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INJECTIVE ENVELOPES OF C^*-DYNAMICAL SYSTEMS*

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Abstract. The injective envelope $I(A)$ of a C^*-algebra A is a unique minimal injective C^*-algebra containing A. As a dynamical system version of the injective envelope of a C^*-algebra we show that for a C^*-dynamical system (A, G, α) with G discrete there is a unique maximal C^*-dynamical system (B, G, β) "containing" (A, G, α) so that $A\times_{\alpha r}G\subset B\times_{\beta r}G\subset I(A\times_{\alpha r}G)$, where $A\times_{\alpha r}G$ is the reduced C^*-crossed product of A by G. As applications we investigate the relationship between the original action α on A and its unique extension $I(\alpha)$ to $I(A)$. In particular, a *-automorphism α of A is quasi-inner in the sense of Kishimoto if and only if $I(\alpha)$ is inner.

1. Introduction. In [10], [12], [13] the author introduced the notion of the *injective envelope* $I(A)$ (resp. *regular monotone completion* $\bar{A}$) of a (not necessarily unital) C^*-algebra A. (Note that a few authors call this $\bar{A}$ the regular completion of A and use the confusing notation $\hat{A}$ instead of $\bar{A}$. But $\hat{A}$ was originally used by Wright [33] to denote the regular σ-completion of A, which is properly contained in $\bar{A}$ in general.) The algebra $I(A)$ is a unique minimal injective C^*-algebra containing A^1 as a C^*-subalgebra with the same unit, where A^1 denotes the C^*-algebra obtained by adjoining a unit to A if A is non-unital and $A\neq\{0\}$, and denotes A itself otherwise. On the other hand, $\bar{A}$ is a unique monotone complete C^*-algebra such that $\bar{A}$ is the monotone closure of A and each $x\in\bar{A}_{sa}$ (the self-adjoint part of $\bar{A}$) is the supremum in $\bar{A}_{sa}$ of the set $\{a\in A^1_{sa}: a\leqq x\}$, where a C^*-algebra B is called *monotone complete* if each bounded increasing net in B_{sa} has a supremum in B_{sa}, and the *monotone closure* of a C^*-subalgebra C of B is the smallest C^*-subalgebra of B containing C which is closed under the formation of suprema in B_{sa} of bounded increasing nets. Moreover, $\bar{A}$ is realized as the monotone closure of A in $I(A)$ and we have canonically $A\subset\bar{A}\subset I(A)$.

The algebra $I(A)$ or $\bar{A}$, being monotone complete AW^*, is more tractable than the original C^*-algebra A and is small enough to inherit some properties of A. For example, $I(A)$ or $\bar{A}$ is an AW^*-factor if and only if A is prime [12, 7.1, 6.3], and if A is unital and simple, then any

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C^*-subalgebra of $I(A)$ containing A is also simple [15, 1.2(i)]. Moreover, each $*$-automorphism α of A extends uniquely to a $*$-automorphism $\bar{\alpha}$ of $\bar{A}$ (resp. $I(\alpha)$ of $I(A)$) with $I(\alpha)|\bar{A} = \bar{\alpha}$ and so we have canonically $\mathrm{Aut}\, A \subset \mathrm{Aut}\, \bar{A} \subset \mathrm{Aut}\, I(A)$ as subgroups, where $\mathrm{Aut}\, A$ denotes the group of all $*$-automorphisms of A.

Throughout the paper (unless stated otherwise) G denotes a fixed *discrete* group, and for C^*-dynamical systems (A, G, α) and (B, G, β) the notation $(A, G, \alpha) \subset (B, G, \beta)$ means that A is a G-invariant C^*-subalgebra of B and $\beta | A = \alpha$. For a C^*-dynamical system (A, G, α), take the injective envelope $I(A \times_{\alpha r} G)$ of the reduced C^*-crossed product $A \times_{\alpha r} G$ of A by G and consider the C^*-subalgebras of $I(A \times_{\alpha r} G)$ which are of the form $B \times_{\beta r} G$ with $(A, G, \alpha) \subset (B, G, \beta)$. The main result of this paper (Theorem 3.4) states that there is a unique maximal C^*-dynamical system $(I_G(A), G, I_G(\alpha))$ among such C^*-dynamical systems (B, G, β). By putting $\bar{\alpha}_t = (\alpha_t)^-$ and $I(\alpha)_t = I(\alpha_t)$, $t \in G$, we obtain C^*-dynamical systems $(\bar{A}, G, \bar{\alpha}) \subset (I(A), G, I(\alpha))$. We have $(I(A), G, I(\alpha)) \subset (I_G(A), G, I_G(\alpha))$ and it follows that $A \times_{\alpha r} G \subset I(A) \times_{I(\alpha) r} G \subset I(A \times_{\alpha r} G)$ and $\bar{A} \times_{\bar{\alpha} r} G \subset (A \times_{\alpha r} G)^-$. This fact is crucial in later discussions.

This paper is arranged as follows. In Section 2, $I_G(A)$ is constructed first as the "injective envelope" of A in the category of operator systems on which G acts as unital complete order isomorphisms and unital completely positive G-module homomorphisms, and then in Section 3 the maximality of $(I_G(A), G, I_G(\alpha))$ in the above sense is established. In Section 7 we show that for a $*$-automorphism α of A its extension $I(\alpha)$ to $I(A)$ is inner if and only if α is quasi-inner in the sense of Kishimoto. In Section 8 some of the conditions in [26, 10.4] which characterize the $*$-automorphism with Connes spectrum equal to the full circle group are shown to hold also in the nonseparable case. Finally in Section 10 a criterion is given for the primeness of reduced C^*-crossed products.

The reader is referred to [2] for the general theory of AW^*-algebras and to [27] for that of automorphisms and crossed products of C^*-algebras.

2. G-injective envelopes.

The statements and proofs of the results in this section parallel closely those in [11], if one replaces operator systems and completely positive maps there by G-modules and G-morphisms defined below, and so most of the proofs are omitted.

The terminologies in [5], [11] will be used without further explanation. For an operator system V we denote the injective envelope of V by $I(V)$ and the group of all unital complete order isomorphisms of V onto itself by $\mathrm{Aut}\, V$. For the same reason for the case of C^*-algebras we have $\mathrm{Aut}\, V \subset \mathrm{Aut}\, I(V)$ as a subgroup.

An operator system V is called a *G-module* if it is made into a left G-module by a group homomorphism $G \ni t \mapsto (x \mapsto t\cdot x) \in \operatorname{Aut} V$. A *$G$-morphism* is a unital completely positive G-module homomorphism between G-modules. A G-morphism is called a *G-isomorphism* (resp. *G-monomorphism*) if it is a complete order isomorphism (resp. complete order injection). A *G-submodule* V of a G-module W is a G-module contained in W such that the inclusion map $V \hookrightarrow W$ is a G-monomorphism. We consider the category of all G-modules and all G-morphisms and define the injectivity of its object as follows. A G-module V is *G-injective* if for any G-monomorphism $\kappa: W \to Z$ and any G-morphism $\phi: W \to V$ there is a G-morphism $\hat{\phi}: Z \to V$ with $\hat{\phi}\circ\kappa = \phi$. A *$G$-extension* of a G-module V is a pair (W, κ) of a G-module W and a G-monomorphism $\kappa: V \to W$. The G-extension (W, κ) is *G-injective* if W is G-injective, and it is *G-essential* (resp. *G-rigid*) if for any G-morphism $\phi: W \to Z$, ϕ is a G-monomorphism whenever $\phi\circ\kappa$ is (resp. for any G-morphism $\phi: W \to W$, $\phi\circ\kappa = \kappa$ implies $\phi = \mathrm{id}_W$, the identity map on W).

DEFINITION 2.1. The *G-injective envelope* of a G-module is a G-extension which is both G-injective and G-essential.

For an operator system $V \subset B(H)$ with H a Hilbert space the space $l^\infty(G, V)$ of all bounded functions of G into V is viewed as an operator system on $l^2(G) \otimes H$, and it becomes a G-module by the action $(t\cdot x)(s) = x(t^{-1}s)$, $t, s \in G$, $x \in l^\infty(G, V)$.

LEMMA 2.2. *With the above notations if V is an injective operator system, then the G-module $l^\infty(G, V)$ is G-injective.*

PROOF. Let $\kappa: W \to Z$ (resp. $\phi: W \to l^\infty(G, V)$) be a G-monomorphism (resp. G-morphism) and define a completely positive map $\psi: W \to V$ by $\psi(x) = \phi(x)(e)$ (e is the identity element of G). As V is injective, there is a completely positive map $\hat{\psi}: Z \to V$ with $\hat{\psi}\circ\kappa = \psi$. Then the map $\hat{\phi}: Z \to l^\infty(G, V)$, $\hat{\phi}(x)(t) = \hat{\psi}(t^{-1}\cdot x)$, $t \in G$, $x \in Z$, is a G-morphism with $\hat{\phi}\circ\kappa = \phi$.

REMARK 2.3. For any G-module $V \subset B(H)$ the map $j: V \to l^\infty(G, B(H))$, $j(x)(t) = t^{-1}\cdot x$, $x \in V$, $t \in G$, is a G-monomorphism with $j(V) \subset l^\infty(G, V) \subset l^\infty(G, B(H))$, and $l^\infty(G, B(H))$ is injective as an operator system (resp. G-injective as a G-module). This shows that each G-module has a G-injective G-extension. Moreover if V is G-injective, then there is an idempotent G-morphism of $l^\infty(G, B(H))$ onto $j(V)$ and so V is injective. Hence V is G-injective if and only if V is injective and there is a G-morphism $\phi: l^\infty(G, V) \to V$ with $\phi\circ j = \mathrm{id}_V$.

We proceed to the proof of the unique existence of the G-injective envelope. Let $V \subset W \subset B(H)$ be two fixed G-modules with W G-injective and containing V as a G-submodule. A *V-projection* on W is an idempotent G-morphism $\phi: W \to W$ with $\phi | V = \mathrm{id}_V$. A *V-seminorm* on W is a seminorm p on W such that $p = \|\phi(\cdot)\|$ for some G-morphism $\phi: W \to W$ with $\phi | V = \mathrm{id}_V$. Define a partial ordering $\prec$ (resp. $\leqq$) on the set of all V-projections (resp. V-seminorms) on W by $\phi \prec \psi$ (resp. $p \leqq q$) if and only if $\phi \circ \psi = \psi \circ \phi = \phi$ (resp. $p(x) \leqq q(x)$ for all $x \in W$).

LEMMA 2.4 (cf. [11, 3.4–3.7]). (i) *Any decreasing net $\{p_i\}$ of V-seminorms on W has a lower bound. Hence Zorn's lemma implies the existence of a minimal V-seminorm on W.*

(ii) *There is a minimal V-projection on W.*

(iii) *A G-injective G-extension of V is G-essential if and only if it is G-rigid.*

PROOF. We sketch only the proof of (i). It is almost the same as the one in [11, 3.4]; but the crucial point here is to show that the completely positive map defining the lower bound is a G-module homomorphism. By 2.3 we may regard W as a G-submodule of $l^\infty(G, B(H))$. If $\phi_i: W \to W \subset l^\infty(G, B(H))$ corresponds to p_i, then a subnet of $\{\phi_i\}$ converges in the point-σ-weak topology to a map $\phi_0: W \to l^\infty(G, B(H))$, which is a G-morphism since the action of G on $l^\infty(G, B(H))$ is σ-weakly continuous. Hence, composing ϕ_0 with an idempotent G-morphism of $l^\infty(G, B(H))$ onto W, we obtain a G-morphism which gives the lower bound.

