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STABILITY BY DECOMPOSITIONS FOR VOLTERRA EQUATIONS

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Abstract. We consider a system of integrodifferential equations of the form

$$x' = A(t)x + \int_0^t C(t, s)x(s)ds \tag{1}$$

which we then write as

$$x' = L(t)x + \int_0^t C_1(t, s)x(s)ds + \left(\frac{d}{dt}\right)\int_0^t H(t, s)x(s)ds\ . \tag{2}$$

A number of Lyapunov functionals are constructed for (2) yielding necessary and sufficient conditions for stability of the zero solution of (1).

1. Introduction. We consider the system

$$x' = A(t)x + \int_0^t C(t, s)x(s)ds \tag{1.1}$$

in which $A(t)$ is an $n \times n$ matrix continuous for $0 \leqq t < \infty$, $C(t, s)$ is an $n \times n$ matrix continuous for $0 \leqq s \leqq t < \infty$, and $n \geqq 1$.

We write (1.1) as

$$x' = L(t)x + \int_0^t C_1(t, s)x(s)ds + \frac{d}{dt}\int_0^t H(t, s)x(s)ds \tag{1.2}$$

and discuss stability and instability of the zero solution of (1.1) via the construction of Lyapunov functionals for the system (1.2).

Evidently, (1.1) can be regarded as a special case of (1.2) and therefore any stability result for (1.2) is also a stability result for (1.1). However, the most interesting stability results of this paper are those obtained by converting (1.1) to (1.2). It turns out that (1.1) can be reduced to (1.2) in several ways and consequently a variety of stability results will be obtained. In most cases, we obtain simple and practical results under mild conditions.

The following terminology is used throughout this paper. For any $t_0 \geqq 0$ and any continuous function ϕ: $[0, t_0] \to R^n$, a solution of (1.2) and hence of (1.1) is a continuous function x: $[0, \infty) \to R^n$, denoted by $x(t, t_0, \phi)$ or $x(t)$, which satisfies (1.2) for $t \geqq t_0$ and such that $x(t) = \phi(t)$ for $0 \leqq t \leqq t_0$. The solution $x = 0$ is called the zero solution.

DEFINITION. The zero solution of (1.2) and hence of (1.1) is said to be

1. *stable* if for every $\varepsilon > 0$ and every $t_0 \geqq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $|\phi(t)| < \delta$ on $[0, t_0]$ implies $|x(t, t_0, \phi)| < \varepsilon$ for $t \geqq t_0$.

2. *uniformly stable* if it is stable and the δ in the definition of stability is independent of t_0.

3. *asymptotically stable* if it is stable and for each $t_0 \geqq 0$ there is a $\beta = \beta(t_0) > 0$ such that $|\phi(t)| < \beta$ on $[0, t_0]$ implies $x(t, t_0, \phi) \to 0$ as $t \to \infty$.

4. *uniformly asymptotically stable* if it is uniformly stable, the β in the definition of asymptotic stability is independent of t_0, and for each $\eta > 0$ there is a $T = T(\eta) > 0$ such that $|\phi(t)| < \beta$ on $[0, t_0]$ implies $|x(t, t_0, \phi)| < \eta$ for $t \geqq t_0 + T$.

An $n \times n$ matrix is said to be stable if all of its characteristic roots have negative real parts. Also, when a function is written without its argument, then it is understood that the argument is t. If D is a matrix or a vector, $|D|$ means the sum of the absolute values of its elements.

Most stability results for the system (1.1) assume that $A(t) = A =$ constant. In this paper we allow $A(t)$ to be a function of t which may be unbounded for $n = 1$. For this reason we discuss the scalar and the vector cases separately.

2. Scalar equations. Consider the scalar equation

$$x' = L(t)x + \int_0^t C_1(t, s)x(s)ds + \frac{d}{dt}\int_0^t H(t, s)x(s)ds \tag{2.1}$$

where $L(t)$ is continuous for $0 \leqq t < \infty$, and $C_1(t, s)$ and $H(t, s)$ are continuous for $0 \leqq s \leqq t < \infty$. Here L, C, H and x are all scalars.

Let

$$P(t) = \int_0^t |C_1(t, s)|\,ds\ , \tag{2.2}$$

$$J(t) = \int_0^t |H(t, s)|\,ds\ , \tag{2.3}$$

and

$$\Phi(t, s) = \int_t^\infty [(1 + J(u))|C_1(u, s)| + (|L(u)| + P(u))|H(u, s)|]du \tag{2.4}$$

assuming of course that $\Phi(t, s)$ exists for $0 \leqq s \leqq t < \infty$.

Let

$$V(t, x(\cdot)) = \left(x - \int_0^t H(t, s)x(s)ds\right)^2 + \nu \int_0^t \Phi(t, s)x^2(s)ds \,, \tag{2.5}$$

where ν is an arbitrary constant. The functional $V(t, x(\cdot))$ plays a central role in the derivation of most of the stability results in this section. Thus, if $x(t) = (t, t_0, \phi)$ is a solution of (2.1), then the derivative $V'_{(2.1)}(t, x(\cdot))$ of $V(t, x(\cdot))$ along $x(t)$ satisfies

$$\begin{aligned} V'_{(2.1)}(t, x(\cdot)) &= 2\left(x - \int_0^t H(t, s)x(s)ds\right)\left(L(t)x + \int_0^t C_1(t, s)x(s)ds\right) \\ &\quad + \nu\frac{d}{dt}\int_0^t \Phi(t, s)x^2(s)ds \\ &= 2L(t)x^2 + 2x\int_0^t C_1(t, s)x(s)ds - 2L(t)x\int_0^t H(t, s)x(s)ds \\ &\quad - 2\int_0^t H(t, s)x(s)ds\int_0^t C_1(t, s)x(s)ds + \nu\frac{d}{dt}\int_0^t \Phi(t, s)x^2(s)ds \,. \end{aligned}$$

Thus,

$$\begin{aligned} &\left| V'_{(2.1)}(t, x(\cdot)) - 2L(t)x^2 - \nu\frac{d}{dt}\int_0^t \Phi(t, s)x^2(s)ds\right| \\ &\qquad \leqq 2\int_0^t |C_1(t, s)|\,|x|\,|x(s)|\,ds + 2|L(t)|\int_0^t |H(t, s)|\,|x|\,|x(s)|\,ds \\ &\qquad\quad + 2\int_0^t |H(t, s)|\,|x(s)|\,ds\int_0^t |C_1(t, s)|\,|x(s)|\,ds \,. \end{aligned}$$

Using the Schwarz inequality, we may write

$$\begin{aligned} \int_0^t |H(t, s)|\,|x(s)|\,ds &= \int_0^t |H(t, s)|^{1/2}|H(t, s)|^{1/2}|x(s)|\,ds \\ &\leqq \left[\int_0^t |H(t, s)|\,ds\int_0^t |H(t, s)|\,x^2(s)ds\right]^{1/2} \\ &= \left[J(t)\int_0^t |H(t, s)|\,x^2(s)ds\right]^{1/2} \end{aligned}$$

so that

$$\begin{aligned} &2\int_0^t |H(t, s)|\,|x(s)|\,ds\int_0^t |C_1(t, s)|\,|x(s)|\,ds \\ &\qquad \leqq 2\left[J(t)\int_0^t |H(t, s)|\,x^2(s)ds\right]^{1/2}\left[P(t)\int_0^t |C_1(t, s)|\,x^2(s)ds\right]^{1/2} \\ &\qquad = 2\left[P(t)\int_0^t |H(t, s)|\,x^2(s)ds\right]^{1/2}\left[J(t)\int_0^t |C_1(t, s)|\,x^2(s)ds\right]^{1/2} \\ &\qquad \leqq P(t)\int_0^t |H(t, s)|\,x^2(s)ds + J(t)\int_0^t |C_1(t, s)|\,x^2(s)ds \,. \end{aligned}$$

