Tôhoku Math. Journ.
37 (1985), 455-462.

STABILITY PROPERTIES OF SOLUTIONS OF LINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

WANG ZHICHENG, LI ZHIXIANG AND WU JIANHONG

(Received August 2, 1984)

Consider the following systems of Volterra equations

$$Z'(t) = A(t)Z(t) + \int_0^t C(t, s)Z(s)ds \,, \tag{1}$$

$$y'(t) = A(t)y(t) + \int_0^t C(t, s)y(s)ds + f(t) \,, \tag{2}$$

$$x'(t) = A(t)x(t) + \int_{-\infty}^t C(t, s)x(s)ds + f(t) \,, \tag{3}$$

where A is an $n\times n$ matrix of functions continuous on $(-\infty, +\infty)$, C is an $n \times n$ matrix of functions continuous for $-\infty < s \leqq t < \infty$, and $f: (-\infty, +\infty) \to R^n$ is continuous. For the fundamental properties of solutions of these equations, we refer to Driver [4] and Burton [2]. Some of those properties may be listed as follows:

(a) There is an $n\times n$ matrix $Z(t)$ satisfying (1) on $[0, \infty)$ and $Z(0) = I$. For each $z_0 \in R^n$, there is a unique solution $z(t, 0, z_0)$ of (1) on $[0, \infty)$ and $z(t, 0, z_0) = Z(t)z_0$.

(b) For (2), given $t_0 \geqq 0$ and a continuous function $\varphi: [0, t_0] \to R^n$, there is a unique solution $y(t, t_0, \varphi)$ satisfying (2) on $[t_0, \infty)$ with $y(t, t_0, \varphi) = \varphi(t)$ for $t \in [0, t_0]$.

(c) For (3) we suppose that $\int_{-\infty}^0 |C(t, s)|ds$ is continuous for $0 \leqq t < \infty$. If $t_0 \in R$ and if $\varphi: (-\infty, t_0] \to R^n$ is a bounded continuous function, there is a unique solution $x(t, t_0, \varphi)$ satisfying (3) on $[t_0, \infty)$ with $x(t, t_0, \varphi) = \varphi(t)$ for $t \leqq t_0$.

(d) There is a unique $n\times n$ matrix $R(t, s)$ satisfying

$$\frac{\partial}{\partial s}R(t, s) = -R(t, s)A(s) - \int_s^t R(t, u)C(u, s)du \,, \qquad R(t, t) = I \tag{4}$$

for $0 \leqq s \leqq t < \infty$. For each $y_0 \in R^n$, the unique solution $y(t, 0, y_0)$ of (2) satisfies

$$y(t, 0, y_0) = Z(t)y_0 + \int_0^t R(t, s)f(s)ds \,, \tag{5}$$

where $Z(t) = R(t, 0)$. If, in addition,

$$\text{(A)} \qquad A(t + T) = A(t) , \qquad C(t + T, s + T) = C(t, s) ,$$

then $R(t + T, s + T) = R(t, s)$.

The purpose of this paper is to discuss the stability properties of solutions of the homogeneous equations

$$(6) \qquad y'(t) = A(t)y(t) + \int_0^t C(t, s)y(s)ds ,$$

$$(7) \qquad x'(t) = A(t)x(t) + \int_{-\infty}^t C(t, s)x(s)ds .$$

DEFINITION 1. The zero solution of (6) is called

(i) stable if for every $\varepsilon > 0$ and any $t_0 \geqq 0$ there exists a $\delta > 0$ such that $|\varphi(t)| < \delta$ on $[0, t_0]$ and $t \geqq t_0$ imply

$$|x(t, t_0, \varphi)| < \varepsilon ;$$

(ii) uniformly stable if it is stable and the δ above is independent of t_0;

