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THE FIRST EIGENVALUE OF HOMOGENEOUS MINIMAL HYPERSURFACES IN A UNIT SPHERE $S^{n+1}(1)$

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1. Introduction. It is well known as a Theorem of Takahashi [8] that a Riemannian n manifold M immersed into an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$ is minimal if and only if each coordinate function is an eigenfunction of Δ on M with eigenvalue n. This implies that the first eigenvalue of M is not greater than n.

Ogiue [12] and Yau [11] independently posed the following problem: "What kind of compact embedded minimal hypersurfaces of $S^{n+1}(1)$ do satisfy the condition that the first eigenvalue is just n?"

It is difficult in general to compute eigenvalues in practice. In [4] a little more restricted problem is considered, that is, they compute the first eigenvalues for some of the compact homogeneous minimal hypersurfaces of $S^{n+1}(1)$. There are 14 kinds of compact homogeneous minimal hypersurfaces of $S^{n+1}(1)$ (cf. Hsiang and Lawson [1]), and some of them are left untouched. We note that a homogeneous hypersurface of $S^{n+1}(1)$ has constant principal curvatures so that it is isoparametric.

The purpose of this paper is to compute the first eigenvalues for some of them and prove the following.

THEOREM. *If M is an n-dimensional compact homogeneous minimal hypersurface in a unit sphere with r distinct principal curvatures, then the first eigenvalue of the Laplacian on M is n unless $r=4$.*

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2. Laplacian of homogeneous hypersurfaces in $S^{n+1}(1)$. Hsiang and Lawson [1] proved that every compact homogeneous hypersurface in $S^{n+1}(1)$ can be obtained as follows.

Let (G, K) be a symmetric pair of compact type of rank 2 with bi-invariant Riemannian metric $\hat{g}$ induced from the Killing form B_G of the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition associated with (G, K). We regard $\mathfrak{p}$ as a Euclidean space with inner product $-B_G$. Choose a maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and denote by Σ the set of all roots of $\mathfrak{g}$. Let Σ_+ be the set of all positive elements in Σ with respect to a fixed linear order. Then it is known that $\mathfrak{k}$ and $\mathfrak{p}$ have the following orthogonal decompositions ([7]):

$$\mathfrak{k} = \mathfrak{k}_0 + \sum \mathfrak{k}_\lambda\ , \quad (\lambda \in \Sigma_+)\ ,$$
$$\mathfrak{p} = \mathfrak{a} + \sum \mathfrak{p}_\lambda\ ,$$

where

$$\mathfrak{k}_\lambda = \{X \in \mathfrak{k} \colon (\operatorname{ad} H)^2 X = -\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\}\ ,$$
$$\mathfrak{p}_\lambda = \{Y \in \mathfrak{p} \colon (\operatorname{ad} H)^2 Y = -\lambda(H)^2 Y \text{ for all } H \in \mathfrak{a}\}\ .$$

Note that $\dim \mathfrak{k}_\lambda = \dim \mathfrak{p}_\lambda$ and denote it by $m(\lambda)$.

Let $S^{n+1}(1)$ be the unit hypersphere of $\mathfrak{p}$ and let $H \in \mathfrak{a}$ be a unit regular element (i.e., $\lambda(H) \neq 0$ for all $\lambda \in \Sigma_+$). Define an embedding Φ_H: $K/L \to N(H) \subset S^{n+1}(1) \subset \mathfrak{p}$ by $\Phi_H(kL) = \operatorname{Ad}(kH)$, where L is the stabilizer of the adjoint action of K at H whose Lie algebra $\mathfrak{l}$ is $\{X \in \mathfrak{k}; \operatorname{ad}(X)(H) = 0\} = \mathfrak{k}_0$.

Because the adjoint action is an isometry and H is a unit regular element in $\mathfrak{p}$, the image $N(H)$ of Φ_H is a hypersurface of $S^{n+1}(1)$.

The homogeneous space K/L is called a regular R-space (cf. [7]). We identify the tangent spaces of $\mathfrak{p}$ with $\mathfrak{p}$ itself and give K/L the Riemannian metric g induced from the embedding Φ_H.

