring in $\boldsymbol{R}^n$.

COROLLARY 1. *For $n \geqq 3$ and $0 < b < 1$, let $R_{G,n}(b)$ denote the Grötzsch ring in $\boldsymbol{R}^n$ consisting of the unit ball B^n minus the slit $[0, b]$ along the x_1-axis. Then*

$$\limsup_n \left\{\operatorname{mod} R_{G,n}(b) - n + \frac{1}{2}\log n\right\} \leqq -1 + \frac{1}{2}\log(2\pi) - \log\log\left(\frac{1+b}{1-b}\right)^{1/2}$$

and

$$\liminf_n \left\{\operatorname{mod} R_{G,n}(b) - n + \frac{1}{2}\log n\right\} \geqq -1 + \frac{1}{2}\log(2\pi) + \log(2K'/\pi K)\ ,$$

where K and K' are the elliptic integrals in (2) with $k = b$.

PROOF. There is a conformal mapping, that is, a Möbius transformation [Ah], of $R_{G,n}(b)$ onto the ring $R_n(a)$ of Theorem 1 with $b = 2a/(1 + a^2)$. Then we have

$$(1 + b)/(1 - b) = ((1 + a)/(1 - a))^2\ ,$$

hence

$$K'(a^2)/K(a^2) = 2K'(b)/K(b)$$

by [LV, pp. 60, 61]. □

REMARK. The bounds for A_n in Lemma 3 may be combined with the estimates in (1) to obtain bounds for $\operatorname{mod} R_n(a)$ (or $\operatorname{mod} R_{G,n}(b)$) in terms of easily understood functions of n and a (or b).

3. Proof of Theorem 2. For the proof of Theorem 2 we require the asymptotic behavior of A_n^n, where A_n is the constant in (1). We achieve this by proving some lemmas.

LEMMA 4. $\lim_{x\to 0}(\Gamma(x + 1))^{1/x} = e^{-\gamma} = 0.5614\cdots$, *where Γ is Euler's Gamma function and γ is Euler's constant*

$$\gamma = \lim_m \left(\sum_{k=1}^{m} \frac{1}{k} - \log m\right) = 0.5772\cdots\ .$$

PROOF.

$$\lim_{x\to 0} \frac{1}{x}\log \Gamma(x + 1) = \Gamma'(1)/\Gamma(1) = -\gamma$$

by l'Hôpital's Rule and [R, p. 11]. □

LEMMA 5. *For $n \geqq 3$ let $J_n = \int_0^{\pi/2} (\sin t)^{(2-n)/(n-1)}dt$. Then* $\lim\,(J_n/(n-1))^{n-1} = 2$.

PROOF. By the change of variable $x = \sin^2 t$ and the fact that

$\Gamma(1/2) = \sqrt{\pi}$, we may write J_n as

$$J_n = \frac{1}{2}\int_0^1 (1-x)^{-1/2} x^{(3-2n)/(2n-2)} dx = (\sqrt{\pi}/2)\Gamma(1/(2n-2))/\Gamma(n/(2n-2))$$

[S, #607, p. 461]. By Legendre's duplication formula [R, p. 24] and the factorial property $\Gamma(z+1) = z\Gamma(z)$ we then have

$$4J_n = 2^{1/(n-1)} \frac{\Gamma^2(1/(2n-2))}{\Gamma(1/(n-1))} .$$

Thus

$$(J_n/(n-1))^{n-1} = 2\frac{(\Gamma(t+1))^{1/t}}{(\Gamma(2t+1))^{1/(2t)}}$$

with $t = 1/(2n-2)$. The limit then follows by use of Lemma 4. □

LEMMA 6. *$A_n^n \sim \sqrt{2\pi} n^{n-1/2} e^{-1}$, where $A_n = I_n^{1/(n-1)} J_n$ is the constant in* (1); *that is,* $\lim n^{1/2-n} A_n^n = \sqrt{2\pi} e^{-1}$.

PROOF. By Lemmas 1 and 5,

$$\lim (n-1) I_n^2 (J_n/(n-1))^{2(n-1)} = 2\pi .$$

Hence

$$\lim A_n^{2(n-1)} n^{3-2n} = 2\pi e^{-2} .$$

Taking square roots and using the fact that $\lim A_n/n = 1$ by Lemma 3, we arrive at the desired asymptotic formula. □

Finally, Theorem 2 follows immediately from (1) and Lemma 6.

COROLLARY 2. *For $0 < b < 1$ let $R_{G,n}(b)$ denote the Grötzsch ring consisting of the unit ball B^n minus the slit $[0, b]$ along the x_1-axis. Then*

$$\limsup_n n^{1/2-n} \operatorname{mod} R_{G,n}(b)^n \leqq (2\sqrt{2\pi}/e)\left(\log \frac{1+b}{1-b}\right)^{-1}$$

and

$$\liminf_n n^{1/2-n} \operatorname{mod} R_{G,n}(b)^n \geqq (2\sqrt{2\pi}/e)(K'/\pi K) ,$$

where K and K' are the elliptic integrals in (2) *with $k = b$.*

PROOF. This Corollary follows from Theorem 2 and the proof of Corollary 1. □

References

[Ah] L. V. AHLFORS, Möbius Transformations in Several Dimensions, University of Minnesota Lecture Notes, Minneapolis, 1981.

[An1] G. D. ANDERSON, Symmetrization and extremal rings in space, Ann. Acad. Sci. Fenn. Ser. AI 438 (1969), 1-24.

[An2] G. D. ANDERSON, Extremal rings in n-space for fixed and varying n, Ann. Acad. Sci. Fenn. Ser. AI 575 (1974), 1-21.

[AV] G. D. ANDERSON AND M. K. VAMANAMURTHY, Estimates for the asymptotic order of a Grötzsch ring constant, Tôhoku Math. J. 34 (1982), 133-139.

[AVV] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, Dimension-free quasiconformal distortion in n-space (to appear).

[G] F. W. GEHRING, Rings and quasiconformal mappings in space, Trans. Amer. Math. Soc. 103 (1962), 353-393.

[LV] O. LEHTO AND K. I. VIRTANEN, Quasiconformal Mappings in the Plane, Grundlehren der math. Wissenschaften 126, Springer-Verlag, New York-Heidelberg-Berlin, 1973.

[R] E. D. RAINVILLE, Special Functions, Macmillan, New York, 1960.

[S] S. M. SELBY, (ed.), Standard Mathematical Tables, 21st edition, Chemical Rubber Co., Cleveland, 1973.

MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824
U.S.A.

AND

UNIVERSITY OF AUCKLAND
AUCKLAND,
NEW ZEALAND