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THE SIGNATURE WITH LOCAL COEFFICIENTS OF LOCALLY SYMMETRIC SPACES

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Abstract. We obtain explicit formulae for the (L^2-) signature with local coefficients of certain locally symmetric spaces and then apply them to derive non-vanishing criteria for the middle dimensional cohomology.

Introduction. Let G be a linear connected semisimple Lie group, Γ a discrete subgroup of G and F a finite dimensional complex irreducible G-module. It is well-known (see e.g. [4: VII, §2]) that the Eilenberg-MacLane cohomology space $H^{\cdot}(\Gamma; F)$ is isomorphic to the relative Lie algebra cohomology space $H^{\cdot}(\mathfrak{g}, K; C^{\infty}(\Gamma\backslash G)\otimes F)$, where $\mathfrak{g}$ is the Lie algebra of G and K a maximal compact subgroup of G. If now Γ is co-compact, as we assume for the time being, one also knows that $L^2(\Gamma\backslash G)$ decomposes as a Hilbert direct sum of irreducible subspaces with finite multiplicities,

$$L^2(\Gamma\backslash G) = \sum_{\pi\in\hat{G}}^{\oplus} N_\Gamma(\pi)\mathscr{H}_\pi ,$$

where $\mathscr{H}_\pi$ is the Hilbert space corresponding to $\pi\in\hat{G}$ and $N_\Gamma(\pi)$ its multiplicity in $L^2(\Gamma\backslash G)$. The above isomorphism then becomes

$$H^{\cdot}(\Gamma; F) \cong \sum_{\pi\in\hat{G}}^{\oplus} N_\Gamma(\pi)H^{\cdot}(\mathfrak{g}, K; \mathscr{H}_\pi\otimes F) ,$$

with only finitely many summands giving a non-zero contribution. Specifically (see [4: I, 5.3]), one has

$$H^{\cdot}(\Gamma; F) \cong \sum_{\pi\in\hat{G}_F}^{\oplus} N_\Gamma(\pi)H^{\cdot}(\mathfrak{g}, K; \mathscr{H}_\pi\otimes F) \tag{0.1}$$

where $\hat{G}_F$ denotes the set of those $\pi\in\hat{G}$ whose infinitesimal character coincides with that of F^*.

Let us now assume that G possesses a non-empty discrete series $\hat{G}_d$. Then $\hat{G}_{d,F} = \hat{G}_d\cap\hat{G}_F$ is also non-empty. Furthermore, if $\pi\in\hat{G}_{d,F}$ then (see [4: II, 5.3]):

$$H^i(\mathfrak{g}, K; \mathscr{H}_\pi\otimes F) \cong \begin{cases} 0, & \text{if } i\neq m \\ \boldsymbol{C}, & \text{if } i = m \end{cases} \tag{0.2}$$

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where $2m = \dim G/K$, necessarily even since $\operatorname{rank} G = \operatorname{rank} K$. From (0.1) and (0.2) it immediately follows that

$$(0.3) \qquad \dim H^m(\Gamma; F) \geqq \textstyle\sum^{\oplus}_{\pi \in \hat{G}_{d,F}} N_\Gamma(\pi) .$$

This inequality can be used to convert the known facts about the multiplicities of discrete series representations into non-vanishing criteria for $H^m(\Gamma; F)$. For instance, the limit formula for multiplicity of DeGeorge and Wallach [6] implies that, given F, there exists a normal subgroup of finite index $\Gamma' \subset \Gamma$ such that $H^m(\Gamma'; F) \neq 0$. On the other hand, if F is a fixed simple G-module whose highest weight is "sufficiently" regular (such that, for example, the regularity condition required in [11: Thm. 3.3] is satisfied by at least one of the representations $\pi \in \hat{G}_{d,F}$), then it can be shown that $H^m(\Gamma; F) \neq 0$ for any torsion-free discrete co-compact subgroup Γ of G.