Since $\mathfrak{k}$ is a semisimple Lie algebra of compact type, $\mathfrak{k}$ has an $\operatorname{Ad}(L)$-invariant decomposition: $\mathfrak{k} = \mathfrak{l} + \mathfrak{m}$. Moreover g is given by

$$g_H(X, Y) = B_G([X, H], [Y, H]) \quad \text{for all} \quad X, Y \in \mathfrak{m}\ .$$

So we can take $\{X_i^\lambda/\lambda(H); \lambda \in \Sigma_+,\ i = 1, \cdots, m(\lambda),\ X_i^\lambda \in \mathfrak{k}_\lambda,\ -B_G(X_i^\lambda, X_j^\mu) = \delta_\mu^\lambda \delta_j^i\}$ as an orthogonal basis of $\mathfrak{m}$ with respect to g_H.

We would like to know what kind of H makes Φ_H an embedded minimal hypersurface. Let $H \in \mathfrak{a}$ be a unit regular element. Then the homogeneous hypersurface $N(H)$ in $S^{n+1}(1)$ is isoparametric so that its principal curvatures $\kappa_i(H)$ and their multiplicities $m(\kappa_i(H))$ are known as follows (cf. [3], [7]): Since $\mathfrak{a}$ is 2-dimensional, we can choose $Z \in \mathfrak{a}$ in such a way that $\{H, Z\}$ is an orthonormal basis for $\mathfrak{a}$. Let $\Sigma_+^* = \{\lambda \in \Sigma_+ \colon \lambda/2 \notin \Sigma_+\}$. Then we have

$$\kappa_i(H) = -\lambda_i(Z)/\lambda_i(H) \quad \text{for} \quad \lambda_i \in \Sigma_+^*\ , \tag{2.1}$$

$$m(\kappa_i(H)) = m(\lambda_i) + m(2\lambda_i)\ , \tag{2.2}$$

where $m(\lambda) = \dim \mathfrak{k}_\lambda$. Moreover the number r of the distinct principal curvatures satisfies

$$r = \sharp\Sigma_+^* \{1, 2, 3, 4, 6\}\ .$$

Therefore, for each $H \in \mathfrak{a}$ which satisfies the condition

$$\sum_{i=1}^{r} m(\kappa_i(H))\kappa_i(H) = 0 \,, \tag{2.3}$$

we get a compact homogeneous minimal hypersurface in $S^{n+1}(1)$.

For such an H we can write down the Δ of $(K/L, g)$ (cf. [4]):

$$\Delta = \sum_{\lambda \in \Sigma_+} \sum_{i=1}^{m(\lambda)} L^2_{X_i^\lambda}/\lambda(H)^2 \,, \tag{2.4}$$

where L_X denotes the Lie derivation on K with respect to the left invariant vector field X.

3. The method of computing the eigenvalues. We review the method in [4]. Let $D(K)$ be the set of all finite dimensional inequivalent unitary representations (ρ, V^ρ) of K and $D(K, L) = \{(\rho, V^\rho) \in D(K);\ V^\rho_L \neq \{0\}\}$, where $V^\rho_L = \{v \in V^\rho;\ \rho(l)v = v \text{ for all } l \in L\}$.

By the theorem of Peter and Weyl, $\{\rho_{ij}(*) = ((\rho(*)v_i, v_j));\ i = 1, \cdots, \dim V^\rho,\ j = 1, \cdots, \dim V^\rho_L,\ (\rho, V^\rho) \in D(K, L)\}$ is a complete orthogonal system of the space $C^\infty_c(K/L)$ of all complex-valued C^∞ functions on K/L, where $\{v_i;\ i = 1, \cdots, \dim V^\rho\}$ is an orthonormal basis of V^ρ and $\{v_j;\ j = 1, \cdots, \dim V^\rho_L\}$ is an orthonormal basis of V^ρ_L with respect to the L^2 norm $((\ ,\ ))$ such that the former is an extension of the latter.

