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RATIONAL FUNCTIONS OF C^*-TYPE ON THE TWO-DIMENSIONAL COMPLEX PROJECTIVE SPACE

TAKASHI KIZUKA

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Introduction. A non-constant rational function f on a smooth algebraic surface X defines a holomorphic mapping of $X \setminus I_f$ onto the Riemann sphere $\boldsymbol{P}$, where I_f denotes the set of all points of indetermination of f. The set $L_c = \{p \in X \setminus I_f \mid f(p) = c\}$ is called the *level curve* of f with value $c \in \boldsymbol{P}$. Following Nishino, we call an irreducible component of L_c a *prime curve* of f (with value c). If a smooth prime curve S is analytically isomorphic to the punctured Gaussian plane $\boldsymbol{C}^*$, we say that S is of $\boldsymbol{C}^*$-*type*. If all the prime curves of f, except for a finite number of them, are of $\boldsymbol{C}^*$-type, we say that f is of $\boldsymbol{C}^*$-*type*. The terms "$\boldsymbol{C}$-*type*" and "$\boldsymbol{P}$-*type*" for prime curves, and for rational functions, are defined similarly, where $\boldsymbol{C}$ is the Gaussian plane. If a rational function f is of $\boldsymbol{C}$-type, or if f is of $\boldsymbol{C}^*$-type, we say that f is of *special type*. In the previous paper [3], we have shown the following fact.

THEOREM 0. *Let C be an algebraic curve in the complex projective plane $\boldsymbol{P}^2$. If the complement $\boldsymbol{P}^2 \setminus C$ has an analytic transcendental automorphism, then C is a smooth cubic curve or there exists a rational function f of special type on $\boldsymbol{P}^2$ whose restriction to $\boldsymbol{P}^2 \setminus C$ is still of special type. In the latter case, C contains at least one prime curve of f.*

This theorem poses the problem to determine all the rational functions of special type on $\boldsymbol{P}^2$. If a rational function f on $\boldsymbol{P}^2$ is of special type, then, for each non-constant rational function ψ on $\boldsymbol{P}$ and for each analytic automorphism S of $\boldsymbol{P}^2$, $\psi \circ f \circ S$ is of special type. So the problem is reduced to that of determining a canonical form of a rational function of special type. If a rational function f of special type on $\boldsymbol{P}^2$ has a prime curve S of degree one (a complex line), we say that f belongs to the family $\mathscr{F}_{\mathrm{I}}$. In this case, regarding the closure $\bar{S}$ of S as the line at infinity, we may regard f as a rational function of special type on $\boldsymbol{C}^2 = \boldsymbol{P}^2 \setminus \bar{S}$. The rational functions of $\boldsymbol{C}$-type on $\boldsymbol{C}^2$ were determined by Jung [1] and the rational functions of $\boldsymbol{C}^*$-type on $\boldsymbol{C}^2$ were determined by Kashiwara (née Saito). If a rational function f of special type on $\boldsymbol{P}^2$ has no prime curve of degree one, we say that f belongs to the family $\mathscr{F}_{\mathrm{II}}$. The rational functions belonging to $\mathscr{F}_{\mathrm{I}}$ are simpler than those belonging to $\mathscr{F}_{\mathrm{II}}$. Recently, Kashiwara [2] has determined all the rational functions of $\boldsymbol{C}$-type belonging to $\mathscr{F}_{\mathrm{II}}$ by her systematic study. In this paper, we resolve the remaining problem of determining all the rational functions of $\boldsymbol{C}^*$-type belonging to $\mathscr{F}_{\mathrm{II}}$.

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Chapter 0. Summary.

§ 1. Reciprocity.

1. A prime curve S of a rational function f with value c ($\neq \infty$) is said to be *of order* ν if the function $f-c$ takes the value zero of order ν on S. A prime curve S with value ∞ is said to be *of order* ν if the function $1/f$ takes the value zero of order ν on S. If the order ν of S is greater than one, we say that S is *multiple*. If each level curve of f is irreducible except for a finite number of level curves, f is called *primitive*. Proposition 1 in Chapter I implies that we have only to determine all the primitive rational functions of $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2$.