This lemma shows as in [11] that for a minimal V-projection ϕ on W the pair $(\mathrm{Im}\,\phi, \kappa)$ is the G-injective envelope of V, where $\mathrm{Im}\,\phi = \phi(W)$ and κ is the inclusion map, and that $\mathrm{Im}\,\phi$ is an injective C^*-algebra equipped with the multiplication $\circ$ given by $x \circ y = \phi(xy)$, where W, being injective, is viewed as a C^*-algebra and xy is the product in W. Hence we obtain the following result.

THEOREM 2.5 (cf. [11, 4.1]). *Every G-module V has a G-injective envelope, written $(I_G(V), \kappa)$, which is unique in the sense that for any G-injective envelope (Z, λ) of V there is a G-isomorphism $\psi: I_G(V) \to Z$ with $\psi \circ \kappa = \lambda$.*

Henceforth we shall identify V with its image $\kappa(V)$ and abbreviate $(I_G(V), \kappa)$ to $I_G(V)$.

REMARK 2.6. As in [11], $I_G(V)$ is characterized as a unique maximal G-essential (resp. minimal G-injective) G-extension of V.

Let V be a G-module and $I(V)$ the injective envelope of V as an operator system. As $\mathrm{Aut}\,V \subset \mathrm{Aut}\,I(V)$, we may regard $I(V)$ together with the inclusion map $V \hookrightarrow I(V)$ as a G-extension of V. Comparing the essentiality as operator systems and the G-essentiality, we see that $I(V)$ is a G-essential G-extension of V, hence that $V \subset I(V) \subset I_G(V)$ as G-submodules. Moreover it follows easily that $I(V)$ is unique among the G-submodules of $I_G(V)$ which become the injective envelope of V.

3. Injective envelopes of C^*-dynamical systems. Let (A, G, α) be a C^*-dynamical system. In this section, to simplify the notation we assume that A is unital and denote again by α the action $I_G(\alpha)$ of G on the G-injective envelope $I_G(A)$ of A induced by α. But the results below (except for the second part of 3.5 (i)) hold also in the non-unital case. We call $(I_G(A), G, \alpha)$ the *injective envelope* of (A, G, α). We have

$$(A, G, \alpha) \subset (\bar{A}, G, \alpha) \subset (I(A), G, \alpha) \subset (I_G(A), G, \alpha) .$$

Following [14] we construct the monotone complete crossed products associated with (A, G, α). Consider $I_G(A)$ as a C^*-subalgebra, containing the unit, of some $B(H)$, represent each element $x \in B(H \otimes l^2(G))$ by a matrix $x = [x_{r,s}]$ $(r, s \in G)$ over $B(H)$, and define operator systems $I_G(A) \bar{\otimes} B(l^2(G))$, $M(I_G(A), G)$ on $H \otimes l^2(G)$ and maps π_α, λ as follows:

$$I_G(A) \bar{\otimes} B(l^2(G)) = \{x \in B(H \otimes l^2(G)) \colon x_{r,s} \in I_G(A) \quad \text{for all} \quad r, s \in G\} ,$$
$$M(I_G(A), G) = \{x \in I_G(A) \bar{\otimes} B(l^2(G)) \colon \alpha_{t^{-1}}(x_{r,s}) = x_{rt,st} \quad \text{for all} \quad r, s \in G\} ,$$
$$\pi_\alpha \colon I_G(A) \to M(I_G(A), G), \ \pi_\alpha(x) = [\delta_{r,s}\alpha_{r^{-1}}(x)], \ x \in I_G(A) ,$$
$$\lambda \colon G \to M(I_G(A), G), \ \lambda(t) = [\delta_{t^{-1}r,s}1], \ t \in G .$$

Similarly, define $A \bar{\otimes} B(l^2(G))$, $M(A, G)$ and so on as subspaces of $B(H \otimes l^2(G))$. Then π_α is a unital $*$-monomorphism with $\lambda(t)\pi_\alpha(x)\lambda(t)^* = \pi_\alpha(\alpha_t(x))$, $t \in G$, $x \in I_G(A)$; $I_G(A) \bar{\otimes} B(l^2(G))$ is a monotone complete C^*-algebra with the multiplication

$$x \circ y = \Big[O\text{-}\sum_t x_{r,t} y_{t,s} \Big] , \quad x, y \in I_G(A) \bar{\otimes} B(l^2(G)) ,$$

where $O\text{-}\sum_t x_{r,t}y_{t,s}$ denotes the order limit in $I_G(A)$ of the finite sums (and need not coincide with the strong limit $s\text{-}\sum_t x_{r,t}y_{t,s}$ in $B(H)$); and $M(I_G(A), G)$ [resp. $M(\bar{A}, G)$, $M(I(A), G)$] is its monotone closed C^*-subalgebra [13], [14]. Moreover, the reduced C^*-crossed product $A \times_{\alpha r} G$ is identified with the C^*-subalgebra of $M(I_G(A), G)$ generated by $\pi_\alpha(A)\lambda(G)$.

Regard $I_G(A) \bar{\otimes} B(l^2(G))$ as a G-module by the action $t \cdot x = \lambda(t)x\lambda(t)^*$, $t \in G$, $x \in I_G(A) \bar{\otimes} B(l^2(G))$. Then $\pi_\alpha(A) \subset A \times_{\alpha r} G \subset M(A, G) \subset M(I_G(A), G)$ are G-submodules of $I_G(A) \bar{\otimes} B(l^2(G))$, and π_α is a G-monomorphism.

LEMMA 3.1. *Keep the above notation.*

(i) *The embedding* $A \hookrightarrow I_G(A)$ *is normal, that is,* $x_i \nearrow x$ *in* A *implies* $x_i \nearrow x$ *in* $I_G(A)$*, where* $x_i \nearrow x$ *in a* C^**-algebra means that* $\{x_i\}$ *is an increasing net with supremum* x.

(ii) *The map* π_α: $I_G(A) \to M(I_G(A), G)$ *is normal.*

(iii) *For another* C^**-dynamical system* (B, G, β) *and a* G*-morphism* $\phi: A \to B$ *(that is, a unital completely positive map with* $\phi(\alpha_t(x)) = \beta_t(\phi(x))$, $t \in G, x \in A$*) the map*

$$\tilde{\phi}: A \bar{\otimes} B(l^2(G)) \to B \bar{\otimes} B(l^2(G)) ,$$
$$\tilde{\phi}(x) = [\phi(x_{r,s})] , \quad x = [x_{r,s}] \in A \bar{\otimes} B(l^2(G))$$

is a unital completely positive map with $\tilde{\phi}(M(A, G)) \subset M(B, G)$ *and* $\tilde{\phi}(A \times_{\alpha r} G) \subset B \times_{\beta r} G$. *Moreover,* $\tilde{\phi}$ *is a* G*-morphism, and it is a* G*-monomorphism if and only if* ϕ *is.*

PROOF. (i) The embedding $A \hookrightarrow I(A) \xrightarrow{j} l^\infty(G, I(A))$ (see 2.3) is normal by [12, 3.1] and the fact that $j(I(A))$ is clearly monotone closed in $l^\infty(G, I(A))$. Moreover, as $l^\infty(G, I(A))$ is G-injective, we may take $I_G(A)$ so that $j(I(A)) \subset I_G(A) \subset l^\infty(G, I(A))$, from which the conclusion follows.

By definition, (ii) and (iii) are clear.

G-injectivity is characterized as follows. A similar result is known [1] when A is W^*, but G is not necessarily discrete.

LEMMA 3.2. *For a* C^**-dynamical system* (A, G, α) *the* G*-module* A *is* G*-injective if and only if* $M(A, G)$ *is injective.*

PROOF. This follows from [14, 3.1(ii)] and 2.3.

LEMMA 3.3. *Let* E *be a unital* C^**-algebra which is also a* G*-module and let* C *and* D *be* G*-invariant* C^**-subalgebras, containing the unit, of* E *with* $C \subset D \subset E$. *Suppose that* D *is a* G*-essential* G*-extension of* C *and that there are a faithful idempotent* G*-morphism* ρ *of* E *onto* D *(that is,* $\rho(x) = 0$ *with* $x \in E^+$ *implies* $x = 0$*) and a* G*-morphism* $\phi: D \to E$ *with* $\phi | C = \mathrm{id}_C$. *Then* $\phi = \mathrm{id}_D$.

PROOF. The map $\rho \circ \phi: D \to D$ is a G-morphism with $\rho \circ \phi | C = \mathrm{id}_C$. By 2.6 we have $C \subset D \subset I_G(C)$ and $\rho \circ \phi$ extends to a G-morphism $(\rho \circ \phi)\hat{}: I_G(C) \to I_G(C)$ with $(\rho \circ \phi)\hat{} | C = \mathrm{id}_C$. Then $(\rho \circ \phi)\hat{} = \mathrm{id}_{I_G(C)}$ and so $\rho \circ \phi | D = \mathrm{id}_D$. As ϕ is unital and completely positive, for $x \in D$ we have $\phi(x^*)\phi(x) \leqq \phi(x^*x)$ and similarly for ρ. Hence $x^*x = \rho \circ \phi(x^*) \rho \circ \phi(x) \leqq \rho(\phi(x^*)\phi(x)) \leqq \rho \circ \phi(x^*x) = x^*x$ and $\rho(\phi(x^*)\phi(x)) = x^*x$. As ρ is a D-module homomorphism [5, 3.1] and is faithful, for $x \in D$ we have

$$\begin{aligned}\rho((\phi(x)-x)^*(\phi(x)-x)) &= \rho(\phi(x^*)\phi(x)) - \rho\circ\phi(x^*)x - x^*\rho\circ\phi(x) + \rho\circ\phi(x^*x)\\ &= x^*x - x^*x - x^*x + x^*x = 0\end{aligned}$$

and $\phi(x) = x$.

THEOREM 3.4. *For C^*-dynamical systems (A, G, α) and (B, G, β) with $(A, G, \alpha)\subset(B, G, \beta)$ we have $A\times_{\alpha r}G\subset B\times_{\beta r}G\subset I(A\times_{\alpha r}G)$ if and only if $(B, G, \beta)\subset(I_G(A), G, \alpha)$. In particular, $A\times_{\alpha r}G\subset \bar{A}\times_{\alpha r}G\subset I(A)\times_{\alpha r}G\subset I_G(A)\times_{\alpha r}G\subset I(A\times_{\alpha r}G)$.*

PROOF. Recall that the injective envelope of an operator system is characterized as a maximal essential extension and similarly for the G-injective envelope (see 2.6).

Necessity: It suffices to show that if $B\times_{\beta r}G$ is an essential extension of $A\times_{\alpha r}G$, then B is a G-essential G-extension of A, that is, a G-morphism $\phi: B\to C$ with C a G-module is a G-monomorphism whenever $\phi|A$ is. Lemma 3.1(iii) shows the existence of a completely positive map $\tilde{\phi}|B\times_{\beta r}G: B\times_{\beta r}G\to C\times_{\iota r}G$, where $\iota_t(x) = t\cdot x$, $t\in G$, $x\in C$. If $\phi|A$ is a G-monomorphism, then $\tilde{\phi}|A\times_{\alpha r}G$ is a complete order injection and so is $\tilde{\phi}|B\times_{\beta r}G$ by hypothesis. Hence ϕ is a G-monomorphism.