Thus,

$$\begin{aligned}&\left|V'_{(2.1)}(t, x(\cdot)) - 2L(t)x^2 - \nu\frac{d}{dt}\int_0^t \Phi(t, s)x^2(s)ds\right|\\ &\quad\leqq \int_0^t |C_1(t, s)|(x^2 + x^2(s))ds + |L(t)|\int_0^t |H(t, s)|(x^2 + x^2(s))ds\\ &\qquad + P(t)\int_0^t |H(t, s)|x^2(s)ds + J(t)\int_0^t |C_1(t, s)|x^2(s)ds\\ &\quad= [P(t) + |L(t)|J(t)]x^2 + \int_0^t [(1 + J(t))|C_1(t, s)|\\ &\qquad + (|L(t)| + P(t))|H(t, s)|]x^2(s)ds\,.\end{aligned}$$

Using (2.4), we obtain

$$\begin{aligned}&\left|V'_{(2.1)}(t, x(\cdot)) - 2L(t)x^2 - \nu\frac{d}{dt}\int_0^t \Phi(t, s)x^2(s)ds\right|\\ &\qquad\leqq [P(t) + |L(t)|J(t)]x^2 - \int_0^t \frac{\partial\Phi(t, s)}{\partial t}x^2(s)ds\,.\end{aligned} \tag{2.6}$$

THEOREM 1. *Let P, J, and Φ be defined by* (2.2)-(2.4). *If* $L(t) < 0$ *and*

(i) $$\sup_{t\geqq 0} J(t) < 1$$

and

(ii) $$J(t)|L(t)| + P(t) + \Phi(t, t) \leqq 2|L(t)|\,,$$

then the zero solution of (2.1) *is stable.*

PROOF. Taking $\nu = 1$ in (2.5) and observing that

$$\frac{d}{dt}\int_0^t \Phi(t, s)x^2(s)ds = \Phi(t, t)x^2 + \int_0^t \frac{\partial\Phi(t, s)}{\partial t}x^2(s)ds\,, \tag{2.7}$$

we obtain from (2.6) and (ii) that $V'_{(2.1)}(t, x(\cdot)) \leqq 0$. Since $V(t, x(\cdot))$ is not positive definite, we still need to prove stability. Let $t_0 \geqq 0$ and $\varepsilon > 0$ be given. We must find $\delta > 0$ so that if $\phi\colon [0, t_0] \to R$ is continuous with $|\phi(t)| < \delta$, then $|x(t, t_0, \phi)| < \varepsilon$ for all $t \geqq t_0$. As $V'_{(2.1)}(t, x(\cdot)) \leqq 0$ for $t \geqq t_0$, we have

$$\begin{aligned}V(t, x(\cdot)) \leqq V(t_0, \phi(\cdot)) &= \left(\phi(t_0) - \int_0^{t_0} H(t_0, s)\phi(s)ds\right)^2 + \int_0^{t_0} \Phi(t_0, s)\phi^2(s)ds\\ &\leqq \delta^2\left[(1 + J(t_0))^2 + \int_0^{t_0}\Phi(t_0, s)ds\right]\\ &\leqq \delta^2\left[4 + \int_0^{t_0}\Phi(t_0, s)ds\right] \overset{\text{def}}{=} \delta^2 N^2\,.\end{aligned}$$

On the other hand,

$$V(t, x(\cdot)) \geqq \left(x(t) - \int_0^t H(t, s)x(s)ds\right)^2 \geqq \left(|x(t)| - \int_0^t |H(t, s)|\,|x(s)|\,ds\right)^2 .$$

Thus,

$$|x(t)| \leqq \delta N + \int_0^t |H(t, s)|\,|x(s)|\,ds .$$

As long as $|x(t)| < \varepsilon$, we have

$$|x(t)| \leqq \delta N + \varepsilon \sup_{t \geqq 0} J(t) \stackrel{\text{def}}{=} \delta N + \varepsilon\beta < \varepsilon$$

for all $t \geqq t_0$ provided that $\delta < \varepsilon(1 - \beta)/N$. Thus, the solution $x = 0$ is stable.

THEOREM 2. *Let P, J, and Φ be defined by* (2.2)-(2.4) *and suppose there is a continuous function h: $[0, \infty) \to [0, \infty)$ such that $|H(t, s)| \leqq h(t - s)$ with $h(u) \to 0$ as $u \to \infty$. If $L(t) > 0$ and there is a positive constant α such that*

$$J(t)|L(t)| + P(t) + \Phi(t, t) + \alpha \leqq 2|L(t)| , \tag{2.8}$$

then the zero solution of (2.1) *is unstable.*

PROOF. Taking $\nu = -1$ in (2.5) and using (2.6)-(2.8) we obtain $V'_{(2.1)}(t, x(\cdot)) \geqq \alpha x^2$ for all $t \geqq t_0$. Now, for any $t_0 \geqq 0$ and any $\delta > 0$ we can find a continuous function ϕ: $[0, t_0] \to R$ with $|\phi(t)| < \delta$ and $V(t_0, \phi(\cdot)) > 0$ so that if $x(t) = x(t, t_0, \phi)$ is a solution of (2.1), then

$$\left(x(t) - \int_0^t H(t, s)x(s)ds\right)^2 \geqq V(t, x(\cdot)) \geqq V(t_0, \phi(\cdot)) + \alpha \int_{t_0}^t x^2(s)ds . \tag{2.9}$$

We will show that $x(t)$ is unbounded. Suppose $x(t)$ is bounded; then as $J(t) < 2$ by (2.8), we have $\int_0^t H(t, s)x(s)ds$ bounded and hence, by (2.9), $x^2(t)$ is in $L^1[0, \infty)$. By the Schwarz inequality we have

$$\left(\int_0^t |H(t, s)|\,|x(s)|\,ds\right)^2 \leqq \int_0^t |H(t, s)|\,ds \int_0^t |H(t, s)|\,x^2(s)ds \leqq 2\int_0^t h(t - s)x^2(s)ds .$$

The last integral is the convolution of an L^1-function with a function tending to zero. Thus, the integral tends to zero as $t \to \infty$ and hence $\int_0^t H(t, s)x(s)ds \to 0$ as $t \to \infty$. Now, by (2.9), we have

$$\left|x(t) - \int_0^t H(t, s)x(s)ds\right| \geqq [V(t_0, \phi(\cdot))]^{1/2}$$

so that for sufficiently large T it follows that $|x(t)| \geqq \gamma$ for some $\gamma > 0$ and all $t \geqq T$. This contradicts $x^2(t)$ being in L^1. Thus, $x(t)$ is unbounded and so the zero solution of (2.1) is unstable. This completes the proof.

Now, consider the scalar equation

$$x' = A(t)x + \int_0^t C(t, s)x(s)ds \,, \tag{2.10}$$

where $A(t)$ and $C(t, s)$ are continuous for $0 \leqq t < \infty$ and $0 \leqq s \leqq t < \infty$, respectively. Suppose that

$$C(t, s) = C_1(t, s) + C_2(t, s) \,,$$

where $C_i(t, s)$, $i = 1, 2$, are continuous. Select a continuous function $H(t, s)$ such that

$$(\partial/\partial t)H(t, s) = C_2(t, s) \,. \tag{2.11}$$

For example, if $\int_t^\infty C_2(u, s)du$ exists, then $H(t, s)$ may be defined by

$$H(t, s) = -\int_t^\infty C_2(u, s)du \,.$$

If we let

$$L(t) = A(t) - H(t, t) \,, \tag{2.12}$$

then (2.10) takes the form (2.1). Thus, Theorems 1 and 2 combined yield the following necessary and sufficient condition for stability of (2.10).