Now, since the Laplacian of the Riemannian manifold $(K/L, g)$ is expressed in terms of the Lie algebra k, we have

$$\rho(\Delta) = \sum_{\lambda \in \Sigma_+} \sum_{i=1}^{m(\lambda)} \rho(X_i^\lambda)^2/\lambda(H)^2 \,, \tag{3.1}$$

$$\Delta\rho_{ij} = ((\rho(\Delta)v_j, v_i)) \,, \quad i = 1, \cdots, \dim V^\rho, \quad j = 1, \cdots, \dim V^\rho_L \,. \tag{3.2}$$

Therefore, it is enough to find all the eigenvalues of the endomorphism $\rho(\Delta)$ on V^ρ_L for all $\rho \in D(K, L)$, because these eigenvalues exhaust all the eigenvalues of Δ for $(K/L, g)$. If g is a bi-invariant metric, then $\rho(\Delta)$ is a scalar operator so that its eigenvalues are easily known. But in our case, it is very difficult in general to know all the eigenvalues of $\rho(\Delta)$, because g is not a bi-invariant metric. Therefore, in [4], $\rho(\Delta)$ is decomposed into the sum of a scalar operator and a nonnegative operator P as follows:

$$\rho(\Delta) = \sum \rho(X_i^\lambda)^2/c + P \,, \tag{3.3}$$

where $c = \{\max_\lambda \lambda(H)^2\}$.

Let Ω be the Casimir operator of K/L. Since K is a simple Lie group, bi-invariant metrics on K are unique up to scalar multiple so that there exists a number a such that $B_K = aB_G|_K$. By definition, $\Omega = \sum (X_i^\lambda)^2/B_K(X_i^\lambda, X_i^\lambda) = \sum \{(X_i^\lambda)^2/aB_G(X_i^\lambda, X_i^\lambda)\} = \sum (X_i^\lambda)^2/a$. Then (3.3) can be written as

$$\rho(\Delta) = a\rho(\Omega)/c + P . \tag{3.4}$$

By virtue of Freudenthal's formula, we know the eigenvalues $q(\Lambda_\rho)$ of $\rho(\Omega)$. So all the eigenvalues of $\rho(\Delta)$ are not smaller than $aq(\Lambda_\rho)/c$. If $q(\Lambda_\rho)$ is not smaller than nc/a, we can conclude that the first eigenvalue of $(K/L, g)$ is just n. Therefore we study the eigenvalues of $\rho(\Delta)$ smaller than nc/a.

4. The computation. Now we realize the Lie algebras $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{l}$ and $\mathfrak{a}$, and compute the first eigenvalue of Δ concretely. Hereafter, we use the notation of [5, pp. 21–37].

(i) The case $r = 1$ and 2.

It is well known that the first eigenvalue of the great n-sphere and the Clifford n-torus is just n.

(ii) The case $r = 3$.

Let $\boldsymbol{F}$ be a division algebra over $\boldsymbol{R}$, i.e., $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$, the real quaternion algebra $\boldsymbol{H}$ or the real Cayley algebra $\boldsymbol{K}$. If we put $H_3(\boldsymbol{F}) = \{u;$ u is a 3×3 matrix with coefficients in $\boldsymbol{F}$, which satisfies $u^* = u\}$, then the subspaces $\mathfrak{p}$ and $\mathfrak{k}$ of $\mathrm{gl}(H_3(\boldsymbol{F}))$ are realized as follows:

Let $R\colon H_3(\boldsymbol{F}) \to \mathrm{gl}(H_3(\boldsymbol{F}))$ and $D\colon SH_3(\boldsymbol{F}) \to \mathrm{gl}(H_3(\boldsymbol{F}))$ be injective linear maps defined respectively by $R(u)v = (uv + vu)/2$ and $D(u)v = (uv - vu)/2$, where $SH_3(\boldsymbol{F}) = \{u \in H_3(\boldsymbol{F});\ T(u) = 0\}$ and

$$T(u) = \begin{cases} \mathrm{tr}(u) + \mathrm{tr}(\bar{u}) & \text{if } \boldsymbol{F} = \boldsymbol{H}, \\ \mathrm{tr}(u) & \text{otherwise}. \end{cases}$$

Then we have

$$\mathfrak{p} = R(\{u \in H_3(\boldsymbol{F});\ \mathrm{tr}(u) = 0\}) ,$$
$$\mathfrak{k} = D(SH_3(\boldsymbol{F})) ,$$

so that $\dim \mathfrak{p} = \dim \mathfrak{k} = 3 \dim \boldsymbol{F} + 2$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Then $\mathfrak{g}$ is a simple Lie algebra of compact type, and $\mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition. Furthermore these Lie algebras exhaust Lie algebras of rank 2 with $r = 3$. The corresponding Lie groups are as follows:

TABLE 1.