Sufficiency: It suffices to show that $A\times_{\alpha r}G\subset I_G(A)\times_{\alpha r}G\subset I(A\times_{\alpha r}G)$. As $A\times_{\alpha r}G\subset I_G(A)\times_{\alpha r}G\subset M(I_G(A), G)$ with $M(I_G(A), G)$ injective, we may take $I(A\times_{\alpha r}G)$ so that $A\times_{\alpha r}G\subset I(A\times_{\alpha r}G)\subset M(I_G(A), G)$. The identity map on $A\times_{\alpha r}G$ extends to a completely positive map $\psi: I_G(A)\times_{\alpha r}G\to I(A\times_{\alpha r}G)$. The map $\rho: M(I_G(A), G)\to\pi_\alpha(I_G(A))$, $\rho(x)=\pi_\alpha(x_{e,e})$, $x=[x_{r,s}]\in M(I_G(A), G)$ is a faithful idempotent G-morphism onto $\pi_\alpha(I_G(A))$. Applying 3.3 to the G-modules $\pi_\alpha(A)\subset\pi_\alpha(I_G(A))\subset M(I_G(A), G)$ and the maps $\phi=\psi|\pi_\alpha(I_G(A))$ and ρ, we see that ϕ is the identity map on $\pi_\alpha(I_G(A))$, hence that ψ is a $\pi_\alpha(I_G(A))$-module homomorphism [5, 3.1]. As $I_G(A)\times_{\alpha r}G$ is generated by $\pi_\alpha(I_G(A))$ and $\lambda(G)$, ψ fixes $I_G(A)\times_{\alpha r}G$ elementwise and so $I_G(A)\times_{\alpha r}G\subset I(A\times_{\alpha r}G)$.

COROLLARY 3.5. (i) *Let (A, G, α) and (B, G, β) be C^*-dynamial systems with $(A, G, \alpha)\subset(B, G, \beta)\subset(I_G(A), G, \alpha)$. Then $A\times_{\alpha r}G$ is prime if and only if $B\times_{\beta r}G$ is prime, and the simplicity of $A\times_{\alpha r}G$ implies that* of $B\times_{\beta r}G$.

(ii) *For a C^*-dynamical system (A, G, α), $\pi_\alpha(\bar{A})\subset\bar{A}\times_{\alpha r}G$ is the monotone closure of $\pi_\alpha(A)$ in $(A\times_{\alpha r}G)^-$ and so $\bar{A}\times_{\alpha r}G\subset(A\times_{\alpha r}G)^-$.*

PROOF. (i) As $A\times_{\alpha r}G\subset B\times_{\beta r}G\subset I(A\times_{\alpha r}G)$, the assertions follow from [12, 6.3, 7.1] and [15, 1.2(i)].

(ii) As in the proof of 3.4 we may assume that $A\times_{\alpha r}G\subset I_G(A)\times_{\alpha r}G\subset I(A\times_{\alpha r}G)\subset M(I_G(A), G)$. As $\pi_\alpha: I_G(A)\to M(I_G(A), G)$ is normal, so is

$\pi_\alpha: I_G(A) \to I(A\times_{\alpha r}G)$; hence $\pi_\alpha(\bar{A})$ is the monotone closure of $\pi_\alpha(A)$ in $I(A\times_{\alpha r}G)$. As $(A\times_{\alpha r}G)^-$ is the monotone closure of $A\times_{\alpha r}G$ in $I(A\times_{\alpha r}G)$, we have $\bar{A}\times_{\alpha r}G\subset(A\times_{\alpha r}G)^-$.

COROLLARY 3.6. *Let (A, G, α) be a C^*-dynamical system with G compact abelian. Then the regular monotone completion $(A\times_\alpha G)^-$ of the C^*-crossed product $A\times_\alpha G$ is realized as a monotone closed C^*-subalgebra of the monotone complete C^*-algebra $\bar{A}\,\bar{\otimes}\,B(L^2(G))$.*

PROOF. Note that as G and its dual $\hat{G}$ are amenable, we may suppress the letter "r" in $A\times_{\alpha r}G$ and so on. Takai's duality theorem [27, 7.9.3] asserts that $(A\times_\alpha G)\times_{\hat{\alpha}}\hat{G}\cong A\otimes C(L^2(G))$. As $\hat{G}$ is discrete, Corollary 3.5(ii) shows that $(A\times_\alpha G)^-$ is realized as the monotone closure of $\pi_{\hat{\alpha}}(A\times_\alpha G)\cong A\times_\alpha G$ in $((A\times_\alpha G)\times_{\hat{\alpha}}\hat{G})^-\cong(A\otimes C(L^2(G)))^-=\bar{A}\,\bar{\otimes}\,B(L^2(G))$ ([15, 3.1(i)], [13, 2.5, 6.7]).

REMARK 3.7. Corollary 3.6 is false for a general locally compact group G. Indeed, consider the C^*-dynamical system $(\boldsymbol{C}, \boldsymbol{Z}, \iota)$, where $\boldsymbol{C}$ is the 1-dimensional C^*-algebra with the trivial action ι. Then $\hat{\boldsymbol{Z}}=\boldsymbol{T}$, $\boldsymbol{C}\times_\iota\boldsymbol{Z}=C(\boldsymbol{T})$, and $C(\boldsymbol{T})^-$, being identified with the non-W^*, AW^*-algebra of bounded Borel functions on $\boldsymbol{T}$ modulo the sets of first category [8], is not a monotone closed C^*-subalgebra (W^*-subalgebra) of the W^*-algebra $\boldsymbol{C}\,\bar{\otimes}\,B(l^2(\boldsymbol{Z}))\cong B(l^2(\boldsymbol{Z}))$.

REMARK 3.8. Here we discuss the difference between injectivity and G-injectivity. Let (A, G, α) be a C^*-dynamical system. If G is not amenable, then we have $I_G(A)\neq I(A)$ in general (that is, $I(A)$ is injective, but not G-injective). Indeed, for the C^*-dynamical system $(\boldsymbol{C}, G, \iota)$ with the trivial action ι the G-module $l^\infty(G)=l^\infty(G,\boldsymbol{C})$ is G-injective, and $I_G(\boldsymbol{C})=\boldsymbol{C}=I(\boldsymbol{C})$ if and only if there is a G-morphism $\phi: l^\infty(G)\to\boldsymbol{C}$ with $\phi\circ j=\mathrm{id}_{\boldsymbol{C}}$ by 2.3, that is, G is amenable. On the other hand, we have $I_G(A)=I(A)$ if $I(A)$ is W^* and G is amenable (see 3.2).

4. A non-injective maximal regular extension.

A *reqular extension* of a unital C^*-algebra A [12, 1.1] is a unital C^*-algebra B containing A as a C^*-subalgebra with the same unit so that each element $x\in B_{sa}$ is the supremum of $\{a\in A_{sa}: a\leqq x\}$. There is a unique maximal regular extension, written $\tilde{A}$, of A, we have $A\subset\bar{A}\subset\tilde{A}\subset I(A)$, and $\tilde{A}$ is a monotone complete C^*-algebra [12, 3.1]. In this section we give an example of a C^*-algebra A for which $\tilde{A}$ is non-injective, that is, $\tilde{A}\neq I(A)$. This $\tilde{A}$ serves also as an example of a non-injective, non-W^*, AW^*-factor of type III, whose existence was first shown in [13, 4.9].

The next lemma follows immediately from [12, 2.6] and [23, p. 83, Lemma 2].

LEMMA 4.1. *Let B be a unital C^*-algebra and A its C^*-subalgebra containing the unit. Then B is a regular extension of A if and only if $j^*(K) \subsetneqq S(A)$ for any weak* closed convex subset $K \subsetneqq S(B)$, where j^* is the transpose of the inclusion map $j: A \hookrightarrow B$ and $S(C)$, with C a C^*-algebra, denotes the state space of C.*

LEMMA 4.2. *Let (A, G, α) be a C^*-dynamical system with A unital and G discrete. If $\boldsymbol{C}1 \subsetneqq A$, then $A \times_{\alpha r} G$ is not a regular extension of $C_r^*(G)$, where $C_r^*(G) = \boldsymbol{C} \times_{\iota r} G \subset A \times_{\alpha r} G$ with $\iota = \alpha | \boldsymbol{C}1$.*

PROOF. We show that $(*)$ there is a weak* closed convex subset K of $S(A \times_{\alpha r} G)$ such that $j^* | K: K \to S(C_r^*(G))$ is one-to-one and onto, where j is as in 4.1. If $A \times_{\alpha r} G$ were a regular extension of $C_r^*(G)$, then Lemma 4.1 would imply that $K = S(A \times_{\alpha r} G)$, hence that $A \times_{\alpha r} G = C_r^*(G)$, a contradiction [27, 7.7.9].

To see $(*)$ let $P(G, A^*)$ be the set of all functions $\Phi: G \to A^*$ such that $\|\Phi(e)\| = 1$ and $\sum_{i,j} \Phi(t_i^{-1} t_j)(\alpha_{t_i^{-1}}(a_i^* a_j)) \geqq 0$ for any finite $t_i \in G$ and $a_i \in A$. By [35, 2.19, 4.24(i)] the map $f \mapsto \Phi_f$, $\Phi_f(t)(a) = f(\pi_\alpha(a)\lambda(t))$, $t \in G$, $a \in A$ gives an affine homeomorphism of $S(A \times_\alpha G)$ with the weak* topology onto $P(G, A^*)$ with the point-weak* topology, and it maps $S(A \times_{\alpha r} G)$ (regarded as a subset of $S(A \times_\alpha G)$) onto the subset $P_r(G, A^*)$ of $P(G, A^*)$ consisting of elements Φ such that $\Phi_i \to \Phi$ in the point-weak* topology for some net $\{\Phi_i\}$ in $P(G, A^*)$ consisting of elements with finite support. Similarly, $P(G) = P(G, \boldsymbol{C}^*)$ and $P_r(G) = P_r(G, \boldsymbol{C}^*)$ are defined and satisfy the above property. Hence we may and shall identify $S(A \times_\alpha G)$ and $P(G, A^*)$, and so on. Let Φ be a state extension to $P(G, A^*) = S(A \times_\alpha G)$ of the function $G \ni t \mapsto 1 \in \boldsymbol{C}$ in $P(G) = S(C^*(G))$. Then $\Phi(t)(1) = 1$ for all $t \in G$ and $K = \{\psi \cdot \Phi: \psi \in P_r(G)\} \subset P_r(G, A^*)$ [35, 4.24(ii)] satisfies $(*)$.

PROPOSITION 4.3. *If G is a countable, non-amenable, ICC (=infinite conjugacy class) group, then the maximal regular extension $C_r^*(G)^\sim$ of the reduced group C^*-algebra $C_r^*(G)$ is a non-injective, non-W^*, σ-finite, monotone complete AW^*-factor of type* III.