THEOREM 3. *Let P, J and Φ be defined by* (2.2)-(2.4), *where H and L are defined by* (2.11) *and* (2.12). *Suppose there is a continuous function $h\colon [0, \infty) \to [0, \infty)$ with $|H(t, s)| \leqq h(t - s)$ and $h(u) \to 0$ as $u \to \infty$, and suppose there is a positive constant α such that*

$$\text{(i)} \qquad \sup_{t \geqq 0} J(t) < 1$$

and

$$\text{(ii)} \qquad J(t)|L(t)| + P(t) + \Phi(t, t) + \alpha \leqq 2|L(t)| \,.$$

Then the zero solution of (2.10) *is stable if and only if $L(t) < 0$.*

Regarding (2.10) as a special case of (2.1), we obtain the following corollaries.

COROLLARY 1. *Suppose that for some $\alpha > 0$,*

$$\int_0^t |C(t, s)|ds + \int_t^\infty |C(u, t)|du - 2|A(t)| \leqq -\alpha \,.$$

Then the zero solution of (2.10) *is stable if and only if $A(t) < 0$.*

PROOF. Let $C_2(t, s) \equiv 0$ in Theorem 3. Then $C_1(t, s) = C(t, s)$, $J(t) \equiv 0$,

$L(t) = A(t)$, and (ii) of Theorem 3 reduces to $P(t) + \Phi(t, t) + \alpha \leqq 2|A(t)|$ with $\Phi(t, t) = \int_t^\infty |C(u, t)|du$. If we choose $h(t) \equiv 0$, then all the conditions of Theorem 3 are satisfied and the proof is complete.

Letting $H(t, s)$ and $L(t)$ satisfy

$$(\partial/\partial t)H(t, s) = C(t, s) \tag{2.13}$$

and

$$L(t) = A(t) - H(t, t) , \tag{2.14}$$

we have:

COROLLARY 2. *Let H and L satisfy* (2.13)-(2.14). *Suppose there is a continuous function $h: [0, \infty) \to [0, \infty)$ with $|H(t, s)| \leqq h(t - s)$ and $h(u) \to 0$ as $u \to \infty$, and suppose there is a positive constant α such that*

$$\text{(i)} \qquad \sup_{t \geqq 0} \int_0^t |H(t, s)|ds < 1$$

and

$$\text{(ii)} \qquad |L(t)| \int_0^t |H(t, s)|ds + \int_t^\infty |L(u)||H(u, t)|du + \alpha \leqq 2|L(t)| .$$

Then the zero solution of (2.10) *is stable if and only if* $L(t) < 0$.

PROOF. Let $C_1(t, s) \equiv 0$ in Theorem 3. Then $C_2(t, s) = C(t, s)$, $P(t) \equiv 0$, and (ii) of Theorem 3 reduces to

$$J(t)|L(t)| + \Phi(t, t) + \alpha \leqq 2|L(t)| \quad \text{with} \quad \Phi(t, t) = \int_t^\infty |L(u)||H(u, t)|du .$$

Thus all the conditions of Theorem 3 are satisfed and this completes the proof.

REMARK 1. Corollary 1 is [3, Theorem 3]. Also, Corollary 2 improves [3, Theorem 4] by relaxing the boundedness condition on $L(t)$. Now, if $L(t)$ is bounded, say $Q_1 \leqq |L(t)| \leqq Q_2$ for some positive constants Q_1 and Q_2, then, for $\alpha = 2Q_2 - RQ_1$ with $0 < R < 2$, Corollary 2 reduces to [3, Theorem 4]. We give below two illustrative examples in which both $A(t)$ and $L(t)$ are unbounded.

EXAMPLE 1. Consider the equation

$$x' = -(t + 2)x - \int_0^t (s + 1)(t - s + 1)^{-2}x(s)ds .$$

As $A(t) = -(t + 2) < 0$, $\int_0^t |C(t, s)|ds = t + 2 - (t + 2)(t + 1)^{-1} - \ln(t + 1)$, and $\int_t^\infty |C(u, t)|du = t + 1$. Then the conditions of Corollary 1 are satisfied and hence the zero solution is stable. However, Corollary 2 does not apply as $H(t, s) = (s + 1)(t - s + 1) + g(s)$ does not satisfy condition (i) for every choice of g.

EXAMPLE 2. Consider the equation

$$x' = [(2t + 1)^{-3}/3 - t - 1/10]x - \int_0^t (t + s + 1)^{-4}x(s)ds .$$

As $A(t)$ changes sign, Corollary 1 does not apply. However, by letting $H(t, s) = (t + s + 1)^{-3}/3$, we have

$$L(t) = A(t) - H(t, t) = -t - 1/10 ,$$

$$\int_0^t |H(t, s)|ds = [(t + 1)^{-2} - (2t + 1)^{-2}]/6 \leqq 1/6 ,$$

and

$$\int_t^\infty |L(u)||H(u, t)|du = (30t + 11)(2t + 1)^{-2}/60 .$$

Thus,

$$\begin{aligned}|L(t)|&\int_0^t |H(t, s)|ds + \int_t^\infty |L(u)||H(u, t)|du \\ &= (10t + 1)[(t + 1)^{-2} - (2t + 1)^{-2}]/10 + (30t + 11)(2t + 1)^{-2}/60 \\ &= (2t + 1)^{-1}/6 + (10t + 1)(t + 1)^{-2}/60 \leqq 1/6 + (10t + 1)/60 \\ &= t/6 + 11/60 < 2|L(t)| .\end{aligned}$$

All conditions of Corollary 2 are satisfied and hence the zero solution is stable.

REMARK 2. While Corollary 1 requires a sign condition on $A(t)$, Corollary 2 requires a strong integral condition on $C(t, s)$. On the other hand, both results can be considered as extreme cases of Theorem 3. Consequently, Theorem 3 does not only extend and unify Theorems 3 and 4 of [3] but also yields a new stability result which is more refined than either of these theorems. This is illustrated by the following example.

EXAMPLE 3. Consider the equation

$$x' = [(\sin t)/2 - 1/4]x + \int_0^t [(t + s + 1)^{-2}/9 - (t - s + 1)^{-3} \sin s]x(s)ds ,$$

where neither Corollary 1 nor Corollary 2 apply. However, by letting

$C_1(t, s) = (t + s + 1)^{-2}/9$, $C_2(t, s) = -(t - s + 1)^{-3}\sin s$, and $H(t, s) = -\int_t^\infty C_2(u, s)du$, we have

$$H(t, s) = [(t - s + 1)^{-2}\sin s]/2\ ,$$

$$J(t) = \int_0^t |H(t, s)|ds \leqq \int_0^t (t - s + 1)^{-2}ds \leqq 1/2\ ,$$

$$P(t) = \int_0^t |C_1(t, s)|ds = [(t + 1)^{-1} - (2t + 1)^{-1}]/9$$
$$\leqq \sup_{t \geqq 0} P(t) = (3 - 2\sqrt{2})/9 < 1/45\ ,$$

$$L(t) = A(t) - H(t, t) = -1/4\ ,$$

and

$$\Phi(t, t) \leqq (3/2)\int_t^\infty |C_1(u, t)|du + (49/180)\int_t^\infty |H(u, t)|du$$
$$= (2t + 1)^{-1}/6 + (49/360)|\sin t| \leqq 109/360\ .$$

Thus, $J(t)|L(t)| + P(t) + \Phi(t, t) \leqq 9/20 < 2|L(t)|$ and hence, by Theorem 3, the zero solution is stable.