In this paper we show that $H^m(\Gamma; F)$ is always non-zero (i.e. for any Γ as above and any finite dimensional G module F), provided that G is such that the compact dual Y of $X = G/K$ has odd Euler characteristic. Our approach is to use the existence of a G-invariant non-degenerate hermitian metric on F to construct a flat hermitian metric on the associated flat vector bundle $\boldsymbol{F}$ over $X_\Gamma = \Gamma \backslash G/K$, and then to obtain an explicit formula for the signature with coefficients in the corresponding local system $\mathscr{F}$. Since $H^{\cdot}(\Gamma; F) \cong H^{\cdot}(X_\Gamma; \mathscr{F})$, the non-vanishing of this signature implies, of course, the non-vanishing of $H^m(\Gamma; F)$.

After replacing the ordinary cohomology by L^2-cohomology, it makes sense to try to extend this technique to discrete subgroups of finite covolume. However, due to the lack, in general, of a reasonably explicit formula for the L^2-signature with local coefficients, the method can only be applied to real-rank one groups. Indeed, in that case such a formula can be derived from the L^2-index theorems in [2]. We shall apply it to the case when $G = SU(2n, 1)$ and $\Gamma \subset G$ is arithmetic. One noteworthy feature here is that the L^2-cohomology space $H^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})$ injects into $H^{\cdot}(\Gamma; F)$ [12: (6.11)] and thus the non-vanishing of the L^2-signature implies, again, that $H^m(\Gamma; F) \neq 0$.

1. Preliminary results. As above, G denotes a connected linear semisimple Lie group, which possesses a Cartan subgroup H, contained in a maximal compact subgroup K of G. For simplicity, we also assume that G is a real form of a simply connected complex Lie group G_C. By (σ, F) we denote a fixed finite dimensional irreducible complex representation of G. The following elementary result will be needed below.

(1.1) LEMMA. *There exists a G-invariant, non-degenerate, hermitian form Q_F on F, which is unique up to multiplication by a real number.*

PROOF. We first note that the dual representation σ^* and the conjugate representation $\bar{\sigma}$ of σ are equivalent. Indeed, since H is compact, $\bar{\sigma}|H \cong \sigma^*|H$. This, in turn, implies that their extensions to H_C are equivalent. Since $G'_C = \{$the set of all regular elements in $G_C\}$ can be represented as a union of inner conjugates of $H'_C = H_C \cap G'_C$, it follows that the extensions of $\bar{\sigma}$ and σ^* to G_C have the same character, therefore $\bar{\sigma} \cong \sigma^*$.

So $\mathrm{Hom}_G(\bar{F}, F^*) = \boldsymbol{C}\tau$, with $\tau\colon \bar{F} \cong F^*$. Let $T\colon F\times F \to \boldsymbol{C}$ be the corresponding sesquilinear form, i.e.,

$$T(u, v) = (\tau\bar{v})(u)\ , \qquad u, v \in F\ .$$

Then $T = S + A$, where

$$\begin{aligned} 2S(u, v) &= T(u, v) + \overline{T(v, u)} \\ 2A(u, v) &= T(u, v) - \overline{T(v, u)} \end{aligned}$$

and we can take $Q_F = S$ if $S \neq 0$, or $Q_F = iA$ otherwise. Both the non-degeneracy and the uniqueness (up to multiplication by a real number) of Q_F follow easily from the fact that $\dim \mathrm{Hom}_G(\bar{F}, F^*) = 1$. q.e.d.

We denote by $\mathfrak{p}$ the orthogonal (with respect to the Cartan-Killing form) of the Lie algebra $\mathfrak{k}$ of K in the Lie algebra $\mathfrak{g}$ of G.