PROOF. Theorem 3.4 says that $C_r^*(G) \subset I_G(\boldsymbol{C}) \times_{\iota r} G \subset I(C_r^*(G))$. As G is non-amenable, Remark 3.8 and Lemma 4.2 show that $I(C_r^*(G))$ is not a regular extension of $C_r^*(G)$, that is, $C_r^*(G)^\sim$ is not injective. As G is countable and ICC and so $C_r^*(G)$ generates a W^*-factor in its regular representation, $C_r^*(G)$ is a separable prime C^*-algebra. As $C_r^*(G)^\sim \subset (IC_r^*(G))$, [12, 6.3, 7.1] and the proof of [12, 3.8] show that $C_r^*(G)^\sim$ is a monotone complete AW^*-factor with a faithful state, hence that it is σ-finite. As $C_r^*(G)^- = C_r^*(G)^\wedge$

is non-W^* [34, Theorem N] and is monotone closed in $C_r^*(G)^{\sim}$, $G_r^*(G)^{\sim}$ is also non-W^*. Hence by [32, Corollary], $C_r^*(G)^{\sim}$ is of type III.

5. G-invariant hereditary C^*-subalgebras. We say that a projection of the regular monotone completion $\bar{A}$ of a C^*-algebra A is *open* [13] if it is a supremum in $\bar{A}_{sa}$ of some positive increasing net in A and that a closed two-sided ideal J of A is *regular* [15] if $J^{\perp\perp} = J$, where $S^{\perp} = \{x \in A: xy = yx = 0 \text{ for all } y \in S\}$ for $S \subset A$ and $S^{\perp\perp} = (S^{\perp})^{\perp}$. Let (A, G, α) be a C^*-dynamical system. As in [27] we write $\mathscr{H}^{\alpha}(A)$ for the set of all non-zero G-invariant hereditary C^*-subalgebras of A and $\mathscr{H}_B^{\alpha}(A)$ for the subset of $\mathscr{H}^{\alpha}(A)$ consisting of B such that the closed two-sided ideal of A generated by B is essential. For $B \in \mathscr{H}^{\alpha}(A)$ denote by $R(B)$ the smallest regular ideal of A containing B and by $\mathscr{R}^{\alpha}(A)$ the set of all non-zero G-invariant regular ideals of A. We say that an element in $I_G(A)$ is G-invariant if it is invariant under the action $I_G(\alpha)$.

The following is a dynamical system version of [13, 6.5].

PROPOSITION 5.1. *Let (A, G, α) be a C^*-dynamical system.*

(i) *For $B \in \mathscr{H}^{\alpha}(A)$ consider the C^*-dynamical system $(B, G, \alpha|B)$. Then the supremum p_B in $\bar{A}$ of each positive increasing approximate unit for B is a G-invariant open projection of $\bar{A}$ such that $\bar{B} = p_B\bar{A}p_B$ and $I_G(B) = p_B I_G(A) p_B$.*

(ii) *The correspondence $B \mapsto p_B$ given by* (i) *maps $\mathscr{H}^{\alpha}(A)$ onto the set of all non-zero G-invariant open projections of $\bar{A}$. By restricting this correspondence to $\mathscr{R}^{\alpha}(A)$ we obtain a bijection of $\mathscr{R}^{\alpha}(A)$ onto the set of all non-zero G-invariant central projections of $\bar{A}$.*

(iii) *For $B \in \mathscr{H}^{\alpha}(A)$ the central support $C(p_B)$ of p_B in $\bar{A}$ coincides with $p_{R(B)}$. Hence B is in $\mathscr{H}_B^{\alpha}(A)$ if and only if $C(p_B) = 1$.*

For the proof of the second equality in (i) we need the next lemma.

LEMMA 5.2. *Let D be a monotone complete C^*-algebra, C its monotone closed C^*-subalgebra containing the unit, and p a projection of C such that the central support $C(p)$ of p in C is 1. Let $\phi: pDp \to pDp$ be a completely positive map with $\phi|pCp = \mathrm{id}_{pCp}$. Then for each family $\{v_i\}$ of non-zero partial isometries of C such that*

$$(*) \qquad p \in \{v_i\}\,, \quad v_i v_i^* \leqq p \quad \textit{for all} \quad i \quad \textit{and} \quad O\text{-}\sum v_i^* v_i = 1\,,$$

the map $\hat{\phi}: D \to D$ given by

$$(**) \qquad \hat{\phi}(x) = O\text{-}\sum_{i,j} v_i^* \phi(v_i x v_j^*) v_j$$

is a unique completely positive map such that

$$(***) \qquad \hat{\phi}|pDp = \phi \quad \textit{and} \quad \hat{\phi}|C = \mathrm{id}_C\,,$$

where O-$\sum$ denotes the order limit of the finite sums.

PROOF OF LEMMA 5.2. As $C(p) = 1$, a standard argument using the comparability theorem [2, p. 80, Corollary] shows the existence of the family $\{v_i\}$ satisfying $(*)$. If $x \in D$ and an index j are fixed and i ranges over a finite subset of indices, then by the Schwarz inequality,

$$\sum_i \phi(v_i x v_j^*)^* \phi(v_i x v_j^*) \leqq \phi\Big(v_j x^* \Big(\sum_i v_i^* v_i\Big) x v_j^*\Big) \leqq \phi(v_j x^* x v_j) \leqq \|x\|^2 ;$$

hence by [13, 1.5], O-$\sum_i v_i^* \phi(v_i x v_j^*) = x_j$, say, exists. A similar argument shows the existence of O-$\sum_j x_j v_j$, that is, the right hand side of $(**)$. Thus $\hat{\phi}$ exists and is clearly completely positive. If $\psi\colon D \to D$ is a completely positive map satisfying $(***)$, then ψ is a C-module homomorphism [5, 3.1] and so for each $x \in D$ and each family $\{v_i\}$ satisfying $(*)$,

$$\begin{aligned}\psi(x) &= \Big(O\text{-}\sum_i v_i^* v_i\Big)\psi(x)\Big(O\text{-}\sum_j v_j^* v_j\Big) = O\text{-}\sum_{i,j} v_i^* \psi(v_i x v_j^*) v_j \\ &= O\text{-}\sum_{i,j} v_i^* \phi(v_i x v_j^*) v_j \,.\end{aligned}$$

Hence the uniqueness of $\hat{\phi}$ follows.

PROOF OF PROPOSITION 5.1. By [13, 6.5] there is a unique open projection p_B of $\bar{A}$ such that $\bar{B} = p_B \bar{A} p_B$. To see the G-invariance of p_B note that each α_t, $t \in G$, maps a positive increasing approximate unit for B to another such. Conversely if p is a G-invariant open projection of $\bar{A}$, then $A \cap pAp$ is a G-invariant hereditary C^*-subalgebra of A with $(A \cap pAp)^- = p\bar{A}p$ [15, 1.1(v)]. Moreover by [15, 1.3(iii)] an ideal J of A is regular if and only if $J = A \cap hA$ for some central projection h of $\bar{A}$. These show (i), except for the second equality, and (ii).

(iii) As $p_{R(B)}$ is a central projection of $\bar{A}$ majorizing p_B, we have $C(p_B) \leqq p_{R(B)}$. Moreover, as $A \cap C(p_B)A$ is a regular ideal containing B, it follows that $R(B) \subset A \cap C(p_B)A$, hence that $p_{R(B)} \leqq C(p_B)$. As a closed two-sided ideal J of A is essential if and only if $J^\perp = \{0\}$, $B \in \mathscr{H}^\alpha(A)$ is in $\mathscr{H}^\alpha_B(A)$ if and only if $R(B) = A$.

We show the equality $I_G(B) = p_B I_G(A) p_B$ in (i). As $A \subset \bar{A} \subset I_G(A)$, we have $I_G(A) = I_G(\bar{A})$ and $I_G(B) = I_G(\bar{B}) = I_G(p_B \bar{A} p_B)$. Hence it suffices to show that if A is monotone complete and p is a G-invariant projection of A, then $I_G(pAp) = pI_G(A)p$. The central support $C(p)$ of p in A, being the supremum of upu^* with u unitaries in A, is also G-invariant and it is immediate to see that $I_G(C(p)A) = C(p)I_G(A)$ (modify the argument in [12, 6.2]). Thus we may also assume that $C(p) = 1$. As $pAp \subset pI_G(A)p$ and $pI_G(A)p$, being a direct summand of $I_G(A)$, is G-injective, we need only show that if $\phi: pI_G(A)p \to pI_G(A)p$ is a G-morphism with $\phi|pAp = \mathrm{id}_{pAp}$, then ϕ is the identity of $pI_G(A)p$. We apply Lemma 5.1 to $I_G(A)$, A, p and ϕ. Take a family $\{v_i\}$ of non-zero partial isometries in A satisfying $(*)$ and define $\hat{\phi}: I_G(A) \to I_G(A)$ by $(**)$. Then $\hat{\phi}|A = \mathrm{id}_A$ and $\hat{\phi}$ is a G-morphism. Indeed, for each $t \in G$ the family $\{\alpha_t(v_i)\}$ also satisfies $(*)$ and so the uniqueness of $\hat{\phi}$ shows that for $x \in I_G(A)$,

$$\hat{\phi}(x) = O\text{-}\sum_{i,j} \alpha_t(v_i^*)\phi(\alpha_t(v_i)x\alpha_t(v_j^*))\alpha_t(v_j) .$$

As ϕ is a G-morphism, it follows that

$$\begin{aligned}\hat{\phi}(I_G(\alpha)_t(x)) &= O\text{-}\sum_{i,j} \alpha_t(v_i^*)I_G(\alpha)_t(\phi(v_ixv_j^*))\alpha_t(v_j) \\ &= I_G(\alpha)_t(O\text{-}\sum_{i,j} v_i^*\phi(v_ixv_j^*)v_j) = I_G(\alpha)_t(\hat{\phi}(x)) .\end{aligned}$$

As $I_G(A)$ is a G-rigid G-extension of A, $\hat{\phi}$ is the identity on $I_G(A)$ and $\phi = \hat{\phi}|pI_G(A)p$ is the identity on $pI_G(A)p$.

6. The center of the G-injective envelope. In what follows, the center of a C^*-algebra A is denoted by $Z(A)$, and for a C^*-dynamical system (A, G, α) and a G-invariant C^*-subalgebra B of A the fixed point subalgebra of B under the action $\alpha|B$ is denoted by B^G. Now we study the algebra $Z(I_G(A))^G$. In the next lemmas (A, G, α) denotes a fixed C^*-dynamical system.

As stated in the proof of 5.1, the following lemma follows from a slight modification of the proof of [12, 6.2].

LEMMA 6.1. *For a G-invariant central projection h of $I_G(A)$, consider the C^*-dynamical system $(hA, G, I_G(\alpha)|hA)$. Then the G-injective envelope $I_G(hA)$ of hA is $hI_G(A)$ together with the inclusion map $hA \hookrightarrow hI_G(A)$.*

LEMMA 6.2. *We have $Z(I(A)) \subset Z(I_G(A))$.*

PROOF. The map $j: I(A) \to l^\infty(G, I(A))$ (see 2.3) is both a G-monomorphism and a unital $*$-monomorphism with $l^\infty(G, I(A))$ G-injective. Hence there is a minimal $j(I(A))$-projection ϕ on $l^\infty(G, I(A))$ so that $I_G(A) = I_G(I(A))$ is identified with $\mathrm{Im}\,\phi$. Noting the multiplication in $\mathrm{Im}\,\phi$ and the fact that j maps $Z(I(A))$ into the center of $l^\infty(G, I(A))$, we see that $j(Z(I(A))) \subset Z(I_G(A))$.