$\boldsymbol{F}$	K	L	G	$\dim(K/L)$
$\boldsymbol{R}$	$SO(3) = B_1$	$\boldsymbol{Z}_2 + \boldsymbol{Z}_2$	$SU(3)$	3
$\boldsymbol{C}$	$SU(3) = A_2$	T^2	$SU(3)\times SU(3)$	6
$\boldsymbol{H}$	$\mathrm{Sp}(3) = C_3$	$\mathrm{Sp}(1)^3$	$SU(6)$	12
$\boldsymbol{K}$	F_4	$\mathrm{Spin}(8)$	E_6	24

We put

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and choose $\mathfrak{a} = \{\sum \xi_i e_i; \sum \xi_i = 0\}$ as a maximal Abelian subalgebra of $\mathfrak{p}$. Then Σ_+^* is given by

$$\Sigma_+^* = \{(\xi_2 - \xi_1)/2, (\xi_3 - \xi_1)/2, (\xi_3 - \xi_2)/2\} .$$

so that

(4.1) $$\lambda_1 = (\xi_2 - \xi_1)/2 , \quad \lambda_2 = (\xi_3 - \xi_1)/2 \quad \text{and} \quad \lambda_3 = (\xi_3 - \xi_2)/2 ,$$

and the multiplicities of the principal curvatures are $m_1 = m_2 = m_3 = \dim \boldsymbol{F}$.

For any $H = (\xi_1, \xi_2, \xi_3) \in \mathfrak{a}$, which satisfies $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1/3 \dim \boldsymbol{F}$, we choose $Z = ((\xi_2 - \xi_3)/\sqrt{3}, (\xi_3 - \xi_1)/\sqrt{3}, (\xi_1 - \xi_2)/\sqrt{3})$. Then we get

$$\begin{aligned} B(H, H) &= \mathrm{tr}(\mathrm{ad}(H), \mathrm{ad}(H)) \\ &= 2 \dim \boldsymbol{F}\{(\xi_1 - \xi_2)^2/4 + (\xi_2 - \xi_3)^2/4 + (\xi_3 - \xi_1)^2/4\} \\ &= 3 \dim \boldsymbol{F}(\xi_1^2 + \xi_2^2 + \xi_3^2) = 1 = B(Z, Z) , \\ B(H, Z) &= 0 , \end{aligned}$$

and hence $\{H, Z\}$ is an orthonormal basis of $\mathfrak{a}$. From (2.1) and (4.1) we get

$$\begin{aligned} \kappa_1(H) &= (\xi_1 + \xi_2 - 2\xi_3)/\sqrt{3}(\xi_2 - \xi_1) , \\ \kappa_2(H) &= (2\xi_2 - \xi_1 - \xi_3)/\sqrt{3}(\xi_3 - \xi_1) , \\ \kappa_3(H) &= (\xi_3 + \xi_2 - 2\xi_1)/\sqrt{3}(\xi_3 - \xi_2) . \end{aligned}$$

We see that an $H \in \mathfrak{a}$ which makes K/L minimal is $(-(3 \dim \boldsymbol{F})^{-1/2}, 0, (3 \dim \boldsymbol{F})^{-1/2})$. Then we have $\kappa_1(H) = -\sqrt{3}$, $\kappa_2(H) = 0$, $\kappa_3(H) = \sqrt{3}$ so that $\sum_{i=1}^3 m_i(\kappa_i(H))\kappa_i(H) = 0$. Therefore we get a homogeneous minimal hypersurface with $r = 3$. With respect to this H, it follows from (4.1) that

(4.2) $$\lambda_1(H)^2 = 1/12 \dim \boldsymbol{F} , \quad \lambda_2(H)^2 = 1/3 \dim \boldsymbol{F} , \quad \lambda_3(H)^2 = 1/12 \dim \boldsymbol{F} .$$

Hence we get

(4.3) $$c = 1/3 \dim \boldsymbol{F} , \quad \text{and} \quad a = B_K/B_G|_K .$$

(ii)-1 The cases of B_1 and A_2 were dealt with in [4].