(iii) asymptotically stable if it is stable and if for each $t_0 \geqq 0$ there is an $\eta > 0$ such that $|\varphi(t)| < \eta$ on $[0, t_0]$ implies

$$x(t, t_0, \varphi) \to 0 \quad \text{as} \quad t \to \infty ;$$

(iv) uniformly asymptotically stable if it is uniformly stable, the η above is independent of t_0, and for every $\varepsilon > 0$ there is a $T(\varepsilon) > 0$ such that $|\varphi(t)| < \eta$ on $[0, t_0]$ and $t \geqq t_0 + T(\varepsilon)$ imply

$$|x(t, t_0, \varphi)| < \varepsilon .$$

The various stability properties for the zero solution of (7) can be defined in the same way as the corresponding type of stability for (6).

THEOREM 1. *The zero solution of* (6) *is*

(i) *stable if and only if for any* $\tau > 0$, *there is an* $M(\tau) > 0$, *such that*

$$W_\tau(t) := \int_{-\infty}^0 \left| \int_0^t R_\tau(t, s)C(s + \tau, u + \tau)ds \right| du \leqq M(\tau)$$

and $|R_\tau(t, 0)| \leqq M(\tau)$ *for* $t \geqq 0$, *where* $R_\tau(t, s)$ *is the unique solution of* (4) *with* $A(s)$ *and* $C(u, s)$ *replaced by* $A(s + \tau)$ *and* $C(u + \tau, s + \tau)$, *respectively.*

(ii) *uniformly stable if and only if there is an* $M > 0$ *such that* $|W_\tau(t)| \leqq M$, *and* $|R_\tau(t, 0)| \leqq M$ *for all* $t \geqq 0$ *and* $\tau \geqq 0$.

(iii) *asymptotically stable if and only if it is stable and both* $W_\tau(t)$ *and* $R_\tau(t, 0)$ *tend to zero as* $t \to \infty$ *for any* $\tau \geqq 0$.

(iv) *uniformly asymptotically stable if and only if it is uniformly stable and both $W_\tau(t)$ and $R_\tau(t, 0)$ tend to zero uniformly in $\tau \geqq 0$.*

PROOF. Given initial values (τ, f), let $y(t) = y(t, \tau, f)$ be the corresponding solution of (6). To prove (i), assume that for any $\tau \geqq 0$, there is an $M(\tau) > 0$ such that $|W_\tau(t)| \leqq M(\tau)$ and $|R_\tau(t, 0)| \leqq M(\tau)$ for $t \geqq 0$. We have

$$\begin{aligned} y'(t+\tau) &= A(t+\tau)y(t+\tau) + \int_0^{t+\tau} C(t+\tau, s)y(s)ds \\ &= A(t+\tau)y(t+\tau) + \int_0^t C(t+\tau, s+\tau)y(s+\tau)ds \\ &\quad + \int_{-\tau}^0 C(t+\tau, s+\tau)f(s+\tau)ds \end{aligned}$$

by (d) above, and

$$\begin{aligned} y(t+\tau) &= R_\tau(t, 0)f(\tau) + \int_0^t R_\tau(t, s)\Big(\int_{-\tau}^0 C(s+\tau, u+\tau)f(u+\tau)du\Big)ds \\ &= R_\tau(t, 0)f(\tau) + \int_{-\tau}^0 \Big(\int_0^t R_\tau(t, s)C(s+\tau, u+\tau)ds\Big)f(u+\tau)du \,. \end{aligned}$$

Then

$$\begin{aligned} |y(t+\tau, \tau, f)| &\leqq |R_\tau(t, 0)f(\tau)| + W_\tau(t)\,\|f\|_\tau \\ &\leqq M(\tau)\,|f(\tau)| + M(\tau)\,\|f\|_\tau \\ &\leqq 2M(\tau)\,\|f\|_\tau \,, \end{aligned}$$

where $\|f\|_\tau = \sup\{f(t): 0 \leqq t \leqq \tau\}$. Thus, $x = 0$ is stable.