LEMMA 6.3 (cf. [10, 4.3], [12, 6.3]). *We have $Z(I(A))^G = Z(I_G(A))^G = (A' \cap I_G(A))^G$, where $A' \cap I_G(A)$ denotes the relative commutant of A in $I_G(A)$.*

PROOF. The inclusions $Z(I(A))^G \subset Z(I_G(A))^G \subset (A' \cap I_G(A))^G$ are clear.

Let u be a unitary in $(A' \cap I_G(A))^G$. Then $\mathrm{Ad}\, u: I_G(A) \to I_G(A)$, $(\mathrm{Ad}\, u)(x) = uxu^*$, $x \in I_G(A)$ is a G-morphism with $\mathrm{Ad}\, u | A = \mathrm{id}_A$, and so $\mathrm{Ad}\, u$ is the identity on $I_G(A)$ and $u \in Z(I_G(A))$. Hence $Z(I_G(A))^G = (A' \cap I_G(A))^G$.

Let h be a projection in $Z(I_G(A))^G$. Then as in the proof of [12, 6.3] there is a unique minimal projection h_1 in $Z(I(A))$ majorizing h. By the uniqueness h_1 is also G-invariant, and noting 6.1, the same argument as in the proof of [12, 6.3] shows that $h = h_1 \in Z(I_G(A))^G$. Hence $Z(I(A))^G = Z(I_G(A))^G$.

PROPOSITION 6.4. *Let (A, G, α) and (B, G, β) be two C^*-dynamical systems with $(A, G, \alpha) \subset (B, G, \beta) \subset (I_G(A), G, I_G(\alpha))$.*

(i) *We have $Z(A) \subset Z(B)$; if in addition $\bar{A} \subset B$, then $Z(B)^G = Z(I_G(A))^G$. In particular, $Z(\bar{A})^G = Z(I(A))^G = Z(I_G(A))^G$.*

(ii) *A is G-prime if and only if B is G-prime.*

(iii) *If A is unital and G-simple, then B is G-simple.*

PROOF. (i) The first inclusion follows from [10, 4.3] and 6.2. If $\bar{A} \subset B$, then $Z(I(A)) = Z(\bar{A}) \subset Z(B)$ [12, 6.3] and by 6.3, $Z(I_G(A))^G = Z(I(A))^G \subset Z(B)^G \subset Z(I_G(A))^G$.

(ii) If J and K are mutually orthogonal non-zero G-invariant closed two-sided ideals of A, then $J^{\perp\perp}$ and $K^{\perp\perp}$ are also such regular ideals of A. Hence A is G-prime if and only if A has no nontrivial G-invariant regular ideal, that is, if and only if $Z(I_G(A))^G = \boldsymbol{C}1$ by (i) and 5.1. Moreover, note that $I_G(A) = I_G(B)$.

(iii) Modify the proof of [15, 1.2] slightly.

7. Quasi-inner *-automorphisms.

In this section we investigate the relationship between a *-automorphism α of a C^*-algebra A and its unique extensions $\bar{\alpha}$ and $I(\alpha)$ to $\bar{A}$ and $I(A)$, respectively.

LEMMA 7.1. *Let (A, G, α) be a C^*-dynamical system with G a locally compact abelian group. Let $I(A \times_\alpha G)$ be the injective envelope of the C^*-crossed product $A \times_\alpha G$, $I(\hat{\alpha})$ the unique extension to $I(A \times_\alpha G)$ of the dual action $\hat{\alpha}$ of $\hat{G}$ on $A \times_\alpha G$ (see [27, 7.8.3]), and Z the center of $I(A \times_\alpha G)$. Denote by $\Gamma(\cdot)$ and $\Gamma_B(\cdot)$ the Connes and Borchers spectra, respectively (see [27, 8.8]).*

(i) *Let $B \in \mathscr{H}_B^\alpha(A)$. Then $\Gamma(\alpha | B) = \mathrm{Ker}(I(\hat{\alpha}) | Z)$, and a $\sigma \in \hat{G}$ belongs to $\Gamma_B(\alpha | B)$ if and only if for any neighborhood Ω of σ there are a non-zero projection h of Z and a $\sigma_1 \in \Omega$ such that the supremum $\vee \{I(\hat{\alpha})_\tau(h): \tau \in \hat{G}\}$ in the projection lattice of $I(A \times_\alpha G)$ equals 1 and $hI(\hat{\alpha})_{\sigma_1}(h) \neq 0$.*

(ii) *If $B_1, B_2 \in \mathscr{H}^\alpha(A)$ with $R(B_1) = R(B_2)$ (in particular, if B_2 is the closed two-sided ideal of A generated by B_1), then $\Gamma(\alpha | B_1) = \Gamma(\alpha | B_2)$ and $\Gamma_B(\alpha | B_1) = \Gamma_B(\alpha | B_2)$.*

(iii) *If in addition G is discrete (hence $\Gamma(I(\alpha))$, $\Gamma(\bar{\alpha})$ and so on make sense), then $\Gamma(I(\alpha))=\Gamma(\bar{\alpha})=\Gamma(\alpha)$ and $\Gamma_B(I(\alpha))=\Gamma_B(\bar{\alpha})=\Gamma_B(\alpha)$.*

PROOF. (i) As $B\in\mathscr{H}_B^\alpha(A)$, the C^*-crossed product $B\times_{\alpha|B}G=C$, say, regarded as a C^*-subalgebra of $A\times_\alpha G$, is an $\hat{\alpha}$-invariant hereditary C^*-subalgebra which generates an essential closed two-sided ideal of $A\times_\alpha G$. By 5.1 we have $I(C)=p_C I(A\times_\alpha G)p_C$ for an $I(\hat{\alpha})$-invariant projection p_C of $I(A\times_\alpha G)$ with central support $C(p_C)=1$. The center of $I(C)$ equals $p_C Z$ and the map $x\mapsto p_C x$ gives a *-isomorphism of Z onto $p_C Z$ [2, p. 37, Corollary 2].

As $(\alpha|B)^\wedge=\hat{\alpha}|C$, it follows from [25, 5.4] or [27, 8.11.8] that for $\sigma\in\hat{G}$ we have $\sigma\notin\Gamma(\alpha|B)$ if and only if $J\cdot\hat{\alpha}_\sigma(J)=\{0\}$ for some non-zero closed two-sided ideal J of C. As $J\cdot\hat{\alpha}_\sigma(J)=\{0\}$ implies $J^{\perp\perp}\cdot\hat{\alpha}_\sigma(J^{\perp\perp})=\{0\}$, the latter condition is equivalent to $J\cdot\hat{\alpha}_\sigma(J)=\{0\}$ for some non-zero regular ideal J of C, which in turn is equivalent to $h\cdot I(\hat{\alpha})_\sigma(h)=0$ for some non-zero projection h of $p_C Z$ [15]. From the first paragraph this is the case if and only if $h\cdot I(\hat{\alpha})_\sigma(h)=0$ for some non-zero projection h of Z. Thus $\sigma\notin\Gamma(\alpha|B)$ if and only if $I(\hat{\alpha})_\sigma|Z\neq\mathrm{id}_Z$.

To see the second assertion we use the following characterization of the Borchers spectrum by Kishimoto [21, 1.1] (with $n=1$). A $\sigma\in\hat{G}$ belongs to $\Gamma_B(\alpha|B)$ if and only if for each neighborhood Ω of σ there are a non-zero closed two-sided ideal J of C which generates an $\hat{\alpha}$-invariant essential closed ideal and a $\sigma_1\in\Omega$ such that $J\cdot\hat{\alpha}_{\sigma_1}(J)\neq\{0\}$. Then the argument proceeds exactly as for $\Gamma(\alpha|B)$. We may take the above J as a regular ideal, and if $I(J)=hp_C I(C)$ with h a projection of Z, then the condition that J generates an $\hat{\alpha}$-invariant essential ideal of C is equivalent to $\vee\{I(\hat{\alpha})_\tau(h):\tau\in\hat{G}\}=1$, and so on.

(ii) By (i) we have $\Gamma(\alpha|B)=\Gamma(\alpha)$ and $\Gamma_B(\alpha|A)=\Gamma_B(\alpha)$ for $B\in\mathscr{H}_B^\alpha(A)$. As $B_i\in\mathscr{H}_B^\alpha(R(B_i))$, $i=1,2$, the conclusion follows.

(iii) By 3.4 we have $A\times_\alpha G\subset I(A)\times_\alpha G\subset I(A\times_\alpha G)$. As $I(\alpha)^\wedge|A\times_\alpha G=\hat{\alpha}$ and $I(\hat{\alpha})$ is a unique extension of $\hat{\alpha}$, it follows that $I(I(\alpha)^\wedge)=I(\hat{\alpha})$ and $I(I(A)\times_\alpha G)=I(A\times_\alpha G)$. Hence (iii) follows from (i) with $B=A$.

REMARK 7.2. From (ii) we see that in [26, 3.3, 3.4] the separability of the C^*-dynamical system can be dropped, that is, for any C^*-dynamical system (A, G, α) with G a locally compact abelian group and any $B\in\mathscr{H}^\alpha(A)$ we have $\Gamma(\alpha)\subset\Gamma(\alpha|B)\subset\Gamma_B(\alpha|B)\subset\Gamma_B(\alpha)$ and $\Gamma_D(\alpha)=(\cup\{\Gamma(\alpha|I): I\in\mathscr{I}^\alpha(A)\})^-$.

THEOREM 7.3. *Let (A, G, α) be a C^*-dynamical system with G discrete abelian and let $(\bar{A}, G, \bar{\alpha})$ and $(I(A), G, I(\alpha))$ be the C^*-dynamical systems canonically associated with it. For $t\in G$ the following conditions are* *equivalent*:

(i) $t \in \Gamma_B(\alpha)^{\perp}$;

(ii) *There are a* $B \in \mathscr{H}_B^{\alpha}(A)$ *and a G-invariant *-derivation* δ *of* B *such that* $\alpha_t | B = \exp \delta$;

(iii) $\bar{\alpha}_t = \operatorname{Ad} u$ *for some unitary* u *in* $\bar{A}^G$;

(iv) $I(\alpha)_t = \operatorname{Ad} u$ *for some unitary* u *in* $I(A)^G$.

PROOF. As $\hat{G}$ is compact, the implication (i) $\Rightarrow$ (ii) follows from [26, 4.3].

(ii) $\Rightarrow$ (iii). By 5.1 we have $\bar{B} = p_B \bar{A} p_B$ for a G-invariant projection p_B of $\bar{A}$ with $C(p_B) = 1$. The *-derivation δ extends uniquely to an inner *-derivation $\bar{\delta} = \operatorname{ad}(ih)$, $h \in \bar{B}_{sa}$, of $\bar{B}$ [16, Theorem 2.1]. If we take the minimal generator for $\bar{\delta}$ as h (see [16, Lemma 3.1]), then the G-invariance of $\bar{\delta}$ and the uniqueness of the minimal generator show that h is G-invariant. Hence $\bar{\alpha}_t | p_B \bar{A} p_B = (\alpha_t | B)^- = (\exp \delta)^- = \exp \bar{\delta} = \operatorname{Ad}(\exp(ih))$ and $\exp(ih)$ is a G-invariant unitary in $p_B \bar{A} p_B$. As $C(p_B) = 1$, it follows from [13, 5.2] that $\bar{\alpha}_t = \operatorname{Ad} u$ for a unique unitary u in $\bar{A}$ such that $p_B u = u p_B = \exp(ih)$. As $\bar{\alpha}_t = \bar{\alpha}_s \circ \bar{\alpha}_t \circ \bar{\alpha}_{s^{-1}} = \operatorname{Ad}(\bar{\alpha}_s(u))$ and $p_B \bar{\alpha}_s(u) = \bar{\alpha}_s(u) p_B = \exp(ih)$ for all $s \in G$, the uniqueness of u shows that $\bar{\alpha}_s(u) = u$ for all $s \in G$.