(ii)-2 The case of C_3.

In this case, $\boldsymbol{F} = \boldsymbol{H}$ and $\mathfrak{k} = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{g}; A, B, C, D \text{ are } 3\times 3 \text{ matrices with coefficients in } \boldsymbol{R} \text{ and } A + D = 0, B + C = 0\right\}$ so that $B_G/12 = \mathrm{tr}_G =$

$\mathrm{tr}_K = B_K/8$. Thus, we get $a = 2/3$. Moreover, from (4, 3), we have $c = 1/12$.

Now, we compute $q(\Lambda_\rho)$ concretely, and compare them with $nc/a = 24/16$. Each $\rho \in D(K)$ corresponds to $(m_1, m_2, \cdots) \in \boldsymbol{Z}^{\mathrm{rank}\,K}$ injectively and for each $(m_1, m_2, \cdots)$, $p_i = p_i(m_1, \cdots)$ are defined. Then $q(\Lambda_\rho)$ can be given in terms of $\{m_i, p_j\}$ as

$$(4.4) \qquad q(\Lambda_\rho) = (m_1p_1 + m_2p_2 + 2m_3p_3 + 2p_1 + 2p_2 + 4p_3)/16 .$$

For details, see [9]. As we need not compute the eigenvalue bigger than 24/16, we mark * in the fourth column in Table 2 for ρ whose $q(\Lambda_\rho)$ is bigger than 24/16. We mark * in the fifth column for ρ if $\rho \notin D(K, L)$. Therefore we must compute the eigenvalues for ρ which is not marked *.

TABLE 2.

m_1	m_2	m_3	p_1	p_2	p_3	$18q(\Lambda)$	$\leqq 24$?	$D(K, L)$?
1	0	0	1	1	1/2	7		*
2	0	0	2	2	1	16		*
3	0	0	3	3	3/2	27	*	
0	1	0	1	2	1	12	adjoint action	
0	2	0	2	4	2	28	*	
0	0	1	1	2	3/2	15		*
0	0	2	2	4	3	36	*	
1	1	0	2	3	3/2	21		*
2	1	0	3	4	2	32	*	
1	0	1	2	3	2	24	*	
0	1	1	2	4	5/2	31	*	

(ii)-3 The case of F_4.
As in (ii)-2, we get

$$q(\Lambda_\rho) = (m_1p_1 + m_2p_2 + m_3p_3/2 + m_4p_4/2 + 2p_1 + 2p_2 + p_3 + p_4)/18 \quad \text{(cf. [9])},$$

$$a = 3/4 \quad \text{(cf. [2])}, \quad c = 1/24 \quad \text{so that} \quad nc/a = 24/18 .$$

Compare $q(\Lambda_\rho)$ with 24/18 in Table 3.

TABLE 3.

m_1	m_2	m_3	m_4	p_1	p_2	p_3	p_4	$18q(\Lambda)$	$\leqq 24$?	$D(K, L)$?
1	0	0	0	2	3	4	2	18		*
2	0	0	0	4	6	8	2	38	*	
0	1	0	0	3	6	8	4	36	*	
0	0	1	0	2	4	6	3	24	*	
0	0	0	1	1	2	3	2	12	adjoint action	
0	0	0	2	2	4	6	4	26	*	
1	0	0	1	3	5	7	4	32	*	

In the cases (ii)-2 and (ii)-3 we see that $q(\Lambda_\rho)$ is not smaller than nc/a for all ρ in $D(K, L)$ except for the adjoint action. In the case of the adjoint action, we have

$$\rho(\Delta) = \sum_{i=1}^{m(\alpha)} \mathrm{Ad}(X_i^\alpha)^2/\alpha(H)^2 + \sum_{i=1}^{m(\beta)} \mathrm{Ad}(X_i^\beta)^2/\beta(H)^2 + \sum_{i=1}^{m(\gamma)} \mathrm{Ad}(X_i^\gamma)^2/\gamma(H) ,$$

where $\Sigma_+^* = \{\alpha, \beta, \gamma\}$ and $m(\alpha) = m(\beta) = m(\gamma) = \dim(K/L)/3$. Clearly we know $V^\rho = \mathfrak{p}$ and $V_L^\rho = \mathfrak{a}$, and we get

$$\mathrm{Ad}(X_\lambda)^2/\lambda(H)^2 H = \lambda(H)\mathrm{Ad}(X_\lambda)Y_\lambda/\lambda(H)^2 = -H_\lambda/\lambda(H) ,$$
$$\mathrm{Ad}(X_\lambda)^2/\lambda(H)^2 Z = \lambda(Z)\mathrm{Ad}(X_\lambda)Y_\lambda/\lambda(H)^2 = -\lambda(Z)H_\lambda/\lambda(H)^2 .$$