We now apply Theorem 3 to the convolution equation

$$x' = Ax + \int_0^t C(t - s)x(s)ds \tag{2.15}$$

in which A is constant and $C(t)$ is continuous for $0 \leqq t < \infty$. Let

$$C(t) = C_1(t) + C_2(t)\ ,$$

where $C_i(t)$, $i = 1, 2$, are continuous for $0 \leqq t < \infty$.

We assume that $C_1(t)$ and $\int_t^\infty C_2(v)dv$ are L^1-functions and let $H(t) = -\int_t^\infty C_2(v)dv$. Then $P(t)$, $J(t)$, and $L(t)$ defined by (2.2), (2.3), and (2.12) reduce to $P(t) = \int_0^t |C_1(v)|dv$, $J(t) = \int_0^t \left|\int_u^\infty C_2(v)dv\right|du$, and $L(t) = A + \int_0^\infty C_2(v)dv$. As $L(t)$ is constant, then, without loss of generality, we may replace $P(t)$ and $J(t)$ in $\Phi(t, t)$ and in (ii) of Theorem 3 by $P(\infty)$ and $J(\infty)$. Thus, letting

$$P = \int_0^\infty |C_1(v)|dv\ , \tag{2.16}$$

$$J = \int_0^\infty \left|\int_t^\infty C_2(v)dv\right|dt\ , \tag{2.17}$$

and

$$L = A + \int_0^\infty C_2(v)dv \,, \tag{2.18}$$

we have $\Phi(t, t) = (1 + J)P + (|L| + P)J$ so that conditions (i) and (ii) of Theorem 3 reduce to $P(J + 1) + (J - 1)|L| < 0$.

THEOREM 4. *Let P, J and L be defined by* (2.16)-(2.18).

(i) *If $P(J + 1) + (J - 1)|L| < 0$, then the zero solution of* (2.15) *is stable if and only if $L < 0$.*

(ii) *If, in addition, $C(t)$ is in $L^j[0, \infty)$, $j = 1$ or $j = 2$, then the zero solution of* (2.15) *is asymptotically stable if and only if $L < 0$.*

(iii) *If, furthermore, $J\int_t^\infty |C(v)|dv$ is in $L^q[0, \infty)$, $0 < q < 2$, then the zero solution of* (2.15) *is uniformly asymptotically stable if and only if $L < 0$.*

PROOF. If we let $h(t) = H(t)$, Part (i) follows from Theorem 3. To prove Part (ii), we use (2.15) to obtain

$$\begin{aligned} |x'(t)| &\leqq |A|\,|x(t)| + \int_0^t |C(t - s)|\,|x(s)|\,ds \\ &\leqq |A|\,|x(t)| + (1/2)\left[\int_0^t |C(t - s)|^2 ds + \int_0^t |x(s)|^2 ds\right]. \end{aligned}$$

If $L < 0$, then the Lyapunov functional $V(t, x(\cdot))$ defined by (2.5) satisfies, as in the proof of Theorem 1, $V'(t, x(\cdot)) \leqq -\alpha x^2$, $\alpha > 0$. Thus, $x^2(t)$ is in $L^1[0, \infty)$. By the above inequality and the hypothesis in (ii), it follows that $x'(t)$ is bounded. Since $x^2(t)$ is in L^1 and $[x^2(t)]' = 2x(t)x'(t)$ is bounded, then $x(t) \to 0$ as $t \to \infty$. Thus $x = 0$ is asymptotically stable. To prove (iii) we consider separately the two cases $J = 0$ and $J \neq 0$. Suppose that $J \neq 0$. Then for any $t_0 \geqq 0$ and any continuous function $\phi\colon [0, t_0] \to R$ we write (2.15) as

$$x'(t) = Ax(t) + \int_0^{t_0} C(t - s)\phi(s)ds + \int_{t_0}^t C(t - s)x(s)ds \,.$$

If we let $y(t) = x(t + t_0)$, we obtain

$$y'(t) = Ay(t) + \int_0^t C(t - s)y(s)ds + \int_0^{t_0} C(t + t_0 - s)\phi(s)ds \,.$$

Let $Z(t)$ be the $n \times n$ matrix (here, $n = 1$) satisfying

$$Z'(t) = AZ(t) + \int_0^t C(t - s)Z(s)ds \quad \text{and} \quad Z(0) = I \,.$$

Then by the variation of parameters formula, we have

$$y(t) = Z(t)\phi(t_0) + \int_0^t Z(t - u)\int_0^{t_0} C(u + t_0 - s)\phi(s)dsdu$$

$$= Z(t)\phi(t_0) + \int_0^t Z(t-u)\int_u^{u+t_0} C(v)\phi(u+t_0-v)dvdu .$$

Let $K = \max_{0 \leqq t \leqq t_0}|\phi(t)|$. Then

$$|y(t)| \leqq K|Z(t)| + K\int_0^t |Z(t-u)| \int_u^\infty |C(v)|dvdu .$$

Letting $G(t) = \int_t^\infty |C(v)|dv$, we have

$$|y(t)| \leqq K|Z(t)| + K\int_0^t |Z(t-u)|G(u)du = K|Z(t)| + K\int_0^t G(t-u)|Z(u)|du$$

$$= K|Z(t)| + K\int_0^t [G(t-u)]^{q/2}[G(t-u)]^{1-q/2}|Z(u)|du .$$

By the Schwarz inequality, we have

$$|y(t)| \leqq K|Z(t)| + K\Big(\int_0^t [G(t-u)]^q du \int_0^t [G(t-u)]^{2-q}|Z(u)|^2 du\Big)^{1/2}$$

As $x = 0$ is asymptotically stable, then $Z(t) \to 0$ as $t \to \infty$. By the hypothesis in (iii), $G(t)$ is an L^q-function and furthermore $G(t) \to 0$ as $t \to \infty$. On the other hand, $|Z|^2$ is an L^1-function and the last integral is the convolution of an L^1-function with a function tending to zero. Thus, the right hand side in the above inequality tends to zero and hence $x(t + t_0) \to 0$ as $t \to \infty$ uniformly in t_0. Consequently, the zero solution is uniformly asymptotically stable.

If $J = 0$, then $C_2(t) \equiv 0$ and $C_1(t) = C(t)$. The condition in (i) reduces to $P < |A|$ and hence the conditions in (ii) and (iii) are satisfied. If we assume $A < 0$ and consider the functional

$$W(t, x(\cdot)) = |x| + \int_0^t \int_t^\infty |C(u-s)|\, du\, |x(s)|ds ,$$

then, for some $\alpha > 0$,

$$W'_{(2.15)}(t, x(\cdot)) \leqq A|x| + \int_0^t |C(t-s)|\,|x(s)|ds + \int_t^\infty |C(u-t)|du|x|$$
$$- \int_0^t |C(t-s)|\,|x(s)|ds = \Big[A + \int_t^\infty |C(v)|dv\Big]|x| = -\alpha|x| .$$

Thus, $x(t)$ is in $L^1[0, \infty)$ and this is equivalent to uniform asymptotic stability of the zero solution of (2.15); see [6].

COROLLARY 3. *Suppose that*

$$\int_0^\infty |C(v)|dv < |A| .$$

Then the zero solution of (2.15) *is uniformly asymptotically stable if and only if* $A < 0$.

COROLLARY 4.

(i) *If* $\int_0^\infty \left| \int_t^\infty C(v)dv \right| dt < 1$, *then the zero solution of* (2.15) *is stable if and only if* $A + \int_0^\infty C(v)dv \leqq 0$.

(ii) *If, in addition,* $C(t) \in L^j[0, \infty)$, $j = 1$ *or* $j = 2$, *then the zero solution of* (2.15) *is asymptotically stable if and only if* $A + \int_0^\infty C(v)dv < 0$.