It is clear that (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i). It follows from [27, 8.9.7] and 7.1 that (iv) $\Rightarrow t \in \Gamma_B(I(\alpha))^{\perp} = \Gamma_B(\alpha)^{\perp}$.

Following Kishimoto [21], [22] we say that a *-automorphism α of a C^*-algebra A is *quasi-inner* if $\Gamma_B(\alpha) = \{1\} \subset \boldsymbol{T} = \hat{\boldsymbol{Z}}$ and it is *properly outer* if $\Gamma_B(\alpha | J) \neq \{1\}$ for each non-zero α-invariant closed two-sided ideal J of A, where $\Gamma_B(\alpha)$ denotes the Borchers spectrum of the action $\boldsymbol{Z} \ni n \mapsto \alpha^n \in \operatorname{Aut} A$. (Note that the word "freely acting" originally used in [21] was renamed "properly outer" in [22].) As in the W^*-case there is for any *-automorphism α of A the largest α-invariant closed two-sided ideal J (resp. K) such that $\alpha | J$ (resp. $\alpha | K$) is quasi-inner (resp. properly outer), $J \cap K = \{0\}$ and $J + K$ is essential in A ([22], see also 7.5 below). Note that the proper outerness in the above sense implies the proper outerness in the sense of Elliott [8] and they are equivalent when A is separable [26, 6.6].

THEOREM 7.4. *For a *-automorphism* α *of a* C^**-algebra* A *the following conditions are equivalent*:

(i) α *is quasi-inner*;

(ii) *There are a* $B \in \mathscr{H}_B^{\alpha}(A)$ *and a *-derivation* δ *of* B *such that* $\alpha | B = \exp \delta$;

(iii) $\bar{\alpha}$ *is inner;*
(iv) $I(\alpha)$ *is inner.*

PROOF. Apply 7.3 to the action $\boldsymbol{Z} \ni n \mapsto \alpha^n \in \mathrm{Aut}\, A$, and note that in this situation the G-invariance of δ in 7.3 (or u) follows automatically.

REMARK 7.5. For a *-automorphism α of a C^*-algebra A let $p(\alpha)$ be the largest $I(\alpha)$-invariant projection in $I(A)$ such that $I(\alpha)|p(\alpha)I(A)p(\alpha)$ is inner. Then $p(\alpha)$ is a central projection in $\bar{A}$ ([13, 5.1], [12, 6.3]) and $A \cap p(\alpha)A$ (resp. $A \cap (1-p(\alpha))A$) is the largest closed two-sided ideal of A such that $\alpha|A \cap p(\alpha)A$ is quasi-inner (resp. $\alpha|A \cap (1-p(\alpha))A$ is properly outer). Indeed, if $\alpha|J$ is quasi-inner for some α-invariant closed two-sided ideal J of A, then $I(\alpha)|p_J I(A) = I(\alpha|J)$ is inner and so $p_J \leqq p(\alpha)$, $J \subset A \cap p(\alpha)A$. Moreover $I(\alpha|A \cap p(\alpha)A) = I(\alpha)|p(\alpha)I(A)$, and similarly for $A \cap (1-p(\alpha))A$.

COROLLARY 7.6. *For a C^*-algebra A the subset q-*Inn A *of* Aut A *consisting of all quasi-inner *-automorphisms of A is a normal subgroup of* Aut A; *indeed, we have*

$$q\text{-Inn}\, A = \mathrm{Aut}\, A \cap \mathrm{Inn}\, \bar{A} = \mathrm{Aut}\, A \cap \mathrm{Inn}\, I(A)\ ,$$

where as before we regard $\mathrm{Aut}\, A \subset \mathrm{Aut}\, \bar{A} \subset \mathrm{Aut}\, I(A)$ *and* Inn $\bar{A}$ *denotes the inner *-automorphism group of $\bar{A}$. Hence if we write* Out A = Aut A/q-Inn A, *then we have*

$$\mathrm{Out}\, A \subset \mathrm{Out}\, \bar{A} \subset \mathrm{Out}\, I(A)\ .$$

COROLLARY 7.7. *If A is a monotone complete C^*-algebra and u is a unitary in $I(A)$ such that $uAu^* = A$, then $u \in A$.*

PROOF. Put $\alpha = \mathrm{Ad}\, u|A \in \mathrm{Aut}\, A$. As $I(\alpha) = \mathrm{Ad}\, u$ is inner, $\alpha = \bar{\alpha}$ is also inner, that is, $\mathrm{Ad}\, u|A = \alpha = \mathrm{Ad}\, v|A$ for some unitary v in A. Hence v^*u belongs to the relative commutant of A in $I(A)$, which equals $Z(A)$ ([10, 4.3], [12, 6.3]), and $u = vv^*u \in A$.

COROLLARY 7.8 (Saitô and Wright [28]). *If A is a simple C^*-algebra and α is a *-automorphism A, then $I(\alpha)$ or $\bar{\alpha}$ is inner if and only if α is inner in the multiplier algebra $M(A)$.*

PROOF. As A is simple, α is inner in $M(A)$ if and only if $\Gamma_B(\alpha) = \Gamma(\alpha) = \{1\}$ ([24] or [27, 8.9.10]). Hence 7.4 applies.

REMARK. See [29] for a slightly more general result.

COROLLARY 7.9. *If A is a C^*-algebra which contains an essential GCR-ideal and α is a *-automorphism of A, then the following conditions are equivalent:*

(i) *α is quasi-inner;*

(ii) *$\alpha(J) = J$ for each regular ideal J of A;*

(iii) *$\alpha | J$ is universally weakly inner for some essential α-invariant closed two-sided ideal J of A.*

PROOF. (i) $\Leftrightarrow$ (ii). By [15, 2.3], A contains an essential GCR-ideal if and only if $\bar{A}$ is a type I AW^*-algebra. (In this case $I(A) = \bar{A}$.) By [19], $\bar{\alpha}$ is inner if and only if it fixes the center of $\bar{A}$ elementwise. By [15] the latter condition is equivalent to (ii).

(i) $\Longrightarrow$ (iii). By 7.4, (i) implies that $\alpha | B = \exp \delta$ for some $B \in \mathscr{H}_B^{\alpha}(A)$ and some *-derivation δ of B. The closed two-sided ideal J of A generated by B is α-invariant and essential. If A^{**} is the enveloping von Neumann algebra of A, then $B^{**} = pA^{**}p$ for some projection p of A^{**} and $J^{**} = C(p)A^{**}$, where $C(p)$ is the central support of p in A^{**}. If α^{**} is the bitranspose of α, then that $\alpha^{**} | pA^{**}p = (\alpha | B)^{**} = \exp \delta^{**}$ is inner implies that so is $(\alpha | J)^{**} = \alpha^{**} | C(p)A^{**}$, that is, (iii).

(iii) $\Longrightarrow$ (i). If J is as in (iii), then clearly $\alpha(K) = K$ for each regular ideal K of J and so $\alpha | J$ is quasi-inner by the equivalence (i) $\Leftrightarrow$ (ii). But as $\bar{J} = R(J)^{-} = \bar{A}$ by 5.1 and $(\alpha | J)^{-} = \bar{\alpha}$, this implies (i).

8. A decomposition of *-automorphisms. Let α be a *-automorphism of a C^*-algebra A and denote, as before, by $\Gamma(\alpha)$ and $\Gamma_B(\alpha)$ the Connes and Borchers spectra of the action $\boldsymbol{Z} \ni n \mapsto \alpha^n \in \operatorname{Aut} A$, respectively. Kishimoto showed in [21, 3.1] that there are the largest α-invariant closed two-sided ideals I_k, $k \in \boldsymbol{N} \cup \{\infty\}$, of A such that $\Gamma(\alpha | I_k) = \Gamma_B(\alpha | I_k) = \boldsymbol{T}_k$, where $\boldsymbol{T}_k$ is the subgroup of $\boldsymbol{T}$ of order k if $k \in \boldsymbol{N}$ and $\boldsymbol{T}_\infty = \boldsymbol{T}$, and that the sequence $\{I_k\}$ is mutually orthogonal and generates an essential ideal of A. If $p_k(\alpha)$ is the $\bar{\alpha}$-invariant central projection of $\bar{A}$ such that $\bar{I}_k = p_k(\alpha)\bar{A}$ and $I(I_k) = p_k(\alpha)I(A)$, then we have $I_k = A \cap p_k(\alpha)A$, since I_k is regular by the maximality and 7.1(ii), and $\{p_k(\alpha)\}$ is an orthogonal sequence with supremum 1. Note also that $p_1(\alpha)$ is the projection $p(\alpha)$ in 7.5.

We characterize the sequence $\{p_k(\alpha)\}$ by the action on $\bar{A}$ or on $I(A)$ of the extended *-automorphisms $\bar{\alpha}$ or $I(\alpha)$. For similar results in the W^*-case see [3], [4]. (Note that as Connes points out in [7], the result in [3] requires a slight modification.)

PROPOSITION 8.1. *For a *-automorphism α of a C^*-algebra A let $p_k(\alpha)$ be as above. Then $p_k(\alpha)$ is the largest projection p in the fixed point algebra $\bar{A}^{\bar{\alpha}}$ (resp. $I(A)^{I(\alpha)}$) such that (*) $\bar{\alpha}^n | q\bar{A}q = \operatorname{Ad} u$ for some $n \in \boldsymbol{Z}$, some non-zero subprojection q of p in $\bar{A}^{\alpha}$ and some unitary u in $q\bar{A}^{\alpha}q$ if and only if $n \equiv 0 \pmod{k}$ (when $k = \infty$, if and only if $n = 0$) (resp. the same property with $\bar{\alpha}$ and $\bar{A}$ replaced by $I(\alpha)$ and $I(A)$). If $p_k(\alpha) = 0$*

for some k, then the property is vacuously satisfied.