(1.2) LEMMA. *There exists a splitting $F = F^+ \oplus F^-$ such that:*

(i) *$F^\pm$ are K-invariant and, for $X \in \mathfrak{p}$, $\sigma(X)$ maps $F^\pm$ to $F^\mp$;*

(ii) *The restriction of Q_F to F^+ (resp. F^-) is positive (resp. negative) definite, F^+ and F^- are orthogonal with respect to Q_F, and letting*

$$\langle u, v\rangle = Q_F(u^+, v^+) - Q_F(u^-, v^-)\ ,$$

where $u = u^+ + u^-$, $v = v^+ + v^-$, with $u^\pm, v^\pm \in F^\pm$ respectively, one obtains a positive definite inner product which is admissible, i.e., it is invariant under K and $\sigma(X)$ is self-adjoint with respect to $\langle\ ,\ \rangle$ for all $X \in \mathfrak{p}$.

PROOF. Let U be the analytic subgroup of G_C corresponding to the (real) Lie subalgebra $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ of $\mathfrak{g}_C$. Then U is compact and so there exists a U-invariant inner product P on F (which is automatically G-admissible). Define $\tau\colon F \to F$ by

$$P(\tau u, v) = Q_F(u, v)\ , \qquad u, v \in F\ .$$

Then τ is invertible and self-adjoint with respect to P. Therefore, all its eigenvalues are real and non-zero. Also, τ commutes with $\sigma(k)$ for

$k \in K$ and anti-commutes with $\sigma(X)$ for $X \in \mathfrak{p}$. Thus, $\tau^2 = \tau^* \tau \in \mathrm{Hom}_G(F, F)$, hence, τ^2 is a scalar operator. It follows that the only eigenvalues of τ are $\pm\lambda$, for some $\lambda > 0$. Taking $F^+ = F_\lambda$, $F^- = F_{-\lambda}$ one obtains the desired splitting. q.e.d.

Let $\mathfrak{h}_C$ be the Cartan subalgebra of $\mathfrak{g}_C$ corresponding to the compact Cartan subgroup $H \subset K$. Let Φ denote the set of all non-zero roots of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$, $\Phi_k \subset \Phi$ the root system of $\mathfrak{k}_C$ with respect to $\mathfrak{h}_C$ and $\Phi_n = \Phi - \Phi_k$ the set of non-compact roots of $\mathfrak{g}_C$. The Weyl groups of Φ and Φ_k will be denoted by W and W_k, respectively. We fix once and for all a positive system of roots $\Psi \subset \Phi$, set $\Psi_k = \Psi \cap \Phi_k$, $\Psi_n = \Psi \cap \Phi_n$, and then denote

$$\rho = \frac{1}{2} \sum_{\alpha \in \Psi} \alpha \ , \quad \rho_k = \frac{1}{2} \sum_{\alpha \in \Psi_k} \alpha \ , \quad \rho_n = \frac{1}{2} \sum_{\alpha \in \Psi_n} \alpha \ .$$

In the dual $\mathfrak{h}_C^*$ of $\mathfrak{h}_C$ we introduce the inner product $\langle \ , \ \rangle$ induced by the Cartan-Killing form of $\mathfrak{g}_C$, in the usual way.

Let $S = S^+ \oplus S^-$ be the spin representation of $\mathrm{Spin}(\mathfrak{p})$. When restricted to $\mathrm{Ad}\, K \subset \mathrm{Spin}(\mathfrak{p})$ and regarded as $\mathfrak{k}_C$-modules, the half-spin representations decompose as follows:

$$S^\pm = \sum\nolimits^{\oplus}_{u \in W^1, \det(u) = \pm 1} V_{u\rho - \rho_k} \ ,$$

where

$$W^1 = \{u \in W;\ u\Psi \supset \Psi_k\}$$

and V_λ denotes the irreducible $\mathfrak{k}_C$-module with highest weight λ. We now define a Dirac operator $D_F \colon F \otimes S \to F \otimes S$ by the formula:

$$D_F = \sum\nolimits_{i=1}^{2m} \sigma(X_i) \otimes c(X_i) \ ,$$

where $\{X_1, \cdots, X_{2m}\}$ is an orthonormal basis of $\mathfrak{p}$ and $c(X)$ denotes the Clifford multiplication by $X \in \mathfrak{p}$. One can easily check that:

(1.3. i) D_F is independent of the choice of an orthonormal basis for $\mathfrak{p}_C$;

(1.3.ii) D_F maps $F^\pm \otimes S$ to $F^\mp \otimes S$ and also $F \otimes S^\pm$ to $F \otimes S^\mp$;

(1.3.iii) $D_F^* = -D_F$, the adjoint being taken with respect to the tensor product of the G-admissible inner product on F and the standard inner product on S.