But it follows from the definition that $H_\lambda = \lambda(H)H + \lambda(Z)Z$, so that we get

$$\begin{aligned} \mathrm{Ad}(\Delta)H &= -\dim(K/L)\{(H - \kappa_\alpha Z) + (H - \kappa_\beta Z) + (H - \kappa_\gamma Z)\}/3 \\ &= -\dim(K/L)\{H + (\kappa_\alpha + \kappa_\beta + \kappa_\gamma)Z/3\} = -\dim(K/L)H , \\ \mathrm{Ad}(\Delta)Z &= -\dim(K/L)\{-(\kappa_\alpha + \kappa_\beta + \kappa_\gamma)H + (\kappa_\alpha^2 + \kappa_\beta^2 + \kappa_\gamma^2)Z\}/3 \\ &= -2\dim(K/L) . \end{aligned}$$

Thus we get $q(\Lambda_\rho) = \{\dim(K/L), 2\dim(K/L)\}$. Therefore we conclude in both cases that the first eigenvalue is just n.

(iii) The case $r = 6$.

The following two Lie algebras exhaust simple Lie algebras of compact type of rank 2 with $r = 6$.

(iii)-1 The case $\mathfrak{k} = \mathfrak{g}_2$ and $\mathfrak{p} = \sqrt{-1}\mathfrak{g}_2$.

The associated symmetric pair of Lie groups is $(G_2 \times G_2, G_2)$, which was dealt with in [4].

(iii)-2 The case $\mathfrak{g} = \mathfrak{g}_2$, $\mathfrak{k} = \mathfrak{su}(2) + \mathfrak{su}(2)$ and $\mathfrak{l} = 0$.

It is known that $D(SU(2)) = \{(\rho_m, V^m)$; m is any nonnegative integer$\}$, where V^m is the vector space of all homogeneous polynomials of degree

m in two complex variables z_1, z_2 and $\rho_m(g)f(z)=f(gz)$, for all $f\in V^m$ (cf. [6]).

It is easily seen that

$$X_1=\begin{pmatrix}0 & i/2\\ i/2 & 0\end{pmatrix},\quad X_2=\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix},\quad X_3=\begin{pmatrix}i/2 & 0\\ 0 & i/2\end{pmatrix}$$

form a basis of $\mathfrak{su}(2)$ such that $[X_i, X_j]=X_k$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. Then we easily get the differential representation of ρ^m:

$$\begin{aligned} d\rho^m(X_1)v_k &= i\{kv_{k-1}+(n-k)v_{k+1}\}/2\ ,\\ d\rho^m(X_2)v_k &= \{kv_{k-1}-(n-k)v_{k+1}\}/2\ ,\\ d\rho^m(X_3)v_k &= i(2k-n)v_k/2\ ,\end{aligned}$$

where $\{v_k=z_1^k z_2^{m-k}\}$ is an orthogonal basis of V^m. Now we define an inclusion

$$\mathfrak{su}(2)+\mathfrak{su}(2)\subset\mathfrak{g}_2\ ,\quad \{X_i\}+\{X_i\}\mapsto\{E_i\}+\{F_i\}\ ,$$

by

$$\begin{aligned} E_1 &= G_{12}+G_{47}/2-G_{56}/2\ , & F_1 &= -(G_{47}+G_{56})/2\ ,\\ E_2 &= -G_{13}-(G_{46}+G_{52})/2\ , & F_2 &= (G_{46}-G_{57})/2\ ,\\ E_3 &= G_{23}+(G_{45}-G_{67})/2\ , & F_3 &= -(G_{45}+G_{67})/2\ ,\end{aligned}$$