Let f be a rational function of special type on $\boldsymbol{P}^2$. Denote by

$$M = M_m \xrightarrow{\sigma_m} M_{m-1} \xrightarrow{\sigma_{m-1}} \cdots \xrightarrow{\sigma_2} M_1 \xrightarrow{\sigma_1} M_0 = \boldsymbol{P}^2$$

the minimal sequence of σ-processes which resolves the indetermination points of f. Set $\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_m$. The pull-back $\sigma^* f$ of f by σ is a rational function on M with no indetermination point. We denote by $\Sigma(f)$ the complete inverse image $\sigma^{-1}(I_f)$ of the set of indetermination points I_f of f under σ in this paper.

Let g be a non-constant primitive rational function on a smooth rational surface V. Consider an algebraic compactification (M, ι) of V, which means that there exist an algebraic curve E on a compact smooth algebraic surface M and a birational biregular isomorphism ι of the complement $M \setminus E$ of E onto V. Suppose that $\iota^* g$ has no indetermination point on M. An irreducible component C of the curve E is called a *basic section* of $\iota^* g$ if the restriction $\iota^* g|_C$ of $\iota^* g$ to C is not constant. Denote by B the union of basic sections of $\iota^* g$. If the function g on V is of C^*-type, then B consists of one or two irreducible components (cf. Chapter I § 1.1). If B is reducible, we say that g is of *direct* C^*-type. If B is irreducible, we say that g is of *torsional* C^*-type. We say also that g is of *proper* C^*-type if g satisfies the following three conditions; (i) g has no indetermination point on V, (ii) g does not take the values 0, ∞ and the regular mapping $g: V \to C^*$ is surjective and (iii) each level curve of g is irreducible, of C^*-type and of order one. The following theorem proved in Chapter III, § 1.3 is used several times in this paper.

THEOREM 1. *Let C be an algebraic curve in P^2. Suppose that there exists a rational function of proper (resp. proper direct) C^*-type on $V = P^2 \setminus C$. If a non-constant primitive rational function g on V does not take the value* 0, ∞, *then g is also of proper (resp. proper direct) C^*-type on V.*

2. Rational functions of special type on P^2 are intimately related to each other as is seen by Theorem 1. The rational functions belonging to the class (D_0), defined in the following, play a pivotal role in this relation of functions. We say that a primitive rational function f of direct C^*-type on P^2 belongs to the *class* (D_0) if f satisfies the following five conditions; (i) I_f consists of only one point, (ii) the level curve L_0 consists of two prime curves both of which are of C-type and of order one, and those two prime curves intersect each other transversally, (iii) the level curve L_∞ is irreducible and of C-type, (iv) the level curve L_1 is irreducible, of C^*-type and multiple and (v) the other level curves are irreducible and of C^*-type.

In Chapter II, we determine the graph of $\Sigma(f)$ of a rational function f bolonging to the class (D_0), using the following two properties; (i) $\Sigma(f)$ is a (reducible) exceptional curve of the first kind and (ii) $\sigma^* f$ is a rational function of P-type on M. Therefore we can construct inductively all the rational functions belonging to the class (D_0). (cf. Example [B] and Example [C] in Chapter II, § 1.2.) In this process, we obtain all the rational functions of $\boldsymbol{C}$-type belonging to $\mathscr{F}_{\mathrm{II}}$. By the same method, we can determine all the rational functions of $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2$. But it is very complicated. In this paper, we use Theorem 1 to determine those functions as is seen in the following paragraph.

Suppose that f is of direct $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2$. By a topological lemma in Chapter III, we know that f has prime curves C_1, C_2, C_3 such that the restriction $f_0|_V$ of a rational function f_0 to $V = \boldsymbol{P}^2 \setminus (\bar{C}_1 \cup \bar{C}_2 \cup \bar{C}_3)$ is of proper $\boldsymbol{C}^*$-type, where $\bar{C}_l$ is the closure of C_l in $\boldsymbol{P}^2$. Let t_l be a homogeneous polynomial defining $\bar{C}_l$. Theorem 1 implies that, for each triple $(\alpha_1, \alpha_2, \alpha_3)$ $(\neq (0, 0, 0))$ of integers satisfying $\alpha_1 \deg(t_1) + \alpha_2 \deg(t_2) + \alpha_3 \deg(t_3) = 0$, the rational function $g = t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3}$ is of special type on $\boldsymbol{P}^2$. By this fact, if f belongs to $\mathscr{F}_{\mathrm{II}}$, f is related to a rational function belonging to the class (D_0). Hence f can be determined concretely. On the other hand, there exists no rational function of torsional $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2$ as is seen in Chapter III, § 3, which completes our study.