One can further check, as in [10] or [4: II, § 6], that:

(1.3.iv) $D_F \colon F \otimes S \to F \otimes S$ is a K-intertwining operator;

(1.3. v) $D_F^2 = -(|\Lambda|^2 - |\rho_k|^2)I + (\sigma \otimes s)(\Omega_k)$, where $\Lambda - \rho$ is the highest weight of the simple $\mathfrak{g}_C$-module F, s is the restriction of the spin representation to $\mathfrak{k}_C$, and Ω_k is the Casimir element associated to $\mathfrak{k}_C$.

We shall also need the following facts (cf. [10: Lemma 8.1]) concerning the decomposition of $F \otimes S$ into irreducible $\mathfrak{k}_C$-modules. If

$$F \otimes S = \sum_\nu V_\nu$$

is this decomposition, then we have:

(1.4. i) $|\Lambda| \geqq |\nu + \rho_k|$ with the equality holding iff $\nu = u\Lambda - \rho_k$ for some $u \in W^1$;

(1.4. ii) The mapping $u \mapsto u\Lambda - \rho_k$ is a bijection of W^1 onto the set of all highest weights ν appearing in the above decomposition and satisfying the equality in (1.4.i);

(1.4.iii) For each $u \in W^1$, $V_{u\Lambda-\rho_k}$ occurs in $F \otimes S$ with multiplicity one, and a weight vector for $u\Lambda - \rho_k$ is the tensor product of a weight vector in F of weight $u(\Lambda - \rho)$ and a weight vector in S of weight $u\rho - \rho_k$.

With these preparations we are now ready to establish:

(1.5) PROPOSITION. *As a virtual representation of* $\mathfrak{k}_C$,

$$F^+ \otimes S - F^- \otimes S = \sum_{u \in W^1} \varepsilon_F(u) V_{u\Lambda-\rho_k} ,$$

where $\varepsilon_F(u) = 1$ *or* -1, *according to whether* $V_{u(\Lambda-\rho)}$, *which occurs in* F *with multiplicity one, is a subrepresentation of* F^+ *or of* F^-.

PROOF. We first show that

$$\operatorname{Ker} D_F = \sum_{u \in W^1} V_{u\Lambda-\rho_k} .$$

Indeed, by (1.3.iii) and (1.3.v),

$$D_F^* D_F | V_\nu = [(|\Lambda|^2 - |\rho_k|^2) - (|\nu + \rho_k|^2 - |\rho_k|^2)]I = (|\Lambda|^2 - |\nu + \rho_k|^2)I$$

and so V_ν occurs in $\operatorname{Ker} D_F = \operatorname{Ker} D_F^* D_F$ iff

$$|\nu + \rho_k| = |\Lambda| ,$$

which, by (1.4.ii), happens precisely when ν is of the form $u\Lambda - \rho_k$, with $u \in W^1$.

Let $D_{F^\pm} = D_F | F^\pm \otimes S: F^\pm \otimes S \to F^\mp \otimes S$. Then clearly

$$F^+ \otimes S - F^- \otimes S = \operatorname{Ker} D_{F^+} - \operatorname{Ker} D_{F^-}$$

and, in view of (1.4.iii), the statement follows now easily from the above decomposition of $\operatorname{Ker} D_F$. q.e.d.

2. Signature with local coefficients and L^2-cohomology. In this section G and K are as in §1, $X = G/K$, Γ is a discrete subgroup of G such that $\Gamma\backslash G$ has finite invariant volume and $X_\Gamma = \Gamma\backslash X$. Again, (σ, F) is a finite dimensional irreducible representation of G (endowed with an admissible inner product) and we let $\boldsymbol{F}$ denote the associated flat vector bundle on X_Γ and $\mathscr{F}$ its sheaf of germs of horizontal sections.