Now assume that $x = 0$ is stable. Then for any $\tau \geqq 0$, there exists a constant $B(\tau) > 0$ such that $\|f\|_\tau \leqq 1$ implies $|y(t+\tau, \tau, f)| \leqq B(\tau)$ for all $t \geqq 0$. Now fix t, τ, and choose f so that $\|f\|_\tau \leqq 1$, $f(\tau)$ a unit vector and

$$\Big|\int_{-\tau}^0 \Big[\int_0^t R_\tau(t, s)C(s+\tau, u+\tau)ds\Big]f(u+\tau)du\Big| \leqq 1 \,.$$

Then

$$\begin{aligned} |R_\tau(t, 0)f(\tau)| &\leqq |y(t+\tau, \tau, f)| \\ &\quad + \Big|\int_{-\tau}^0 \Big[\int_0^t R_\tau(t, s)C(s+\tau, u+\tau)ds\Big]f(u+\tau)du\Big| \\ &\leqq B(\tau) + 1 \,. \end{aligned}$$

So

$$|R_\tau(t, 0)| \leqq B(\tau) + 1 \,.$$

Furthermore, for all $t \geqq 0$ and $\|f\|_\tau \leqq 1$, we have

$$|y(t+\tau, \tau, f) - R_\tau(t, 0)f(\tau)| \leqq 2B(\tau) + 1 .$$

Then

$$|W_\tau(t)| \leqq 2B(\tau) + 1 \quad \text{for} \quad t \geqq 0 .$$

To prove part (ii), we note that B can be chosen independently of $\tau \geqq 0$. Parts (iii) and (iv) follow in a similar manner.

THEOREM 2.
(i) *The following statements are equivalent*:
(a) *the zero solution of* (6) *is stable*;
(b) *the zero solution of* (7) *is stable*;
(c) *for any* $\tau \geqq 0$, *and for every* $F \in BC(-\infty, +\infty)$, *there are* $M(\tau)$ *and* $M^*(\tau, F)$ *such that* $|R_\tau(t, 0)| \leqq M(\tau)$ *and* $|x(t, \tau, F)| \leqq M^*(\tau, F)$ *for all* $t \geqq 0$, *where* $x(t, \tau, F)$ *is the solution of* (7).
(ii) *The following statements are equivalent*:
(a) *the zero solution of* (6) *is uniformly stable*;
(b) *the zero solution of* (7) *is uniformly stable*;
(c) $R_\tau(t, 0)$ *is bounded on* R^+ *uniformly in* $\tau \geqq 0$ *and for each* $F \in BC(-\infty, +\infty)$ *the solution* $x(t, \tau, F)$ *of* (7) *is bounded on* R^+ *uniformly in* $\tau \geqq 0$.

The proof differs very little from that of Miller's Theorem 2 in [5] and therefore is ommitted.

THEOREM 3. *The following statements are equivalent*:
(a) *the zero solution of* (6) *is uniformly asymptotically stable*;
(b) *the zero solution of* (7) *is uniformly asymptotically stable*;
(c) $R_\tau(t, 0)$ *tend to zero uniformly in* τ *as* $t \to \infty$ *and for each* $F \in BC(-\infty, +\infty)$, *the solution* $x(t, \tau, F)$ *of* (7) *tends to zero as* $t \to \infty$ *uniformly in* τ.

LEMMA 1. *If* (A) *holds and*

$$\text{(B)} \qquad \int_{-\infty}^{t} |C(t, s)|ds \quad \text{is continuous in} \quad t \in (-\infty, +\infty) ,$$

then the solution $y(t, \tau, f)$ *of* (6) *has the property that for any* $L > 0$ *there is a* $B(L) > 1$ *such that* $\tau \geqq 0$ *and* $t \in [\tau, \tau + L]$ *imply*