§ 2. Statement of results.

1. Here, we give a summary of Chapter II to explain our recurrence formulas. Suppose that a primitive rational function f belongs to the class (D_0). Let $\sigma: M \to \boldsymbol{P}^2$ be the minimal resolution of the points of indetermination of f by a finite sequence of σ-processes. We denote by F_c the level curve of $\sigma^* f$ with value c. If the level curve F_1 of $\sigma^* f$ with value 1 contains $n-1$ irreducible components with the self-intersection numbers smaller than -2, we say that f belongs to D_n. The class (D_0) divides into subclasses D_n $(n = 1, 2, 3, \cdots)$. We have the following in Chapter II.

PROPOSITION 0. *If f belongs to D_n, then F_1 is represented by the diagrams in Figure* 1.

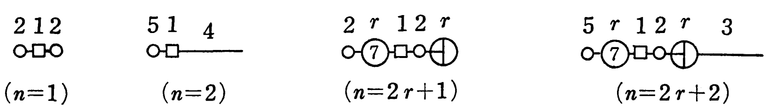


FIGURE 1.

In this diagram, a circle or a square represents a non-singular rational curve. A circle represents an irreducible component of $\Sigma(f)$. A square represents the proper image of $\bar{L}_1$ (the closure of the level curve L_1 of f with value 1 in $\boldsymbol{P}^2$) under the mapping σ^{-1}. The numbers attached to circles and squares are those obtained by the multiplication of -1 to the self-intersection numbers of the corresponding curves. A short segment connecting circles or squares represents a transversal intersection of the corresponding curves.

A long line with a non-negative integer p attached in Figure 2(i) is the abbreviation for the diagram in Figure 2(ii). A sign with a non-negative integer r attached in Figure 2(iii) is the abbreviation for the diagram in Figure 2(iv). A sign with a non-negative integer r attached in Figure 2(v) is the abbreviation for the diagram in Figure 2(vi).

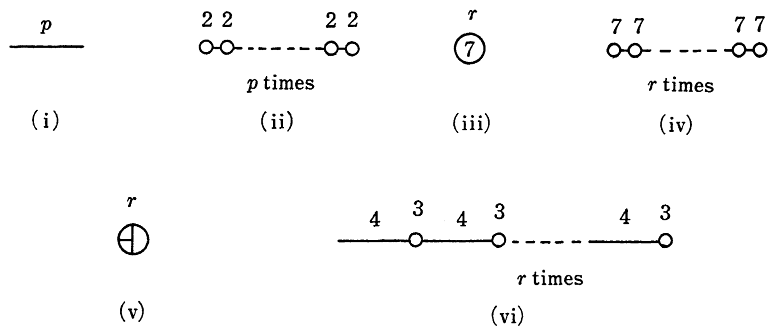


FIGURE 2.

We shall see in Chapter II, § 2.1, that, in the case where the graph of $\Sigma(f)$ is linear, $\Sigma(f) \cup F_\infty \cup F_0 \cup F_1$ is represented by the diagrams in Figure 3.

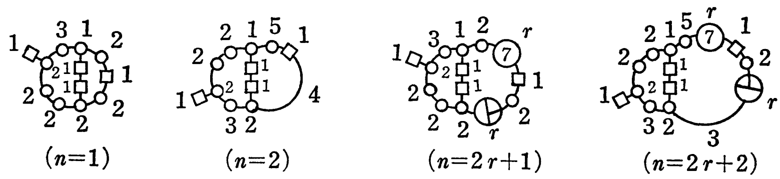


FIGURE 3.

In each diagram, a square represents the proper image of the closure of one of the prime curves of f with values 0, 1, ∞ under the mapping σ^{-1}. The connected square in the center of each diagram represents the level curve F_0 of $\sigma^* f$ with value 0. The components intersecting F_0 are the basic sections of $\sigma^* f$. The right-hand side of each diagram is the level curve F_1 with value 1 and the left-hand side is the level curve F_∞ with the value ∞.