We assume X_Γ equipped with a riemannian metric coming from a G-invariant riemannian metric on X. Together with the (not necessarily flat) metric on $\boldsymbol{F}$ determined by the admissible inner product on F, it gives rise, in a natural way, to an L^2 structure on $\mathscr{F}$-valued differential forms. Let $\mathscr{A}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})$ denote the complex of smooth $\mathscr{F}$-valued forms ω such that ω and $d\omega$ are square integrable. Its cohomology,

$$H^{\cdot}_{(2)}(X_\Gamma; \mathscr{F}) = H^{\cdot}(\mathscr{A}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F}))$$

is, by definition, the L^2-cohomology of X_Γ with coefficients in $\mathscr{F}$. Since $\operatorname{rank} G = \operatorname{rank} K$, by a result of Borel and Casselman [3: Theorem A], $H^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})$ is finite dimensional. In particular, it coincides with the space $\mathscr{H}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})$ of square integrable $\mathscr{F}$-valued harmonic forms.

Now the flat hermitian metric on $\boldsymbol{F}$, given by the form Q_F of (1.1), and the riemannian metric on X_Γ determine a hermitian pairing on $\mathscr{A}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})$, which in turn induces a hermitian form $Q_F^{(m)}$ on the (finite dimensional) middle cohomology space $H^m_{(2)}(X_\Gamma; \mathscr{F})$. Using the fact that $H^m_{(2)}(X_\Gamma; \mathscr{F}) = \mathscr{H}^m_{(2)}(X_\Gamma; \mathscr{F})$, it is easy to check that $Q_F^{(m)}$ is non-degenerate. Its signature will be denoted $\operatorname{Sign}_{(2)}(X_\Gamma; \mathscr{F})$.

As in [8: § 2], one can construct an elliptic differential operator on X_Γ whose L^2-index (cf. [9] or [2]) is equal to $\operatorname{Sign}_{(2)}(X_\Gamma; \mathscr{F})$. To see this, let us first notice that $H^{\cdot}_{(2)}(X_\Gamma; \mathscr{F}) = \mathscr{H}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})$ is the space of L^2-solutions of the operator $d + d^*$ acting on the completion $\widetilde{\mathscr{A}}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})$ of $\mathscr{A}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})$. This completion can be canonically identified with $(L^2(\Gamma\backslash G) \otimes \Lambda^{\cdot}\mathfrak{p} \otimes F)^k$, the subspace of K-invariant elements in the Hilbert tensor product $L^2(\Gamma\backslash G) \otimes \Lambda^{\cdot}\mathfrak{p} \otimes F$. The inner product on $\mathfrak{p}$ determines a star operator $*\colon \Lambda^k\mathfrak{p}_C \to \Lambda^{2m-k}\mathfrak{p}_C$ and one then defines an involution $\tau_\mathfrak{p}$ on $\Lambda^{\cdot}\mathfrak{p}_C$ by

$$\tau_\mathfrak{p}(\omega) = i^{k(k-1)+m} * \omega\ , \quad \omega \in \Lambda^k\mathfrak{p}_C\ .$$

On the other hand, the splitting $F = F^+ \oplus F^-$ of Lemma (1.2) determines an involution τ_F on F, namely $\tau_F | F^\pm = \pm I$. Setting

$$\tau = I \otimes \tau_\mathfrak{p} \otimes \tau_F$$

one obtains an involution on $(L^2(\Gamma\backslash G) \otimes \Lambda^{\cdot}\mathfrak{p} \otimes F)^K$.