$$|y(t, \tau, f)| \leqq B(L)\|f\|_\tau .$$

PROOF. For $\tau \geqq 0$ and $t \in [\tau, \tau + L]$,

$$y'(t, \tau, f) = A(t)y(t) + \int_0^t C(t, s)y(s)ds ,$$

$$\begin{aligned} y(t, \tau, f) &= f(\tau) + \int_\tau^t A(s)y(s)ds + \int_\tau^t du \int_0^u C(u, s)y(s)ds \\ &= f(\tau) + \int_\tau^t A(s)y(s)ds + \int_\tau^t du \int_0^\tau C(u, s)y(s)ds \\ &\quad + \int_\tau^t du \int_\tau^u C(u, s)y(s)ds \\ &= f(\tau) + \int_\tau^t du \int_0^\tau C(u, s)f(s)ds + \int_\tau^t ds \int_s^t C(u, s)y(s)du \\ &\quad + \int_\tau^t A(s)y(s)ds \, . \end{aligned}$$

Thus

$$\begin{aligned} |y(t, \tau, f)| &\leqq \left(1 + \int_\tau^{\tau+L} du \int_0^\tau |C(u, s)|ds\right)\|f\|_\tau \\ &\quad + \int_\tau^t \left(|A(s)| + \int_s^t |C(u, s)|du\right)|y(s)|ds \\ &\leqq \left(1 + \int_\tau^{\tau+L} du \int_0^u |C(u, s)|ds\right)\|f\|_\tau \\ &\quad + \int_\tau^t \left(|A(s)| + \int_\tau^t |C(u, s)|du\right)|y(s)|ds \\ &\leqq (1 + LM_1)\|f\|_\tau + M_2 \int_\tau^t |y(s)|ds \, , \end{aligned}$$

where

$$M_1 = \sup_{t \geqq 0} \int_{-\infty}^t |C(t, s)|ds \, , \qquad M_2 = \sup_s |A(s)| + L \sup_{0 \leqq \tau \leqq s \leqq t \leqq \tau + 2L} |C(t, s)| \, .$$

From Bellman's inequality, we get

$$|y(t, \tau, f)| \leqq (1 + LM_1)\|f\|_\tau \exp(M_2(t-\tau)) \leqq (1 + LM_1)\|f\|_\tau \exp(LM_2) \, .$$

COROLLARY 1. *Suppose* (A) *and* (B) *hold.*

(i) *The following statements are equivalent:*

(a) *the solution* $y(t, \tau, f)$ *of* (6) *is uniformly bounded, that is, for any* $\alpha > 0$, *there is a* $B(\alpha) > 0$, *such that* $\tau \geqq 0$, $\|f\|_\tau \leqq \alpha$, *and* $t \geqq \tau$ *imply*

$$|y(t, \tau, f)| \leqq B(\alpha) \; ; \tag{8}$$

(b) *the solution* $y(t, \tau, f)$ *of* (6) *is uniformly bounded for* $\tau \in \{kT: k = 1, 2, \cdots\}$, *that is,* (8) *holds only for* $\tau \in \{kT: k = 1, 2, \cdots\}$.

(ii) *The following statements are equivalent:*

(a) *the zero solution of* (6) *is stable* (*resp. uniformly stable, resp. uniformly asymptotically stable*);

(b) *the zero solution of* (6) *is stable* (*resp. uniformly stable, resp. uniformly asymptotically stable*) *for* $\tau \in \{kT: k = 1, 2, \cdots\}$.