If f belongs to D_n and if the graph of $\Sigma(f)$ is linear, then we say that f belongs to D_n^0. There exists only one rational function belonging to D_n^0 up to the projective transformations of $\boldsymbol{P}^2$. In Chapter II, using the diagrams in Figure 3, we see that a function belonging to D_l^0 has a connection with a function belonging to D_{l-1}^0. Therefore, a function f belonging to D_n^0 can be written as $f = v_{n+1}v_{n-1}/v_n^3$ for a homogeneous coordinate $(X:Y:Z)$ of $\boldsymbol{P}^2$, where v_l is a homogeneous polynomial of (X, Y, Z) defined by the following recurrence formula;

$$v_{l+1} = (v_l^3 + u^{m_l})/v_{l-1} \quad (m_l = \deg(v_l);\ l = 0, 1, \cdots, n)\ ,$$
$$v_{-1} = X\ , \quad v_0 = Y\ , \quad u = XYZ - X^3 - Y^3\ .$$

Let $\{b_l\}$ be the so-called Fibonacci sequence satisfying $b_{-1} = 0$, $b_0 = 1$ and $b_{l+1} = b_l + b_{l-1}$. We know that $m_n = b_{2n}$.

We divide the complement $D_n \setminus D_n^0$ into D_n^{k+} and D_n^{k-} according to the shape of $\Sigma(f)$ where the integer k is the number of the diverging components of $\Sigma(f)$ (for the definition of a *diverging component*, see Chapter I, §2.1). A function belonging to $D_n^{l\pm}$ has a connection with a function belonging to $D_n^{l-1\pm}$. By induction, a function f belonging to $D_n^{k\pm}$ has a connection with a function belonging to D_n^0 so that f can be represented as $f = w_k w_{k-1}/v_n^{\deg(u_k)}$ for a homogeneous coordinate $(X:Y:Z)$ of $\boldsymbol{P}^2$, where u_l and w_l are homogeneous polynomials of (X, Y, Z) defined by the following recurrence formulas;

$$u_l = u_{l-1}v_n^{\mu_l(n)+a_l\deg(w_{l-1})} - P_{a_l}(w_{l-1}^{m_n}, v_n^{\deg(w_{l-1})})w_{l-1}^{s_l(n)}\ ,$$

where $P_{a_l}(z_1, z_2)$ is a homogeneous polynomial in (z_1, z_2) of degree a_l with $P_{a_l}(1, 0) \neq 0$ $(l = 1, 2, \cdots, k)$, and

$$w_l = (v_n^{\deg(u_l)} + u_l^{m_n})/w_{l-1}\ .$$

When f belongs to D_n^{k+}, we define $s_l(n)$ and w_0 as follows; $s_l(n) = [b_{2n} + 3(-1)^l b_{2n-3}]/2$ and $w_0 = v_{n+1}$. When f belongs to D_n^{k-}, we define $s_l(n)$ and w_0 as follows; $s_l(n) = [b_{2n} + 3(-1)^{l-1}b_{2n-3}]/2$ and $w_0 = v_{n-1}$. In both cases, $\mu_l(n) = [s_l(n)\deg(w_{l-1}) - \deg(u_{l-1})]/m_n$ $(l = 1, 2, \cdots, k)$. When f does not belong to D_1^{k-}, we define $u_0 = u$. When f belongs to D_1^{k-}, we define u_0 as $u_0 = X$. (In the case where $n = 1$, we may suppose that $a_l > 0$, $l = 1, 2, \cdots, k-1$. When f belongs to D_1^{1-}, a_1 must be positive.)

2. Set $R_{n,k} = w_{k-1}^{m_n}/v_n^{\deg(w_{k-1})}$, where w_{k-1} and v_n are those defined above. The rational function $R_{n,k}$ is of $\boldsymbol{C}$-type on $\boldsymbol{P}^2$. If the rational function f does not belong to D_1^{1-}, then $R_{n,k}$ belongs to $\mathscr{F}_{\mathrm{II}}$. Otherwise, $R_{1,1}$ belongs to $\mathscr{F}_{\mathrm{I}}$. Conversely, a rational function R of $\boldsymbol{C}$-type belonging to $\mathscr{F}_{\mathrm{II}}$ can be expressed in the form $R = \Lambda(R_{n,k})$ for a homogeneous coordinate $(X:Y:Z)$ of $\boldsymbol{P}^2$, where $\Lambda(z)$ is a non-constant rational function in one variable z. Set $\psi_{n,k} = u_{k-1}v_n^{\mu_k(n)}/w_{k-1}^{s_k(n)}$. If f dose not belong to D_1^{1-}, then the

rational function $\psi_{n,k}$ is of C^*-type on P^2 and belongs to $\mathscr{F}_{\mathrm{II}}$. Otherwise, $\psi_{1,1}$ is of C-type on P^2 and belongs to $\mathscr{F}_{\mathrm{I}}$. The mapping τ defined by $\tau(p) = (R_{n,k}(p), \psi_{n,k}(p))$, $p \in P^2 \setminus \{v_n w_{k-1} = 0\}$, is a biregular birational mapping of $P^2 \setminus \{v_n w_{k-1} = 0\}$ onto $C^* \times C$.