We now recall that, via the identification $\widetilde{\mathscr{A}}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F}) \cong (L^2(\Gamma\backslash G) \otimes \Lambda^{\cdot}\mathfrak{p} \otimes F)^K$, the exterior differential has the expression

$$d = \textstyle\sum_{i=1}^{2m} [R_\Gamma(X_i) \otimes \varepsilon(X_i) \otimes I + I \otimes \varepsilon(X_i) \otimes \sigma(X_i)]\ ,$$

where $\{X_i\}_{i=1}^{2m}$ is an orthonormal basis of $\mathfrak{p}$, R_Γ denotes the right quasiregular representation of F on $L^2(\Gamma\backslash G)$, and $\varepsilon(X)$ the exterior multiplication by $X \in \mathfrak{p}$. Correspondingly,

$$d^* = \sum_{i=1}^{2m} [-R_\Gamma(X_i) \otimes \iota(X_i) \otimes I + I \otimes \iota(X_i) \otimes \sigma(X_i)] ,$$

with $\iota(X)$ denoting the interior multiplication by X. Since, for $X \in \mathfrak{p}$,

$$\iota(X) = \tau_{\mathfrak{p}}\varepsilon(X)\tau_{\mathfrak{p}} \quad \text{and} \quad \sigma(X) = -\tau_F\sigma(X)\tau_F ,$$

one has

$$(d + d^*)\tau = -\tau(d + d^*) ,$$

i.e., $d + d^*$ switches the ± 1-eigenspaces $\tilde{\mathscr{A}}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})^{\pm}$ of τ. We let $L^{\pm}_{\Gamma,F}$ be the operator $d + d^*$ from $\tilde{\mathscr{A}}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})^{\pm}$ to $\tilde{\mathscr{A}}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})^{\mp}$. The space of L^2-solutions for $L^{\pm}_{\Gamma,F}$ is $\mathscr{H}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})^{\pm}$ = the ± 1-eigenspace of the restriction of τ to $\mathscr{H}^{\cdot}_{(2)}(X_\Gamma; \mathscr{F})$. In particular, $L^{+}_{\Gamma,F}$ has a well-defined L^2-index, which will be denoted $\mathrm{Index}_{(2)} L^{+}_{\Gamma,F}$. Using Lemma (1.2) one sees easily, as in [1: § 6], that

$$\mathrm{Index}_{(2)} L^{+}_{\Gamma,F} = \mathrm{Sign}_{(2)}(X_\Gamma; \mathscr{F}) . \tag{2.1}$$

We now want to apply [2: Prop. 7.4] to the operator $L^{+}_{\Gamma,F}$ and express its index as a combination of spinor numbers. The representation-theoretic symbol of $L^{+}_{\Gamma,F}$ is the virtual $\mathfrak{k}_C$-module $(\Lambda\mathfrak{p} \otimes F)^+ - (\Lambda\mathfrak{p} \otimes F)^-$, where $(\Lambda\mathfrak{p} \otimes F)^{\pm}$ = the ± 1-eigenspace of $\tau_{\mathfrak{p}} \otimes \tau_F$. Let $\Lambda^{\pm}\mathfrak{p}_C$ denote the ± 1-eigenspace of $\tau_{\mathfrak{p}}$. It is well-known that, in the representation ring of $\mathrm{Spin}(\mathfrak{p})$ and therefore a fortiori in $R(\mathfrak{k}_C)$,

$$\Lambda^+\mathfrak{p}_C - \Lambda^-\mathfrak{p}_C = (-1)^m S \otimes (S^+ - S^-) .$$

Thus,

$$\begin{aligned}(\Lambda\mathfrak{p} \otimes F)^+ - (\Lambda\mathfrak{p} \otimes F)^- &= (\Lambda^+\mathfrak{p}_C - \Lambda^-\mathfrak{p}_C) \otimes (F^+ - F^-) \\ &= (-1)^m(F^+ \otimes S - F^- \otimes S) \otimes (S^+ - S^-) .\end{aligned}$$

Taking now into account (1.5), it follows from [2: Prop. 7.4] that

$$\mathrm{Sign}_{(2)}(X_\Gamma; \mathscr{F}) = (-1)^m \sum_{u \in W^1} \varepsilon_F(u) \mathrm{Index}_{(2)} D^{+}_{u\Lambda-\rho_k,\Gamma} \tag{2.2}$$

where $D^{+}_{\lambda,\Gamma}$ denotes, as in [2], the twisted Dirac operator with coefficient in the irreducible $\mathfrak{k}_C$-module V_λ.