LEMMA 2. *Suppose* (A) *and* (B) *hold. Let* $W(t) = W_0(t)$, $R(t, 0) = R_0(t, 0)$. *Then*

(i) *for* $kT \leqq \tau \leqq (k+1)T$, $t \geqq (k+1)T - \tau$,

$$|R_\tau(t, 0)| \leqq (|R(t^*, 0)| + W(t^*)) \max_{0 \leqq s \leqq T} B(s) ,$$

$$|W_\tau(t)| \leqq 2(|R(t^*, 0)| + W(t^*)) \max_{0 \leqq s \leqq T} B(s) ,$$

where $t^* = t - ((k+1)T - \tau)$;

(ii) *for* $kT \leqq \tau \leqq (k+1)T$, $t \geqq \tau - kT$,

$$|R(t, 0)| \leqq (|R_\tau(t^{**}, 0)| + W_\tau(t^{**})) \max_{0 \leqq s \leqq T} B(s) ,$$

$$|W(t)| \leqq 2(|R_\tau(t^{**}, 0)| + W_\tau(t^{**})) \max_{0 \leqq s \leqq T} B(s) ,$$

where $t^{**} = t - (\tau - kT)$.

PROOF. For $0 \leqq t \leqq (k+1)T - \tau$, by Lemma 1 we have

$$|y(t + \tau, \tau, f)| \leqq B(t) \|f\|_\tau \leqq \max_{0 \leqq s \leqq T} B(s) \|f\|_\tau .$$

Let $\psi(s) = y(s, \tau, f)$, $0 \leqq s \leqq (k+1)T$. Then

$$\|\psi\|_{(k+1)T} \leqq \max_{0 \leqq s \leqq T} B(s) \|f\|_\tau .$$

For $t \geqq (k+1)T - \tau$, $t = (k+1)T - \tau + t^*$,

$$\begin{aligned} y(t + \tau, \tau, f) &= y(t^* + (k+1)T, \tau, f) = y(t^* + (k+1)T, (k+1)T, \psi) \\ &= R(t^*, 0)\psi((k+1)T) \\ &\quad + \int_{-(k+1)T}^{0} \left\{ \int_0^{t^*} R(t^*, s) C(s, u) ds \right\} \psi(u + (k+1)T) du , \end{aligned}$$

$$\begin{aligned} |y(t + \tau, \tau, f)| &\leqq (|R(t^*, 0)| + W(t^*)) \|\psi\|_{(k+1)T} \\ &\leqq (|R(t^*, 0)| + W(t^*)) \max_{0 \leqq s \leqq T} B(s) \|f\|_\tau . \end{aligned}$$

For fixed t, τ and any $\varepsilon > 0$, choose f so that $|f(\tau)| = 1$, $\|f\|_\tau \leqq 1$ and

$$\left| \int_{-\tau}^{0} \left(\int_0^t R_\tau(t, s) C_\tau(s, u) ds \right) f(\tau + u) du \right| \leqq \varepsilon ,$$

where

$$C_\tau(s, u) = C(s + \tau, u + \tau) , \qquad R_\tau(t, s) = R(t + \tau, s + \tau) .$$

Then, since

$$y(t+\tau, \tau, f) = R_\tau(t, 0)f(\tau) + \int_{-\tau}^{0}\left(\int_0^t R_\tau(t, s)C_\tau(s, u)ds\right)f(\tau+u)du ,$$

we have

$$|y(t, \tau, f)| \geqq |R_\tau(t, 0)f(\tau)| - \varepsilon ,$$

and hence

$$|R_\tau(t, 0)f(\tau)| \leqq \varepsilon + (|R(t^*, 0)| + W(t^*))\max_{0\leqq s\leqq T} B(s) .$$

This implies

$$|R_\tau(t, 0)| \leqq (|R(t^*, 0)| + W(t^*))\max_{0\leqq s\leqq T} B(s) ,$$

$$\left|\int_{-\tau}^{0}\left(\int_0^t R_\tau(t, s)C_\tau(s, u)ds\right)f(\tau+u)du\right| \leqq |y(t+\tau, \tau, f)| + |R_\tau(t, 0)f(\tau)|$$
$$\leqq (|R(t^*, 0)| + W(t^*))\max_{0\leqq s\leqq T} B(s)(\|f\|_\tau + |f(\tau)|) .$$