(iii) *If, furthermore,* $\int_t^\infty |C(v)|dv \in L^q[0, \infty)$, $0 < q < 2$, *then the zero solution of* (2.15) *is uniformly asymptotically stable if and only if* $A + \int_0^\infty C(v)dv < 0$.

PROOF. We need only to show that $A + \int_0^\infty C(v)dv = 0$ implies stability in (i). In fact, by letting $C_1(t) \equiv 0$ in Theorem 4, we have $C_2(t) = C(t)$, $P = 0$, and (i) of Theorem 4 reduces to $(J - 1)|L| < 0$. If $L \neq 0$, the result follows from Theorem 4. If $L = 0$, the result also follows from [3, Theorem 5].

In [4] Grossman and Miller gave a characterization of the uniform asymptotic stability of the zero solution of the convolution system

$$x' = Ax + \int_0^t C(t-s)x(s)ds \tag{2.19}$$

in terms of the location of the zeros of $\det(s - A - \hat{C}(s))$ in the complex plane; $\hat{C}(s)$ denotes the Laplace transform of $C(t)$.

THEOREM (Grossman-Miller). *Suppose* $C \in L^1[0, \infty)$. *Then the zero solution of* (2.19) *is uniformly asymptotically stable if and only if* $\det[sI - A - \hat{C}(s)] \neq 0$ *for* $\operatorname{Re} s \geqq 0$.

Letting $F(s) = s - A - \hat{C}(s)$, we have $F(0) = -A - \int_0^\infty C(t)dt$. Thus, in the case $n = 1$, if $A + \int_0^\infty C(t)dt \geqq 0$, then $F(0) \leqq 0$. Since $F(s) \to \infty$ as $s \to \infty$ along the real axis, then $F(s)$ has zeros in the right halfplane. By the above theorem, the zero solution of (2.19) is not uniformly asymptotically stable. In other words, $A + \int_0^\infty C(t)dt < 0$ is a necessary condition for uniform asymptotic stability. But, when $C(t) \geqq 0$, it follows from Corollary 3 that $A + \int_0^\infty C(v)dv < 0$ is a sufficient condition for uniform asymptotic stability. Thus, when $C(t) \geqq 0$, Grossman-Miller's result is equivalent to the condition $A + \int_0^\infty C(v)dv < 0$. This result was also obtained by Brauer [2] under stronger conditions on $C(t)$. In general, such a result is much easier to apply than Grossman-Miller's result. In fact, apart from kernels such as $C(t) = ke^{-\alpha t}$, $\alpha > 0$, the location of the zeros of $F(s)$ is not an easy task, especially when $A > 0$.

When $C(t) \leqq 0$ or $C(t)$ changes sign, the condition $A + \int_0^\infty C(v)dv < 0$ is no longer sufficient for uniform asymptotic stability. In this case, we may apply Corollary 3 if $A < 0$ and $|A|$ sufficiently large or Corollary 4 if $A + \int_0^\infty C(v)dv < 0$ and $\int_0^\infty \left|\int_t^\infty C(v)dv\right| dt < 1$. The last condition is mild if $\left|A + \int_0^\infty C(v)dv\right|$ is small; see Example 2 of [3]. Here is another example.

EXAMPLE 4. Consider the equation

$$x' = \int_0^t C(t-s)x(s)ds ,$$

where

$$C(t) = \begin{cases} b \sin t & \text{if} \quad 0 \leqq t \leqq 2\pi \\ 0 & \text{if} \quad t \geqq 2\pi , \end{cases}$$

with $b > 0$.

As $A + \int_0^\infty C(v)dv = 0$ and $\int_0^\infty \left|\int_t^\infty C(v)dv\right| dt = 2\pi b$, we conclude, by Corollary 4, that the zero solution is stable if $b < 1/(2\pi)$. However, for $b = \alpha(\alpha^2 + 1)/(1 - e^{-2\pi\alpha})$, $\alpha > 0$, the function $x(t) = \exp(\alpha t)$ is a solution on $[2\pi, \infty)$. Since $b \to 1/(2\pi)$ as $\alpha \to 0$, and $b \to \infty$ as $\alpha \to \infty$, then the zero solution is unstable for every $b > 1/(2\pi)$.

If $\left|A + \int_0^\infty C(v)dv\right|$ is large, then the condition $\int_0^\infty \left|\int_t^\infty C(v)dv\right| dt < 1$ may be severe. In this case, Theorem 3 provides us with a stability criterion under mild donditions.

EXAMPLE 5. Consider the equation

$$x' = -x - (1/a)\int_0^t e^{-(t-s)/a}x(s)ds , \quad a > 0 .$$

Here, $A + \int_0^\infty G(v)dv = -2$, $J = a$, and $A + \int_0^\infty |C(v)|dv = 0$. Thus, Corollary 3 does not apply and Corollary 4 yields uniform asymptotic stability only if $a < 1$. However, the zero solution is uniformly asymptotically stable for all large values of a. If we choose $C_2(t) = -[\exp(-2t)]/a$ and $C_1(t) = [-\exp(-t/a) + \exp(-2t)]/a$, then $L = -1 - 1/(2a)$, $J = 1/(4a)$, $P = 1 - 1/(2a)$, and $P(J+1) + (J-1)|L| = -1/(2a)$. By Theorem 3, the zero solution is uniformly asymptotically stable.

REMARK 3. Example 5 can easily be handled by Grossman-Miller's result. The purpose here is to show that, by an appropriate decomposition of the kernel $C(t)$, Theorem 3 can yield a well refined criterion for uniform asymptotic stability. This criterion will be quite useful when the kernel is of the form $C(t) = -k(\alpha t + 1)^{-p}$ as Grossman-Miller's result is very hard to apply. Below we give an effective decomposition of kernels of the form $C(t) = (at - b)(ct + d)^{-p}$, where $a \geqq 0$, $b > 0$, $c > 0$, and $d > 0$. We ask that $p > 3/2$ if $a = 0$ and $p > 5/2$ if $a \neq 0$.

If $a = 0$, then $C(t) = -k(\alpha t + 1)^{-p}$ for some $k > 0$ and $\alpha > 0$. In this case, we let

$$C_2(t) = \begin{cases} C(t) & \text{if } 0 \leqq t \leqq \beta\,, \quad \beta \geqq 0 \\ -\gamma(\alpha t + 1)^{-q} & \text{if } t \geqq \beta\,, \end{cases}$$

where $\gamma = k(\alpha\beta + 1)^{q-p}$ and $q > \max(2, p)$. We choose β and q so that $J < 1$, $L = A + \int_0^\infty C_2(v)dv < 0$, and $P(J + 1) + (J - 1)|L| < 0$.

If $a \neq 0$, then $C(t)$ changes sign. We may then write $C(t)$ as

$$C(t) = k_1/(\alpha t + 1)^{p-1} - k_2/(\alpha t + 1)^p\,, \quad 0 < k_1 < k_2$$

and let $C_1(t) = k_1/(\alpha t + 1)^{p-1}$ and $C_2(t) = -k_2/(\alpha t + 1)^p$.

EXAMPLE 6. Consider the equation

$$x' = (1/5)x - (1/3)\int_0^t (t - s + 1)^{-2}x(s)ds\,.$$

Here, $C(t) = -(1/3)(t + 1)^{-2}$. Thus, by choosing $\beta = 6$ and $q = 4$, we have $A + \int_0^\infty C_2(v)dv = -32/315$, $P = 2/63$, and $J = (42 \ln 7 - 17)/126 < 1$. Thus, all conditions of Theorem 4 are satisfied and the zero solution is uniformly asymptotically stable.