PROOF. We prove only the statement for $\bar{\alpha}$, since the case of $I(\alpha)$ is treated similarly. By 7.3, $(*)$ is equivalent to the condition $\Gamma_B(\bar{\alpha}\,|\,q\bar{A}q)^\perp = k\boldsymbol{Z}$ $(=\{0\}$ if $k=\infty)$ for each non-zero subprojection q of p in $\bar{A}^{\bar{\alpha}}$. If q is a non-zero subprojection of $p_k(\alpha)$ in $\bar{A}^{\bar{\alpha}}$, then

$$\begin{aligned}\boldsymbol{T}_k &= \Gamma(\alpha\,|\,I_k) = \Gamma(\bar{\alpha}\,|\,p_k(\alpha)\bar{A}) \subset \Gamma(\bar{\alpha}\,|\,q\bar{A}q) \subset \Gamma_B(\bar{\alpha}\,|\,q\bar{A}q) \subset \Gamma_B(\bar{\alpha}\,|\,p_k(\alpha)\bar{A})\\ &= \Gamma_B(\alpha\,|\,I_k) = \boldsymbol{T}_k\end{aligned}$$

by 7.1 and 7.2 and so $\Gamma_B(\bar{\alpha}\,|\,q\bar{A}q)^\perp = \boldsymbol{T}_k^\perp = k\boldsymbol{Z}$. Hence $p_k(\alpha)$ satisfies $(*)$. If a projection p in $\bar{A}^{\bar{\alpha}}$ satisfies $(*)$ and $p\cdot p_j(\alpha) \neq 0$, then as $p\cdot p_j(\alpha) \leqq p$ and $p\cdot p_j(\alpha) \leqq p_j(\alpha)$, we have $\Gamma_B(\bar{\alpha}\,|\,p\cdot p_j(\alpha)\bar{A}p\cdot p_j(\alpha))^\perp = k\boldsymbol{Z} = j\boldsymbol{Z}$ and $j = k$. Thus $p\cdot p_j(\alpha) = 0$ for each $j \neq k$ and $p \leqq 1 - \sum_{j\neq k} p_j(\alpha) = p_k(\alpha)$.

In some equivalent conditions in [26, 10.4] for aperiodic *-automorphisms we can drop the separability of the C^*-algebra.

PROPOSITION 8.2. *For a *-automorphism α of a C^*-algebra A the following conditions are equivalent:*

(i) $\Gamma(\alpha) = \boldsymbol{T}$.

(ii) *There is no $B \in \mathscr{H}^\alpha(A)$ such that $\alpha^n|B = \exp\delta$ for some $n \neq 0$ and some α-invariant *-derivation δ of B.*

(iii) *For each $n \in \boldsymbol{N}$ the *-automorphism α^n is properly outer.*

(iv) *For each $\varepsilon > 0$, each $n \in \boldsymbol{N}$ and each $B \in \mathscr{H}^\alpha(A)$ there is an $x \in B^+$ with $\|x\| = 1$ such that $\|x\alpha^k(x)\| < \varepsilon$ for $1 \leqq k \leqq n$.*

PROOF. By 7.2 we have $\Gamma(\alpha) = \cap\{\Gamma_B(\alpha\,|\,B) : B \in \mathscr{H}^\alpha(A)\}$. Hence (i) is equivalent to $\Gamma_B(\alpha\,|\,B) = \boldsymbol{T}$ for each $B \in \mathscr{H}^\alpha(A)$, which in turn is equivalent to $\Gamma_B(\alpha\,|\,B)^\perp = \{0\}$ for each $B \in \mathscr{H}^\alpha(A)$. For if $\Gamma_B(\alpha\,|\,B) \neq \boldsymbol{T}$, then $\Gamma_B(\alpha\,|\,B)$ is a finite union of finite subgroups of $\boldsymbol{T}$ [27, 8.8.5] and so $\Gamma_B(\alpha\,|\,B)^\perp \neq \{0\}$. Moreover, the reverse implication is clear. Thus 7.3 shows that (i) $\Leftrightarrow$ (ii).

(iii) $\Rightarrow$ (i). If $\Gamma(\alpha) \neq \boldsymbol{T}$, then $k \in \Gamma_B(\alpha\,|\,B)^\perp$ for some $B \in \mathscr{H}^\alpha(A)$ and $k \neq 0$ and so α^k is not properly outer by 7.3.

(i) $\Rightarrow$ (iii). If α^n is not properly outer for some $n \in \boldsymbol{N}$, then the central projection p in $\bar{A}$ inducing the inner part of $\bar{\alpha}^n$ is non-zero and $\bar{\alpha}^n|p\bar{A} = \mathrm{Ad}\,u$ for some unitary u in $p\bar{A}$. The maximality of p and the fact that $\bar{\alpha}^n|\bar{\alpha}(p)\bar{A} = \mathrm{Ad}\,\bar{\alpha}(u)$ and similarly for $\bar{\alpha}^{-1}$ show that $\bar{\alpha}(p) = p$. Now we use the argument in [26, 10.1]. It follows readily that $u, \bar{\alpha}(u), \cdots, \bar{\alpha}^{(n-1)}(u)$ are unitaries in $p\bar{A}$ implementing $\bar{\alpha}^n|p\bar{A}$ and that they commute mutually. If we put $v = u\bar{\alpha}(u)\cdots\bar{\alpha}^{(n-1)}(u)$, then $\bar{\alpha}^{n^2}|p\bar{A} = \mathrm{Ad}\,v$ and $\bar{\alpha}(v) = v$. By 7.3, $n^2 \in \Gamma_B(\bar{\alpha}\,|\,p\bar{A})^\perp = \Gamma_B(\alpha\,|\,A\cap pA)^\perp$ and $A\cap pA$ is a non-zero α-invariant regular ideal of A. Hence $\Gamma(\alpha) \neq \boldsymbol{T}$.

(iii) $\Leftrightarrow$ (iv). This follows from the fact that Kishimoto's result [21, 2.1] shows that [26, 7.1] holds also in the nonseparable case (see the proof of [26, 10.4]).

COROLLARY 8.3. *For a *-automorphism α of a C^*-algebra A let I_∞ be as above. Then I_∞ is the largest α-invariant hereditary C^*-subalgebra B of A such that $\alpha^n | B$ is properly outer for each $n \in \boldsymbol{N}$.*

9. Tensor products and *-automorphisms. In this section we show two results on *-automorphisms of minimal C^*-tensor products. For C^*-algebras A and B we denote by $A \otimes B$ the minimal C^*-tensor product of A and B.

The following is an analogue of the result of Kallman [18] and that of Wassermann [31] in the setting of quasi-inner and properly outer *-automorphisms.

THEOREM 9.1. *Let A and B be C^*-algebras and let $\alpha \otimes \beta$ be the *-automorphism of $A \otimes B$ induced by *-automorphisms α of A and β of B. Let $p(\alpha), p(\beta)$ and $p(\alpha \otimes \beta)$ be the projections of $I(A), I(B)$ and $I(A \otimes B)$ inducing the inner parts of $I(\alpha), I(\beta)$ and $I(\alpha \otimes \beta)$ respectively* (*see* 7.5).

(i) *We have $p(\alpha \otimes \beta) = p(\alpha) \otimes p(\beta)$ in $I(A) \otimes I(B) \subset I(A \otimes B)$.*

(ii) *$\alpha \otimes \beta$ is quasi-inner if and only if both α and β are quasi-inner.*

(iii) *$\alpha \otimes \beta$ is properly outer if and only if either α or β is properly outer.*

PROOF. As $A \otimes B \subset I(A) \otimes I(B) \subset I(A \otimes B)$ [13, 6.7] and $I(\alpha \otimes \beta) | A \otimes B = \alpha \otimes \beta = I(\alpha) \otimes I(\beta) | A \otimes B$, we have $I(\alpha \otimes \beta) = I(I(\alpha) \otimes I(\beta))$. This and 7.4 show that replacing α, β, A and B by $I(\alpha), I(\beta), I(A)$ and $I(B)$, we may assume that A and B are injective C^*-algebras and so $\alpha | p(\alpha)A$ and $\beta | p(\beta)B$ are inner. Then $\alpha \otimes \beta = \sum_{i,j} (\alpha \otimes \beta) | (p_i \otimes q_j)(A \otimes B)$, where $p_1 = p(\alpha)$, $p_2 = 1 - p(\alpha)$, $q_1 = p(\beta)$ and $q_2 = 1 - p(\beta)$, and $(\alpha \otimes \beta) | (p_1 \otimes q_1)(A \otimes B)$ is inner. If the sufficiency of (iii) were proved, then all the remaining assertions would follow. Hence it suffices to show that if α is properly outer, then so is $\alpha \otimes \beta$. The required property is equivalent to $I(\alpha \otimes \beta)$ being freely acting (see [18, 13]). Let $x \in I(A \otimes B)$ and $xy = I(\alpha \otimes \beta)(y)x$ for all $y \in I(A \otimes B)$. Regard B as a C^*-subalgebra of some $B(K)$ with K a Hilbert space and regard $A \otimes B \subset A \bar{\otimes} B(K)$ (see Section 3 or [13]). As $A \bar{\otimes} B(K)$ is injective [13, 3.10], we may take the injective envelope $I(A \otimes B)$ so that $A \otimes B \subset I(A \otimes B) \subset A \bar{\otimes} B(K)$.

If $L_g\colon A \mathbin{\overline{\otimes}} B(K) \to A$ is the left slice map defined for $g \in B(K)_*$ [13], then for each $a \in A$,

$$L_g(x)a = L_g(x(a \otimes 1)) = L_g(I(\alpha \otimes \beta)(a \otimes 1)x) = \alpha(a)L_g(x)\ ,$$

and $L_g(x) = 0$ for each g; hence $x = 0$ as desired. (Note that the product of two elements in $I(A \otimes B)$ need not coincide with that as elements of $A \mathbin{\overline{\otimes}} B(K)$, but so do they if one of the elements belongs to $A \otimes B$, since $I(A \otimes B)$ is obtained as the image of a minimal $(A \otimes B)$-projection on $A \mathbin{\overline{\otimes}} B(K)$, which is an $(A \otimes B)$-module homomorphism. Hence the above calculation is justified.)

The following is an analogue of the result of Sakai [30].

THEOREM 9.2. *Let A be a C^*-algebra and let σ be the flip automorphism of the two-fold tensor product $A \otimes A$, that is, the *-automorphism defined by $\sigma(x \otimes y) = y \otimes x$, $x, y \in A$. Then σ is quasi-inner if and only if the injective envelope $I(A)$ is a type* I *W^*-factor. This is the case if and only if $C(H) \subset A \subset B(H)$ for some Hilbert space H* [15].

PROOF. As in 9.1 we may assume that A is injective.

Sufficiency: Suppose that $A = B(H)$ for some Hilbert space H. Then $C(H \otimes H) = C(H) \otimes C(H) \subset A \otimes A \subset B(H \otimes H)$ and so $I(A \otimes A) = B(H \otimes H)$ [15, 3.1]. If we define the unitary U in $B(H \otimes H)$ by $U(\xi \otimes \eta) = \eta \otimes \xi$, $\xi, \eta \in H$, then $\mathrm{Ad}\,U | A \otimes A = \sigma$ and $I(\sigma) = \mathrm{Ad}\,U$; hence σ is quasi-inner.

Necessity: Suppose that σ is quasi-inner, that is, $I(\sigma) \in \mathrm{Aut}\, I(A \otimes A)$ is inner. As in [30, Lemma 2] we see that A is an AW^*-factor. Indeed, let Z be the center of A. Then $Z \otimes Z$ is contained in the center of $I(A \otimes A)$ by [10, 4.3]; hence for each $x, y \in Z$ we have $x \otimes y = I(\sigma)(x \otimes y) = \sigma(x \otimes y) = y \otimes x$. But this shows that Z is 1-dimensional. Next we show that A contains a minimal projection. Let $\{\pi_i, H_i\}$ be a family of inequivalent irreducible *-representations of A such that the direct sum $\{\pi, H\}$ of the family is faithful. We identify A with its image $\pi(A)$ and regard $A \subset B(H)$, $A \otimes A \subset A \mathbin{\overline{\otimes}} B(H) \subset B(H \otimes H)$. If e_i is the projection onto H_i, then we have

$$A'' = \bigoplus e_i B(H) e_i \quad \text{and} \quad A' = \bigoplus \mathbf{C} e_i \ (C^*\text{-sum [2, p. 52]})\ ,$$

where the double prime (resp. prime) denotes the double commutant (resp. commutant). As in 9.1 we take the injective envelope $I(A \otimes A)$ so that $A \otimes A \subset I(A \otimes A) \subset A \mathbin{\overline{\otimes}} B(H)$. By assumption there is a unitary u in $I(A \otimes A)$ such that $I(\sigma)(x) = (\mathrm{Ad}\, u)(x) = u \circ x \circ u^*$ for $x \in I(A \otimes A)$, where $\circ$ denotes the multiplication in $I(A \otimes A)$. Note that for the reason stated in 9.1 we have $x \circ y = xy$ if x or y belongs to $A \otimes A$.