where $G_{ij}=E_{ij}-E_{ji}$ and E_{ij} is a standard basis of 7×7 matrix with coefficients in $\boldsymbol{R}$. Then we have

$$[E_i, E_j]=E_k\ ,\quad [F_i, F_j]=F_k\ ,\quad [E_s, F_t]=0\ ,$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and $s, t=1, 2, 3$. Moreover a maximal Abelian subspace $\mathfrak{a}$ is given by $\mathfrak{a}=\{\xi_1G_{72}-\xi_2G_{14}+\xi_3G_{67};\ \xi_1+\xi_2+\xi_3=0\}$, and H, which makes $N(H)$ a minimal hypersurface in $S^{n+1}(1)$, is

$$H=\{(\sqrt{3}-1)G_{72}-2G_{14}+(\sqrt{3}+1)G_{67}\}/2\sqrt{6}\in\mathfrak{a}\ .$$

All the root vectors with respect to the above $\mathfrak{a}$ are

$$\{E_1+3F_1,\ E_1-F_1,\ E_2+3F_2,\ E_2-F_2,\ E_3-3F_3,\ E_3+F_3\}\ .$$

Thus we get

$$\begin{aligned} \mathrm{ad}(H)^2(E_1+3F_1) &= -(2+\sqrt{3})(E_1+3F_1)/12\ ,\\ \mathrm{ad}(H)^2(E_1-F_1) &= -(6-3\sqrt{3})(E_1-F_1)/12\ ,\\ \mathrm{ad}(H)^2(E_2+3F_2) &= -(2-\sqrt{3})(E_2+3F_2)/12\ ,\\ \mathrm{ad}(H)^2(E_2-F_2) &= -(6+3\sqrt{3})(E_2-F_2)/12\ ,\end{aligned}$$

$$\operatorname{ad}(H)^2(E_3 - 3F_3) = -(E_3 - 3F_3)/6 ,$$
$$\operatorname{ad}(H)^2(E_3 + F_3) = -(E_3 + F_3)/2 .$$

Therefore from (2.4), we have

$$\Delta = -\{E_3^2 - 2E_3F_3 + 5F_3^2 + 4(E_1^2 + 2E_1F_1 + 5F_1^2) + 4(E_2^2 + 2E_2F_2 + 5F_2^2) + 8\sqrt{3}(E_2F_2 - E_1F_1 + F_2^2 - F_1^2)\} .$$

If we note that $D(SU(2)\otimes SU(2)) = \{(\rho^n, V^n)\otimes(\rho^m, V^m)\}$, then after a long computation, (3.2) can be written as

$$\begin{aligned} -d(\rho^n\otimes\rho^m)v_k\otimes u_l &= \{(k - n/2 - l + m/2)^2 + (2l - m)^2 + 4(nk - k^2) + 20(lm - l^2) + 10m \\ &\quad + 2n\}v_k\otimes u_l + 4\{k(m - l)v_{k-1}\otimes u_{l+1} + l(n - k)v_{k+1}\otimes u_{l-1}\} \\ &\quad - 4\sqrt{3}\{klv_{k-1}\otimes u_{l-1} + (n - k)(m - l)v_{k+1}\otimes u_{l+1} + l(l - 1)v_K\otimes u_{l-2} \\ &\quad + (m - l)(m - l - 1)v_k\otimes u_{l+2}\} , \end{aligned}$$

where $V^n = \{v_k\}$ and $V^m = \{u_l\}$. The stabilizer L is given by

$$L = \left\{\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \pm\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\otimes\pm\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix},\right.$$
$$\left.\pm\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\otimes\pm\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\ \pm\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\otimes\pm\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\} .$$

So by an easy computation we see that $V_L^\rho \neq \{0\}$ if and only if $n + m$ is even. Moreover, we see that if $n + m \equiv 0 \pmod 4$, then $V_L^\rho = \{v_k\otimes u_l + v_{n-k}\otimes u_{m-l};\ k + l \text{ is even}\}$ and if $n + m \equiv 2 \pmod 4$, then $V_L^\rho = \{v_k\otimes u_l - v_{n-k}\otimes u_{m-l};\ k + l \text{ is odd}\}$. $q(\Lambda_{n,m})$ is not smaller than 6 for each pair (n, m), and hence we see that the first eigenvalue is just n.

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