EXAMPLE 7. Consider the equation

$$x' = Ax + \int_0^t [(3t - 3s + 1)^{-2} - \gamma(3t - 3s + 1)^{-3}]x(s)ds\,,$$

$0 < \gamma < 18$.

Letting $C_1(t) = (3t + 1)^{-2}$ and $C_2(t) = -\gamma(3t + 1)^{-3}$, we have $J = \gamma/18$, $P = 1/3$, and $A + \int_0^\infty C_2(v)dv = A - \gamma/6$. If $A < (1/3)[\gamma/2 + (\gamma + 18)/(\gamma - 18)]$, then all conditions of Theorem 4 are satisfied and the zero solution is uniformly asymptotically stable. Observe that the right-hand side of the above inequality is an upper bounded for A which is maximum for $\gamma = 18 - 6\sqrt{2}$. Thus, for this choice of γ, we can take $A = 1/2$.

3. Vector equations.

We consider the system

$$x' = Lx + L_1(t)x + \int_0^t C_1(t, s)x(s)ds + \frac{d}{dt}\int_0^t H(t, s)x(s)ds \tag{3.1}$$

where L is a constant $n \times n$ matrix, $L_1(t)$ is an $n \times n$ matrix continuous for $0 \leqq t < \infty$, $C_1(t, s)$ and $H(t, s)$ are $n \times n$ matrices continuous for $0 \leqq s \leqq t < \infty$, and $n \geqq 1$.

If B is any positive definite $n \times n$ matrix, then there is a positive constant k such that

$$k|x|^2 \leqq x^T Bx \quad \text{for all} \quad x \in R^n . \tag{3.2}$$

Let D be an $n \times n$ symmetric matrix satisfying

$$L^T D + DL = -I . \tag{3.3}$$

For a detailed discussion of the existence of such a matrix D, see [1] and [3].

Let

$$P = \sup_{t \geqq 0} \int_0^t |C_1(t, s)| ds , \tag{3.4}$$

$$J = \sup_{t \geqq 0} \int_0^t |H(t, s)| ds , \tag{3.5}$$

$$m = \sup_{t \geqq 0} |L_1(t)| , \tag{3.6}$$

and

$$\Phi(t, s) = \int_t^\infty [(1 + J)|C_1(u, s)| + (|L| + P + m)|H(u, s)|]du \tag{3.7}$$

assuming that $\Phi(t, s)$ exists for $0 \leqq s \leqq t < \infty$.

THEOREM 5. *Let D, P, J, m and Φ be defined by* (3.3)-(3.7) *and suppose that for some constant α,*

(i) $J < 1$

and

(ii) $|D|[P + 2m + J(m + |L|) + \Phi(t, t)] \leqq \alpha < 1$.

In addition, suppose there is a continuous function $h: [0, \infty) \to [0, \infty)$ such that $|H(t, s)| \leqq h(t - s)$ and $h(u) \to 0$ as $u \to \infty$. Then the zero solution of (3.1) *is stable if and only if D is positive definite.*

PROOF. Let

$$V(t, x(\cdot)) = \left(x - \int_0^t H(t, s)x(s)ds\right)^T D\left(x - \int_0^t H(t, s)x(s)ds\right) + |D| \int_0^t \Phi(t, s)|x(s)|^2 ds .$$

The derivative of $V(t, x(\cdot))$ along a solution $x(t) = x(t, t_0, \phi)$ of (3.1) is given by

$$V'_{(3.1)}(t, x(\cdot)) = \left[x^T L^T + x^T L_1^T(t) + \left(\int_0^t C_1(t, s)x(s)ds\right)^T\right] D\left(x - \int_0^t H(t, s)x(s)ds\right)$$
$$+ \left[x^T - \left(\int_0^t H(t, s)x(s)ds\right)^T\right] D\left[Lx + L_1(t)x + \int_0^t C_1(t, s)x(s)ds\right]$$
$$+ |D|\frac{d}{dt}\int_0^t \Phi(t, s)\,|x(s)|^2 ds\ .$$

Using (3.3), we may write

$$V'_{(3.1)}(t, x(\cdot)) \leqq -|x|^2 + 2|D|\,|L_1(t)|\,|x|^2 + 2|D|\int_0^t |C_1(t, s)|\,|x|\,|x(s)|\,ds$$
$$+ 2|D|\,|L|\int_0^t |H(t, s)|\,|x|\,|x(s)|\,ds$$
$$+ 2|D|\,|L_1(t)|\int_0^t |H(t, s)|\,|x|\,|x(s)|\,ds$$
$$+ 2|D|\int_0^t |C_1(t, s)|\,|x(s)|\,ds \int_0^t |H(t, s)|\,|x(s)|\,ds$$
$$+ |D|\frac{d}{dt}\int_0^t \Phi(t, s)\,|x(s)|^2 ds\ .$$

Applying the Schwarz inequality as in the proof of Theorem 1, we obtain

$$V'_{(3.1)}(t, x(\cdot)) \leqq -|x|^2 + 2m|D|\,|x|^2 + |D|\int_0^t |C_1(t, s)|\,(|x|^2 + |x(s)|^2)ds$$
$$+ |D|(|L| + m)\int_0^t |H(t, s)|\,(|x|^2 + |x(s)|^2)ds$$
$$+ |D|\,J\int_0^t |C_1(t, s)|\,|x(s)|^2 ds + |D|\,P\int_0^t |H(t, s)|\,|x(s)|^2 ds$$
$$+ |D|\,\Phi(t, t)\,|x|^2 + |D|\int_0^t [(\partial/\partial t)\Phi(t, s)]|x(s)|^2 ds\ .$$

By (3.4)-(3.7), we have

$$V'_{(3.1)}(t, x(\cdot)) \leqq \{-1 + |D|\,[2m + P + (m + |L|)J + \Phi(t, t)]\}\,|x|^2$$
$$+ |D|\int_0^t [(1 + J)|C_1(t, s)| + (|L| + P + m)|H(t, s)|]|x(s)|^2 ds$$
$$+ |D|\int_0^t [(\partial/\partial t)\Phi(t, s)]|x(s)|^2 ds$$
$$\leqq (-1 + \alpha)|x|^2 = -\gamma\,|x|^2\,,\quad \gamma > 0\ .$$

Suppose that D is positive definite and let $\varepsilon > 0$ and $t_0 \geqq 0$ be given. We must find $\delta > 0$, $\delta < \varepsilon$, so that if $\phi\colon [0, t_0] \to R^n$ is continuous with $|\phi(t)| < \delta$, then $|x(t, t_0, \phi)| < \varepsilon$ for all $t \geqq t_0$. As $V'_{(3.1)}(t, x(\cdot)) \leqq 0$, then

$$V(t, x(\cdot)) \leqq V(t_0, \phi(\cdot)) \leqq |D|\Big[|\phi(t_0)| + \int_0^{t_0} |H(t_0, s)|\,|\phi(s)|\,ds\Big]^2$$
$$+ |D| \int_0^{t_0} \Phi(t_0, s)|\phi(s)|^2 ds$$
$$\leqq \delta^2 N^2 \quad \text{for some constant} \quad N = N(t_0)\ .$$

Using (3.2), we may write

$$V(t, x(\cdot)) \geqq \Big(x - \int_0^t H(t, s)x(s)ds\Big)^T D\Big(x - \int_0^t H(t, s)x(s)ds\Big)$$
$$\geqq r^2\Big(|x| - \int_0^t |H(t, s)|\,|x(s)|\,ds\Big)^2 \quad \text{for some} \quad r > 0\ .$$

Thus,

$$|x(t)| \leqq (\delta N/r) + \int_0^t |H(t, s)|\,|x(s)|\,ds\ .$$

As long as $|x(t)| < \varepsilon$, we have $|x(t)| \leqq (\delta N/r) + \varepsilon J < \varepsilon$ for all $t \geqq t_0$ provided that $\delta < \varepsilon r(1 - J)/N$. Thus, the zero solution of (3.1) is stable.