Hence, with U as above and $x \in A \otimes A$ we have $UxUu = \sigma(x)\circ u = u\circ x = ux$ and so $Uu \in (A \otimes A)' = A' \bar{\otimes} A' = \bigoplus_{i,j} \boldsymbol{C}(e_i \otimes e_j)$. Hence $u = U(\bigoplus \lambda_{ij}(e_i \otimes e_j))$, $\lambda_{ij} \in \boldsymbol{C}$. We have $\lambda_{ij} \neq 0$ for some i, j. Let ζ_1 and ζ_2 be unit vectors in e_iH and e_jH respectively and let $g \in B(H)_*$ be defined by $g = (\cdot\zeta_2, \zeta_1)$. Computation shows that $L_g(u) = \lambda_{ij}(\cdot, \zeta_1)\zeta_2 \in A$. Hence A contains the minimal projection $(\cdot, \zeta_1)\zeta_1$ and it is a type I W^*-factor.

REMARK. By a similar technique we can show that for any C^*-algebra A the projection $p(\sigma)$ of $I(A \otimes A)$ inducing the inner part of $I(\sigma)$ is given by $p(\sigma) = \sum h_i \otimes h_i$, where h_i runs through all central projections h in $I(A)$ such that $hI(A)$ is a type I W^*-factor, hence that σ is properly outer if and only if $I(A)$ has no non-zero atomic part.

10. Prime reduced C^*-crossed products. In [20, 3.1] Kishimoto gave a criterion for the simplicity of the reduced C^*-crossed product of a C^*-algebra by a discrete (not necessarily abelian) *-automorphism group (see also [9], [21, 2.3]). Now we present a primeness version of his result.

Let A be a C^*-algebra and B its C^*-subalgebra. Following Choda and Watatani [4] we say that a *-automorphism α of A is *B-subfreely acting* on A if $ab = b\alpha(a)$ for all $a \in A$ with $b \in B$ implies $b = 0$.

THEOREM 10.1. *Let (A, G, α) be a C^*-dynamical system with G any discrete group. For $t \in G$ put $G(t) = \{s \in G: st = ts\}$ and let $\bar{A}^{G(t)}$ be the fixed point subalgebra of $\bar{A}$ under the action $\bar{\alpha}|G(t)$. If A is G-prime and $\bar{\alpha}_t$ is $\bar{A}^{G(t)}$-subfreely acting on $\bar{A}$ for each $t \in G\setminus\{e\}$ (in particular if α_t is properly outer for each $t \in G\setminus\{e\}$), then the reduced C^*-crossed product $A \times_{\alpha r} G$ is prime. Conversely, if in addition G is finite, then the primeness of $A \times_{\alpha r} G = A \times_\alpha G$ implies that A is G-prime and $\bar{\alpha}_t$ is $\bar{A}^{G(t)}$-subfreely acting on $\bar{A}$ for each $t \in G\setminus\{e\}$. The same is true if one replaces $\bar{\alpha}$ and $\bar{A}$ by $I(\alpha)$ and $I(A)$.*

LEMMA 10.2. *Let B be a monotone complete C^*-algebra and C its C^*-subalgebra. Let $D = m\text{-cl}_B C$ be the monotone closure of C in B.*

(i) *The supremum in B of any positive increasing approximate unit for C is a projection of B which serves as a unit for D.*

(ii) *If E is a hereditary C^*-subalgebra of C, then there is a unique projection p of D such that $m\text{-cl}_B E = pDp$. If in particular E is a closed two-sided ideal of C, then the projection p is a central projection of D.*

PROOF. As in [13] we write $x_i \to x$ (O) in B if a net $\{x_i\}$ in B order-converges to $x \in B$, and we freely use the computation rules for order convergence in [13, 1.2] or [17, 2.1].

(i) If $\{a_i\}$ is a positive increasing approximate unit for C, then $a_i \nearrow p$ (O) in B for some $p \in D^+$. For each $x \in C$ we have $x = xp$, since $xa_i \to x$ in norm and $xa_i \to xp$ (O). In particular, $a_i = a_i p \to p^2$ (O) and so $p^2 = p$. Moreover, $x = xp$ for all $x \in D$ since $D = m\text{-cl}_B C$.

(ii) By (i) the supremum p in B of a positive increasing approximate unit $\{b_i\}$ for E is a projection of D. As $E \ni b_i x b_i \to pxp \in pCp \subset pDp$ (O) for each $x \in C$, it follows that $pCp \subset m\text{-cl}_B E$, hence that $pDp = p(m\text{-cl}_B C)p = m\text{-cl}_B\, pCp \subset m\text{-cl}_B E$ [13, 2.4]. The reverse inclusion is clear since pDp contains E and is monotone closed in B.

If E is a closed two-sided ideal of C, then for each $x \in C_{sa}$ we have $E \ni xb_i x \to xpx \in m\text{-cl}_B E = pDp$ and $(1-p)xpx(1-p) = 0$. Hence $px(1-p) = 0$, $px = pxp = (pxp)^* = (px)^* = xp$ and so p commutes with each element of $m\text{-cl}_B C = D$.

Let (B, G, β) be a C^*-dynamical system with B monotone complete and G discrete. As in Section 3 define the monotone complete crossed product $M(B, G)$ as a monotone closed C^*-subalgebra of the monotone complete C^*-algebra $B \mathbin{\overline{\otimes}} B(l^2(G))$, and the maps π and λ.

LEMMA 10.3. *Keep the above notation.*

(i) *For* $x = [x_{r,s}] \in M(B, G)$ *consider the following conditions:*

(a) x *belongs to the center of* $M(B, G)$;

(b) x *commutes with* $\pi(B)\lambda(G)$ *elementwise;*

(c) $x_{tr,ts} = x_{r,s}$ *for all* $r, s, t \in G$ *and* $ax_{r,s} = x_{r,s}\beta_{s^{-1}r}(a)$ *for all* $r, s \in G$ *and* $a \in B$;

(d) $x_{r,e} \in B^{G(r)}$ *for all* $r \in G$ *and* $ax_{r,e} = x_{r,e}\beta_r(a)$ *for all* $r \in G$ *and* $a \in B$.

Then we have (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d).

(ii) *If* β_t *is* $B^{G(t)}$*-subfreely acting on* B *for each* $t \in G \setminus \{e\}$ *and* G *acts ergodically on the center of* B, *then* $M(B, G)$ *is a monotone complete* AW^**-factor.*

(iii) *If there is a* $t \in G \setminus \{e\}$ *such that the conjugacy class of* t *is finite and* β_t *is not* $B^{G(t)}$*-subfreely acting on* B, *then* $M(B, G)$ *is not an* AW^**-factor.*

PROOF. (i) We omit the proof of (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c), since the corresponding proof for the W^*-crossed product works also in this situation.

(c) $\Rightarrow$ (d). Note that $\beta_t(x_{r,e}) = x_{rt^{-1},t^{-1}} = x_{trt^{-1},e} = x_{r,e}$ for all $t \in G(r)$.

(ii) If $x \in M(B, G)$ is central, then (d) shows that $x_{r,e} = 0$ for all $r \in G \setminus \{e\}$ and $x_{e,e}$ is a G-invariant central element of B. Thus x is a scalar multiple of the unit.

(iii) Let $\{s_j t s_j^{-1}: 1 \leqq j \leqq n\}$ be the finite conjugacy class of t, where $s_j t s_j^{-1} \neq s_k t s_k^{-1}$ if $j \neq k$. By hypothesis there is a non-zero $b \in B^{G(t)}$ such that $ab = b\beta_t(a)$ for all $a \in B$. Put $x=\sum_j \pi(\beta_{s_j}(b))\lambda(s_jts_j^{-1})$. For each $r \in G$ and j we have $rs_jt(rs_j)^{-1} = s_kts_k^{-1}$ for some k and $s_k^{-1}rs_j \in G(t)$, so that $\beta_{rs_j}(b) = \beta_{s_k}(b)$. Hence

$$\begin{aligned}\lambda(r)x &= \sum_j \lambda(r)\pi(\beta_{s_j}(b))\lambda(r^{-1})\lambda(rs_jts_j^{-1}) \\ &= \left[\sum_j \pi(\beta_{rs_j}(b))\lambda(rs_jt(rs_j)^{-1})\right]\lambda(r) = x\lambda(r)\end{aligned}$$

and for each $a \in \beta$,

$$\begin{aligned}\pi(a)x &= \sum_j \pi(\beta_{s_j}(\beta_{s_j^{-1}}(a)b))\lambda(s_jts_j^{-1}) \\ &= \sum_j \pi(\beta_{s_j}(b\beta_t \circ \beta_{s_j^{-1}}(a)))\lambda(s_jts_j^{-1}) = x\pi(a)\ .\end{aligned}$$

Thus x is a nontrivial central element of $M(B, G)$.

PROOF OF THEOREM 10.1. We prove only the statement for $\bar{\alpha}$ and $\bar{A}$, since the proof for $I(\alpha)$ and $I(A)$ proceeds similarly. The G-primeness of A is equivalent to the G-primeness of $\bar{A}$, or to saying that G acts ergodically on the center of $\bar{A}$ (see 6.5(ii)). The proper outerness of α_t is equivalent to that of $\bar{\alpha}_t$ (see 7.5), which implies that $\bar{\alpha}_t$ is $\bar{A}^{G(t)}$-subfreely acting on $\bar{A}$, since on a monotone complete C^*-algebra proper outerness is equivalent to being freely acting. Moreover by 3.6(i), $A\times_{\alpha r}G$ is prime if and only if $\bar{A}\times_{\alpha r}G$ is prime.

Hence, by replacing (A, G, α) by $(\bar{A}, G, \bar{\alpha})$ we may assume that A is monotone complete. Then $A\times_{\alpha r}G$ is identified with the C^*-subalgebra of $M(A, G)$ generated by $\pi(A)\lambda(G)$ and Lemma 10.3(ii) shows that if A is G-prime and α_t is $A^{G(t)}$-subfreely acting on A for each $t \in G \setminus \{e\}$, then $M(A, G)$ is a monotone complete AW^*-factor. If $A\times_{\alpha r}G$ is not prime, then there is a nontrivial regular ideal J of $A\times_{\alpha r}G$ and $m\text{-cl}\, J = p(m\text{-cl}\, A\times_{\alpha r}G)$ for some nontrivial central projection p of $M(A, G)$ by 10.2 and 10.3(i), a contradiction.

Clearly the primeness of $A\times_{\alpha r}G$ implies the G-primeness of A whether G is finite or not. If G is finite, then $A\times_{\alpha r}G = M(A, G)$ and the second assertion follows from 10.3(iii).

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