3. The signature formula and applications.

A. We retain the notation of the previous two sections and assume in addition that Γ is co-compact and torsion-free. Then $X_\Gamma = \Gamma\backslash G/K$ is a compact smooth manifold and so the L^2-cohomology and the ordinary cohomology coincide. In order to compute $\mathrm{Sign}(X_\Gamma; \mathscr{F}) = \mathrm{Sign}_{(2)}(X_\Gamma; \mathscr{F})$, one can use the Atiyah-Singer index theorem [1]. A faster way to obtain an explicit formula for this signature is to apply the Hirzebruch proportionality principle, not directly to the operator $L^+_{\Gamma,F}$ though, but to the twisted Dirac operators which appear in (2.2), and thus reduce the computation to well-known results for compact groups. Recall that U denotes the maximal compact subgroup of G_C containing K, with Lie algebra $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$, and let $Y = U/K$ be the compact dual of $X = G/K$. We view both X_Γ and Y as being endowed with the riemannian metrics arising from the Cartan-Killing form on $\mathfrak{g}_C = \mathfrak{u}_C$. Using the approach alluded to above, one obtains (cf. e.g. [6: 4.7]), for the Dirac operator $D^+_{\lambda,\Gamma}$,

$$\text{(A.1)} \qquad \operatorname{Index} D^+_{\lambda,\Gamma} = \frac{\operatorname{vol}(X_\Gamma)}{\operatorname{vol}(Y)} \prod_{\alpha \in \Psi} \frac{\langle \lambda + \rho_k, \alpha \rangle}{\langle \rho, \alpha \rangle} .$$

Applying (2.2) we get:

(3.1) THEOREM. $\operatorname{Sign}(X_\Gamma; \mathscr{F}) = (-1)^m \dfrac{\operatorname{vol}(X_\Gamma)}{\operatorname{vol}(Y)} \dim F \sum_{u \in W^1} \varepsilon_F(u) \det(u)$.

Let us now remark that $|W^1| = |W|/|W_k|$, which in turn is equal to the Euler characteristic $\chi(Y)$ of Y. Since $\varepsilon_F(u)\det(u) = \pm 1$, for each $u \in W^1$, it follows that:

(3.2) COROLLARY. *If* $\chi(Y) \not\equiv 0 \pmod 2$ *then* $\operatorname{Sign}(X_\Gamma; \mathscr{F}) \neq 0$; *in particular* $H^m(\Gamma; F) = H^m(X_\Gamma; \mathscr{F}) \neq 0$, *for any torsion-free co-compact discrete subgroup* $\Gamma \subset G$ *and any finite dimensional representation* F *of* G.

B. In this subsection $G = SU(2n, 1)$, with $n \geqq 2$, and Γ is any neat, arithmetic subgroup. According to [2: Thm. 7.1], if $\lambda + \rho_k$ is Φ-regular then

$$\text{(B.1)} \qquad \operatorname{Index}_{(2)} D^+_{\lambda,\Gamma} = \frac{\operatorname{vol}(X_\Gamma)}{\operatorname{vol}(Y)} \prod_{\alpha \in \Psi} \frac{\langle \lambda + \rho_k, \alpha \rangle}{\langle \rho, \alpha \rangle} + c_n(\Gamma) \dim V_\lambda .$$