Now, suppose that $x = 0$ is stable but D is not positive definite. Then there is an $x_0 \neq 0$ such that $x_0^T D x_0 \leqq 0$. If k is any non-zero constant, then $x_1 = kx_0 \neq 0$ and $x_1^T D x_1 \leqq 0$. Let $\varepsilon = 1$ and $t_0 = 0$. Since $x = 0$ is stable, then there is a $\delta > 0$ and $x_1 \neq 0$ such that $|x_1| < \delta$ and $|x(t, 0, x_1)| < \varepsilon$ for all $t \geqq 0$. Let $x(t) = x(t, 0, x_1)$. Then along the solution $x(t)$, we have $V(0, x_1) = x_1^T D x_1 \leqq 0$ and $V'(t, x(\cdot)) \leqq -\gamma|x|^2$ so that for some $t_1 > 0$, $V(t_1, x(\cdot)) < 0$ and

$$V(t, x(\cdot)) \leqq V(t_1, x(\cdot)) - \gamma \int_{t_1}^t |x(s)|^2 ds = -\eta - \gamma \int_{t_1}^t |x(s)|^2 ds\ ,$$

where $\eta = -V(t_1, x(\cdot)) > 0$ and $t \geqq t_1$. Thus,

$$\Big[x(t) - \int_0^t H(t, s)x(s)ds\Big]^T D\Big[x(t) - \int_0^t H(t, s)x(s)ds\Big] \tag{3.8}$$
$$\leqq V(t, x(\cdot)) \leqq -\eta - \gamma \int_{t_1}^t |x(s)|^2 ds\ .$$

Since $|x(t)| < 1$ for all $t \geqq 0$ and

$$\eta + \gamma \int_{t_1}^t |x(s)|^2 ds \leqq |D|\Big[|x(t)| + \int_0^t |H(t, s)|\,|x(s)|\,ds\Big]^2 \leqq |D|(1 + J)^2 \leqq 4|D|\ ,$$

then $|x(t)|^2$ is in $L^1[0, \infty)$. By the Schwarz inequality,

$$\Big[\int_0^t |H(t, s)|\,|x(s)|\,ds\Big]^2 \leqq \int_0^t |H(t, s)|\,ds \int_0^t |H(t, s)|\,|x(s)|^2 ds \leqq J \int_0^t h(t - s)|x(s)|^2 ds\ .$$

The last integral tends to zero as it is the convolution of an L^1-function with a function tending to zero. Thus, $\int_0^t |H(t, s)||x(s)|ds \to 0$ as $t \to \infty$ and hence, by (3.8), $x^T(t)Dx(t) \leqq -\eta/2$ for all sufficiently large t. This implies that $|x(t)|^2 \geqq \mu$ for some $\mu > 0$ and all sufficiently large t, a contradiction to $|x(t)|^2$ being L^1. Thus, the assumption that D is not positive definite is false and the proof is now complete.

Let us apply Theorem 5 to the system (1.1) assuming that $A(t) = A =$ constant and the matrix D satisfies

$$A^TD + DA = -I\,. \tag{3.9}$$

Taking $L_1(t) = H(t, s) \equiv 0$ in Theorem 5, we obtain the following, which is [3, Theorem 8]:

COROLLARY 5. *Suppose that $A(t) = A =$ constant, D satisfies* (3.9), *and there is a constant α such that*

$$|D|\left[\int_0^t |C(t, s)|\,ds + \int_t^\infty |C(u, t)|\,du\right] \leqq \alpha < 1\,.$$

Then the zero solution of (1.1) *is stable if and only if D is positive definite.*

Now, suppose that $A(t)$ is not constant and $\int_t^\infty A(v)dv$ exists and in $L^1[0, \infty)$. Letting $C_1(t, s) = C(t, s) - A(t - s)$ and $H(t) = -\int_t^\infty A(v)dv$, Equation (1.1) takes the form

$$x' = -H(0)x + A(t)x + \int_0^t C_1(t, s)x(s)ds + \frac{d}{dt}\int_0^t H(t - s)x(s)ds$$

with

$$P = \sup_{t\geqq 0}\int_0^t |C(t, s) - A(t - s)|\,ds\,, \tag{3.10}$$

$$J = \int_0^\infty\left|\int_t^\infty A(v)dv\right|dt\,, \tag{3.11}$$

$$m = \sup_{t\geqq 0}|A(t)|\,, \tag{3.12}$$

and

$$\int_0^\infty [A^T(t)D + DA(t)]dt = -I\,. \tag{3.13}$$

THEOREM 6. *Let P, J, m and D be defined by* (3.10)-(3.13) *and suppose there is a constant α such that*

$$(3.14)\qquad |D|\left\{2\left[m + J\left(m + \left|\int_0^\infty A(t)dt\right|\right)\right] + (1 + J)\left[P + \int_t^\infty |C(u, t) - A(u - t)|du\right]\right\} \leqq \alpha < 1\ .$$

Then the zero solution of (1.1) *is stable if and only if D is positive definite.*

PROOF. As $H(t, s) = H(t - s)$, then $\int_t^\infty |H(u, t)|du = \int_0^\infty |H(v)|dv = J$. Also, $L = -H(0) = \int_0^\infty A(v)dv$ and (ii) of Theorem 5 reduces to (3.14). But, (3.14) implies that $2|D|J\left|\int_0^\infty A(t)dt\right| < 1$ and (3.13) implies that $2|D|\left|\int_0^\infty A(t)dt\right| \geqq n$. Thus, $J < (1/n) \leqq 1$ and hence (i) of Theorem 5 is satisfied. Taking $h(t) = H(t)$, we have all the conditions of Theorem 5 satisfied. This completes the proof.

In the special case where $C(t, s) = C(t - s)$, Equation (1.1) reduces to

$$(3.15)\qquad x' = A(t)x + \int_0^t C(t - s)x(s)ds$$

and P of (3.10) reduces to

$$(3.16)\qquad P = \int_0^\infty |C(v) - A(v)|dv\ .$$

THEOREM 7. *Let J, m, D and P be defined by* (3.11)-(3.13) *and* (3.16) *respectively, and suppose that*

$$(3.17)\qquad (m + P)(J + 1) + J\left|\int_0^\infty A(t)dt\right| < 1/(2|D|)\ .$$

Then the zero solution of (3.15) *is asymptotically stable if and only if D is positive definite.*

PROOF. As $C(t, s) = C(t - s)$, then

$$\int_t^\infty |C(u, t) - A(u - t)|du = \int_0^\infty |C(v) - A(v)|dv = P\ .$$

Thus, (3.17) implies (3.14) and the stability or instability of (3.15) follows from Theorem 6. To show asymptotic stability, we observe from (3.17) that m and P are bounded. Hence, by (3.16) and the assumption on $A(t)$, it follows that $C(t)$ is in $L^1[0, \infty)$. As $x = 0$ is stable, then by (3.15), $x'(t)$ is bounded. Using the functional $V(t, x(\cdot))$ in the proof of Theorem 5, we see that along a solution $x(t) = x(t, t_0, \phi)$ of (3.15), $V'(t, x(\cdot)) \leqq -\delta|x|^2$, $\delta > 0$. Thus, $|x(t)|^2$ is in $L^1[0, \infty)$ and since $(|x(t)|^2)' = (x^T)'x + x^Tx'$ is bounded, then $x(t) \to 0$ as $t \to \infty$. This completes the proof.