Set $\Psi(z) = P(z)/z^l$, where $l \in Z^+ \cup \{0\}$ and $P(z)$ is an arbitrary polynomial in z. Set $\varphi_0 = (R_{n,k})^p \{\psi_{n,k} - \Psi(R_{n,k})\}^q$, where $p \in Z$, $q \in Z^+$ and $(p, q) = 1$. Except when f belongs to D_1^{1-}, the rational function φ_0 is of C^*-type on P^2 and belongs to $\mathscr{F}_{\mathrm{II}}$. Conversely, we get the following in Chapter III, § 2.

THEOREM 2. *A rational function φ of C^*-type belonging to $\mathscr{F}_{\mathrm{II}}$ can be expressed as $\varphi = \Lambda(\varphi_0)$ for a homogeneous coordinate $(X : Y : Z)$ of P^2. Here $\Lambda(z)$ is a non-constant rational function in one variable z.*

3. Suppose that a rational function f belongs to the class (D_0). If f belongs to $D_n^{k\pm}$, we call the graph of $\Sigma(f) \cup F_\infty \cup F_0 \cup F_1$ simply the graph of $D_n^{k\pm}$. The following is the list of the graph of $D_n^{k\pm}$.

(a) The graph of $D_1^{k\pm}$.

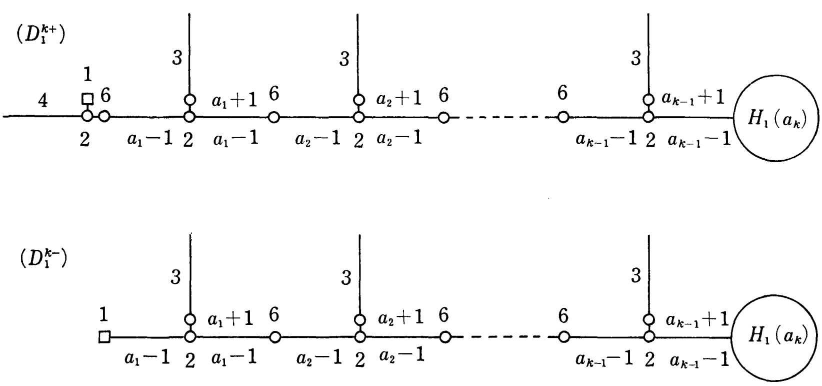
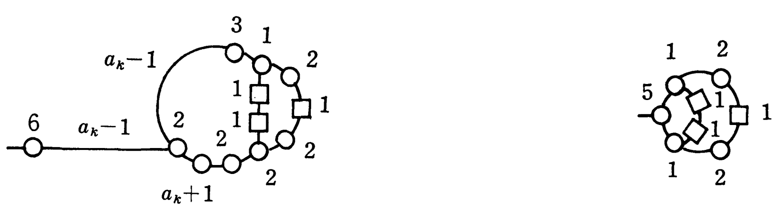


FIGURE 4.

In the above, the circle labeled $H_1(a_k)$ represents the following diagrams:



$(a_k > 0)$ $(a_k = 0)$

FIGURE 5.

In the graph of D_1^{1-}, the case $a_1 = 0$ does not occur.

(b) The graph of D_n^{k+} $(n \geqq 2)$.

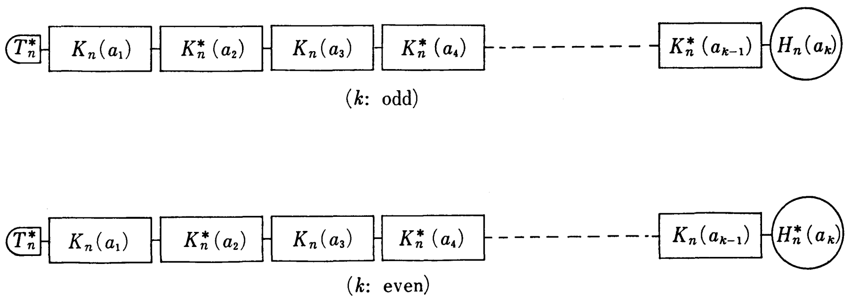


FIGURE 6.