The constant $c_n(\Gamma)$ in this formula is itself an amalgam of various constants. Rather than tracing each of its constituents through the literature, it is more convenient to compute it by comparing the above index formula with Kato's dimension formula for spaces of automorphic forms [7]. Indeed, using the computations in [5], one can show that, for $\lambda_q = (2q-1)\rho_n$ with $q \geqq 2$, one has

$$\operatorname{Index}_{(2)} D^+_{\lambda_q,\Gamma} = N_\Gamma(\pi_{\lambda_q+\rho_k}) = \dim \mathscr{C}_{(2n+1)q}(\Gamma) ,$$

where $\mathscr{C}_p(\Gamma)$ is the space of Γ-cusp forms of weight p. Kato's formula for the dimension of $\mathscr{C}_p(\Gamma)$ then gives

$$\text{(B.2)} \qquad c_n(\Gamma) = 2^{2n-1}\zeta(1-2n) m_\infty(\Gamma) ;$$

here $\zeta(s)$ is the Riemann zeta function and for the (rather intricate) definition of $m_\infty(\Gamma)$ we refer to [7].

(3.3) THEOREM. $$\operatorname{Sign}_{(2)}(X_\Gamma;\mathscr{F}) = \frac{\operatorname{vol}(X_\Gamma)}{\operatorname{vol}(Y)}\dim F \sum_{u\in W^1}\varepsilon_F(u)\det(u) + 2^{4n-1}\zeta(1-2n)m_\infty(\Gamma)\operatorname{Sign} Q_F\,.$$

PROOF. From (2.2), (B.1) and (B.2) it follows that

$$\begin{aligned}\operatorname{Sign}_{(2)}(X_\Gamma;\mathscr{F}) = {} & \frac{\operatorname{vol}(X_\Gamma)}{\operatorname{vol}(Y)}\dim F \sum_{u\in W^1}\varepsilon_F(u)\det(u)\\ & + 2^{2n-1}\zeta(1-2n)m_\infty(\Gamma)\sum_{u\in W^1}\varepsilon_F(u)\dim V_{u\Lambda-\rho_k}\,.\end{aligned}$$

Using (1.5), one sees that

$$\begin{aligned}\sum_{u\in W^1}\varepsilon_F(u)\dim V_{u\Lambda-\rho_k} &= \dim F^+\otimes S - \dim F^-\otimes S\\ &= (\dim F^+ - \dim F^-)\dim S = 2^{2n}\operatorname{Sign} Q_F\,,\end{aligned}$$

which completes the proof.

(3.4) COROLLARY. *Let Γ be a neat arithmetic subgroup of $SU(2n,1)$ such that*

$$\frac{\operatorname{vol}(X_\Gamma)}{\operatorname{vol}(Y)} > 2^{4n-1}|\zeta(1-2n)|m_\infty(\Gamma)\,.$$

Then $H^{2n}(\Gamma;\mathscr{F})\neq 0$, for any finite dimensional representation F of $SU(2n,1)$.

PROOF. Since $|W^1|$ $(=2n+1)$ is odd, $\sum_{u\in W^1}\varepsilon_F(u)\det(u)$ is a non-zero integer. Thus

$$\begin{aligned}\left|\frac{\operatorname{vol}(X_\Gamma)}{\operatorname{vol}(Y)}\dim F\sum_{u\in W^1}\varepsilon_F(u)\det(u)\right| &\geqq \frac{\operatorname{vol}(X_\Gamma)}{\operatorname{vol}(Y)}\dim F\\ &\geqq \frac{\operatorname{vol}(X_\Gamma)}{\operatorname{vol}(Y)}|\operatorname{Sign} Q_F| > |2^{4n-1}\zeta(1-2n)m_\infty(\Gamma)\operatorname{Sign} Q_F|\,.\end{aligned}$$

From this and (3.3) it follows that $\operatorname{Sign}_{(2)}(X_\Gamma;\mathscr{F})\neq 0$, and therefore $H^{2n}_{(2)}(X_\Gamma;\mathscr{F})\neq 0$. But $H^{2n}_{(2)}(X_\Gamma;\mathscr{F})$ injects into $H^{2n}(X_\Gamma;\mathscr{F})$ (cf. [12: (6.9), (6.11)]) and thus $H^{2n}(\Gamma;F) = H^{2n}(X_\Gamma;\mathscr{F})\neq 0$ too.

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