COROLLARY 6. *Let J, m and P be defined by* (3.11), (3.12), *and* (3.16) *respectively, and suppose that $n = 1$ and*

$$(m + P)(J + 1) + (J - 1)\left|\int_0^\infty A(t)dt\right| < 0 \,. \tag{3.18}$$

Then the zero solution of (3.15) *is asymptotically stable if and only if* $\int_0^\infty A(t)dt < 0$.

PROOF. For $n = 1$ we have by (3.13), $2|D|\left|\int_0^\infty A(t)dt\right| = 1$. Thus, (3.17) reduces to (3.18) and this completes the proof.

EXAMPLE 6. Consider the equation

$$x' = -(1/2)\alpha^2 e^{-\alpha t}x - (1/2)\alpha^2 \int_0^t e^{-\alpha(t-s)}x(s)ds \,, \quad 0 < \alpha < 1/3 \,.$$

Here, $C(t) = A(t) = -\alpha^2 e^{-\alpha t}/2$. Thus, $P = 0$, $m = \alpha^2/2$, $J = 1/2$, and $\int_0^\infty A(t)dt = -\alpha/2 < 0$. Hence

$$(m + P)(J + 1) + (J - 1)\left|\int_0^\infty A(t)dt\right| = \alpha(3\alpha - 1)/4 < 0 \,,$$

and, by Corollary 6, the zero solution is asymptotically stable.

Now, we present another interesting application of Theorem 5. Consider the scalar equation

$$x' = Ax + \int_0^t [k(t - s) + C(t, s)]x(s)ds \tag{3.19}$$

in which A is a constant, $C(t, s)$ is continuous for $0 \leqq s \leqq t < \infty$, and $k: [0, \infty) \to (-\infty, \infty)$ is differentiable with k' in $L^1[0, \infty)$.

We differentiate (3.19) to obtain

$$x'' = Ax' + k(0)x + \int_0^t k'(t - s)x(s)ds + \frac{d}{dt}\int_0^t C(t, s)x(s)ds \,.$$

Let $x' = y$. Thus,

$$y' = k(0)x + Ay + \int_0^t k'(t - s)x(s)ds + \frac{d}{dt}\int_0^t C(t, s)x(s)ds \,.$$

Then we have

$$z' = Lz + \int_0^t C_1(t - s)z(s)ds + \frac{d}{dt}\int_0^t H(t, s)z(s)ds \,,$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad C_1(t) = \begin{pmatrix} 0 & 0 \\ k'(t) & 0 \end{pmatrix}, \quad H(t, s) = \begin{pmatrix} 0 & 0 \\ C(t, s) & 0 \end{pmatrix},$$

and

$$L = \begin{pmatrix} 0 & 1 \\ k(0) & A \end{pmatrix}. \tag{3.20}$$

Thus,

$$P = \int_0^\infty |k'(v)|\,dv \tag{3.21}$$

and

$$J = \sup_{t \geqq 0} \int_0^t |C(t, s)|\,ds . \tag{3.22}$$

If $A < 0$ and $k(0) < 0$, then L is a stable matrix and there is a symmetric positive definite matrix D such that

$$L^T D + DL = -I . \tag{3.23}$$

THEOREM 8. *Let L, P, J and D be defined by* (3.20)-(3.23). *If $A < 0$, $k(0) < 0$, and there is a constant α such that*

(i) $$J < 1$$

and

(ii) $$|D|\left[P(J + 1) + P + |L|J + (P + |L|)\int_t^\infty |C(u, t)|\,du\right] \leqq \alpha < 1 ,$$

then the zero solution of (3.19) *is stable.*

PROOF. If we observe that the condition $|H(t, s)| \leqq h(t - s)$ is needed only to prove the converse of Theorem 5, then Theorem 8 is an immediate consequence of Theorem 5.

REMARK 4. There is no integrability condition on the kernel in (3.19). Thus, if we take $k(t) = k =$ constant and $C(t, s) = C(t - s)$, then (3.19) reduces to

$$x' = Ax + \int_0^t [k + C(t - s)]x(s)ds . \tag{3.24}$$

In this case $P = 0$,

$$J = \int_0^\infty |C(v)|\,dv , \tag{3.25}$$

and Theorem 8 reduces to the following:

COROLLARY 7. *Let L, D and J be defined by* (3.20), (3.23), and (3.25) *respectively. If $A < 0$, $k < 0$, and*

$$2|L||D|J < 1 ,$$

then the zero solution of (3.24) *is stable.*

EXAMPLE 7. Consider the equation

$$x' = -x + \int_0^t [-\alpha + \beta(t - s + 1)^{-2}]x(s)ds ,$$

where α and β are positive constants.

As

$$L = \begin{pmatrix} 0 & 1 \\ -\alpha & -1 \end{pmatrix} ,$$

a simple calculation yields

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a = (\alpha + 1 + 1/\alpha)/2$, $b = 1/(2\alpha)$ and $c = (1 + 1/\alpha)/2$. Thus, by Corollary 7, the zero solution is stable if $\beta(\alpha^2 + 4\alpha + 8 + 8/\alpha) < 1$.

The following result is an extension of Theorem 4 to the convolution system

$$x' = Ax + \int_0^t C(t - s)x(s)ds , \tag{3.26}$$

where A is an $n \times n$ constant matrix and $C(t)$ is an $n \times n$ matrix continuous for $0 \leqq t < \infty$.

Let

$$C(t) = C_1(t) + C_2(t) ,$$

where $C_i(t)$, $i = 1, 2$, are continuous for $0 \leqq t < \infty$. We assume that $C_1(t)$ and $\int_t^\infty C_2(v)dv$ are L^1-functions and let

$$P = \int_0^\infty |C_1(v)|dv , \tag{3.27}$$

$$J = \int_0^\infty \left| \int_t^\infty C_2(v)dv \right| dt , \tag{3.28}$$

and

$$L = A + \int_0^\infty C_2(v)dv . \tag{3.29}$$

Let D be an $n \times n$ symmetric matrix satisfying

$$L^T D + DL = -I . \tag{3.30}$$

THEOREM 9. *Let P, J, L and D be defined by* (3.27)-(3.30).

(i) *If $P(J+1) + J|L| < 1/(2|D|)$, then the zero solution of* (3.26) *is stable if and only if D is positive definite.*

(ii) *If, in addition, $C(t)$ is in $L^j[0, \infty)$, $j = 1$ or $j = 2$, then the solution of* (3.26) *is asymptotically stable if and only if D is positive definite.*

(iii) *If, furthermore, $J \neq 0$ and $\int_t^\infty |C(v)|dv$ is in $L^q[0, \infty)$, $0 < q < 2$, then the zero solution of* (3.26) *is uniformly asymptotically stable if and only if D is positive definite.*

PROOF. Let $H(t) = -\int_t^\infty C_2(v)dv$. Then (3.26) may be written as

$$x' = Lx + \int_0^t C_1(t-s)x(s)ds + \frac{d}{dt}\int_0^t H(t-s)x(s)ds .$$

This is (3.1) with $L_1(t) \equiv 0$, $C_1(t, s) = C_1(t-s)$ and $H(t, s) = H(t-s)$. Thus $m = 0$, $\int_t^\infty |C_1(u, t)|du = \int_0^\infty |C_1(v)|dv = P$, and $\int_t^\infty |H(u, t)|du = \int_0^\infty |H(v)|dv = J$. Hence, (ii) of Theorem 5 reduces to $P(J+1) + J|L| < 1/(2|D|)$. This condition with (3.30) implies that $J < 1$. If we let $h(t) = H(t)$, then Part (i) of the theorem follows from Theorem 5. Parts (ii) and (iii) are proved exactly in the same way as the corresponding parts of Theorem 4.

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