(c) The graph of D_n^{k-} $(n \geqq 2)$: Figure 7.

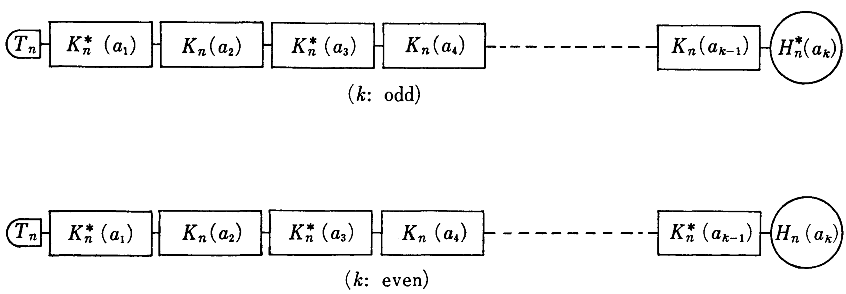


FIGURE 7.

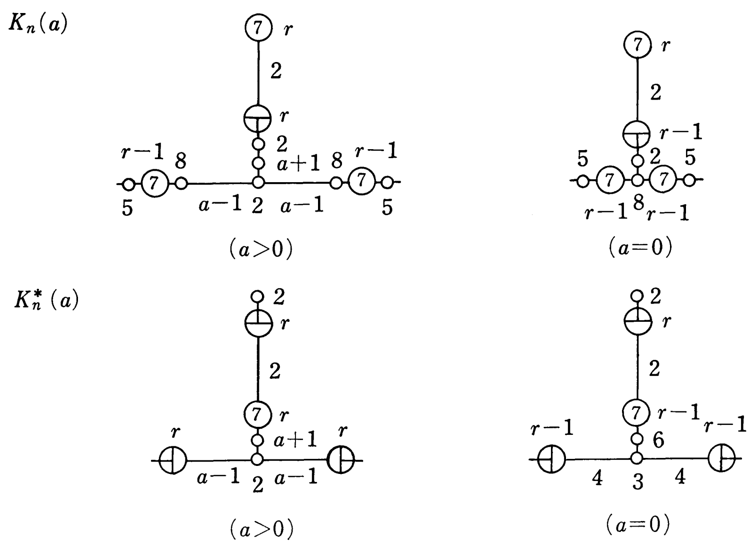


FIGURE 8.

In Figures 6 and 7, the marks labeled $K_n(a_l)$, $K_n^*(a_l)$, $H_n(a_k)$, $H_n^*(a_k)$, T_n, T_n^*, represent the diagrams in Figures 8 through 13.

(i) Case $n = 2r + 1$ ($r = 1, 2, \cdots$): Figure 8 through 10.

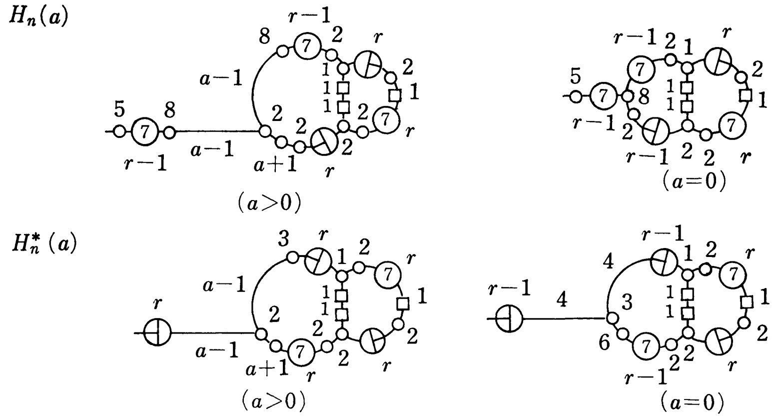


FIGURE 9.

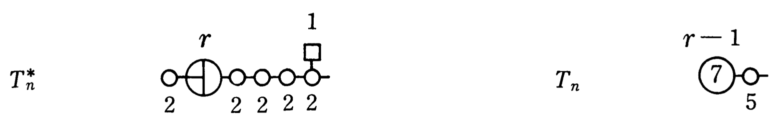


FIGURE 10.