Thus

$$|W_\tau(t)| \leqq 2(|R(t^*, 0)| + W(t^*))\max_{0\leqq s\leqq T} B(s) .$$

(ii) For $0 \leqq t \leqq \tau - kT$,

$$y(t+kT, kT, f) \leqq B(t)\|f\|_\tau \leqq \max_{0\leqq s\leqq T} B(s)\|f\|_\tau .$$

Let $\psi = y(t, kT, f)$, $0 \leqq t \leqq \tau$. Then

$$\begin{aligned} y(t+kT) &= y(\tau + t^{**}, kT, f) = y(\tau + t^{**}, \tau, \psi) \\ &= R_\tau(t^{**}, 0)y(\tau) + \int_{-\tau}^{0}\left(\int_0^{t^{**}} R_\tau(t^{**}, s)C_\tau(s, u)ds\right)\psi(u+\tau)du \\ &\leqq (|R_\tau(t^{**}, 0)| + W_\tau(t^{**}))\|\psi\|_\tau \\ &\leqq (|R_\tau(t^{**}, 0)| + W_\tau(t^{**}))\max_{0\leqq s\leqq T} B(s)\|f\|_{kT} \end{aligned}$$

for $t \geqq \tau - kT$. But

$$y(t+kT, kT, f) = R(t, 0)f(kT) + \int_{-kT}^{0}\left(\int_0^t R(t, s)C(s, u)ds\right)f(u+kT)du .$$

Using the same proof as above, one obtains

$$|R(t, 0)| \leqq (|R_\tau(t^{**}, 0)| + W_\tau(t^{**}))\max_{0\leqq s\leqq T} B(s) ,$$

$$|W(t)| \leqq 2(|R_\tau(t^{**}, 0)| + W_\tau(t^{**}))\max_{0\leqq s\leqq T} B(s) ,$$

and the lemma is proved.

COROLLARY 2. *For* (6) *or* (7),

(i) *the zero solution is uniformly stable if and only if there is a* $t_0 \geqq 0$, *such that the zero solution is stable at* $\tau = t_0$;

(ii) *the zero solution is uniformly asymptotic stable if and only if there is a* $t_0 \geqq 0$, *such that the zero solution is uniformly asymptotically stable at* $\tau = t_0$.

COROLLARY 3. *If* (A) *and* (B) *hold, then the following are equivalent*:

(i) *the zero solution of* (6) *is uniformly stable*;

(ii) $Z(t)$ *and* $W(t) = \int_{-\infty}^{0} \left| \int_{0}^{t} R(t, u)C(u, s)du \right| ds$ *are bounded on* R^+;

(iii) *the zero solution of* (7) *is uniformly stable.*

COROLLARY 4. *If* (A) *and* (B) *hold, then the following are equivalent*:

(i) $Z(t) \to 0$, $W(t) \to 0$ *as* $t \to \infty$;

(ii) *the zero solution of* (6) *is uniformly asymptotically stable*;

(iii) *the zero solution of* (7) *is uniformly asymptotically stable.*

The authors would like to thank Professor T. A. Burton for his helpful discussions.

REFERENCES

[1] T. A. BURTON, Volterra Integral and Differential Equations, Academic Press, New York, 1983.

[2] T. A. BURTON, Lecture Notes on Periodic Solutions of Volterra Equations, to appear.

[3] T. A. BURTON AND W. E. MAHFOUD, Stability criteria for Volterra equations, Trans. Amer. Math. Soc., 279 (1983), 143–174.

[4] R. D. DRIVER, Existence and stability of a delay-differential system, Arch. Rational Mech. Anal., 10 (1962), 401–426.

[5] R. K. MILLER, Asymptotic stability properties of linear Volterra integrodifferential equations, J. Differential Equations, 10 (1971), 485–506.

INSTITUTE OF APPLIED MATHEMATICS
HUNAN UNIVERSITY
CHANGSHA, HUNAN 1801
PEOPLE'S REPUBLIC OF CHINA