(ii) Case $n = 2r + 2$ ($r = 0, 1, \cdots$): Figure 11 through 13.

When $r = 0$, T_n is empty. The square in the graph of T_n^* is the proper image of $\bar{L}_\infty$ under the mapping σ^{-1}. In the case where f belongs to D_n^{k-}, the proper image of $\bar{L}_\infty$ under the mapping σ^{-1} intersects a component in the part labeled $K_n^*(a_1)$, as is illustrated in Figure 14.

Chapter I. Curves with the property (P). In this chapter, we state several elementary facts on level curves of rational functions of $\boldsymbol{P}$-type.

§ 1. Definition.

1. Let f be a non-constant rational function on a compact smooth algebraic surface X.

PROPOSITION 1. *If the set I_f is not empty, then there exists a pair of a primitive rational function f_0 on X and a rational function φ on $\boldsymbol{P}$ such that $f = \varphi \circ f_0$. If another pair of a primitive rational function f_1 on X and a rational function λ on $\boldsymbol{P}$ satisfies the condition $f = \lambda \circ f_1$,*

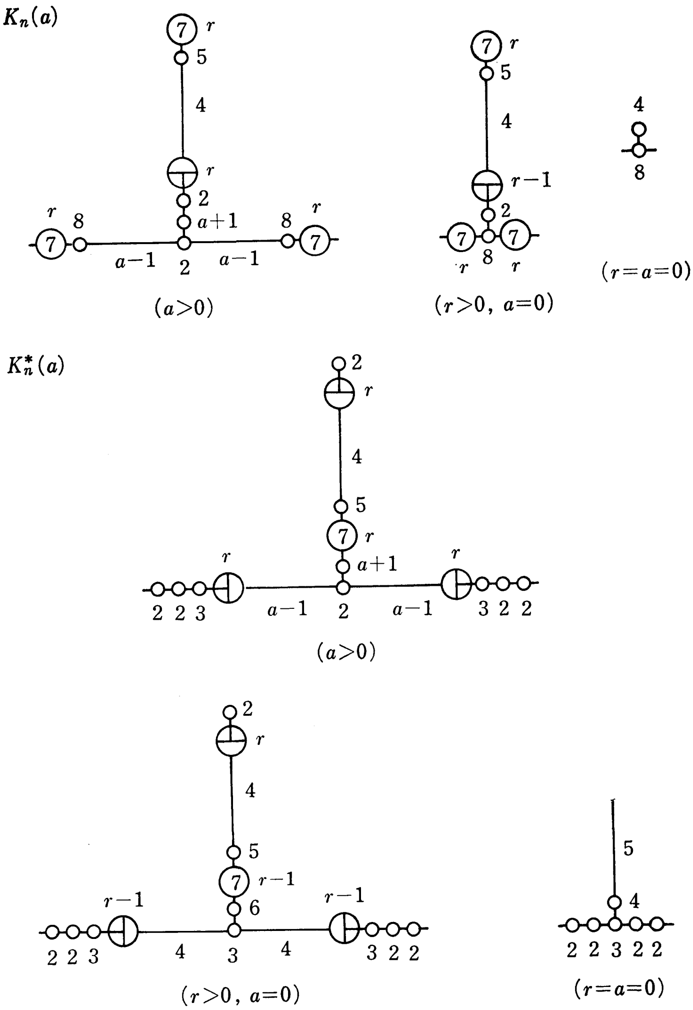


FIGURE 11.

then there exists an analytic automorphism (a linear fractional transformation) T of $\boldsymbol{P}$ such that $f_1 = T \circ f_0$ and $\varphi = \lambda \circ T$.

PROOF. We recall the "Stein fractorization". Let $\sigma: M \to X$ be a resolution of the indetermination points of f by a finite sequence of σ-processes. The mapping σ of the smooth surface M onto X is holomorphic and the pull-back $\sigma^* f$ of f by σ has no indetermination point. If two points p_1, p_2 on M are contained in the same connected component of $(\sigma^* f)^{-1}(\sigma^* f(p_1))$, then we write the fact as $p_1 \sim p_2$. The relation $\sim$ is an equivalence relation. Let π be the canonical projection of M onto the quotient space

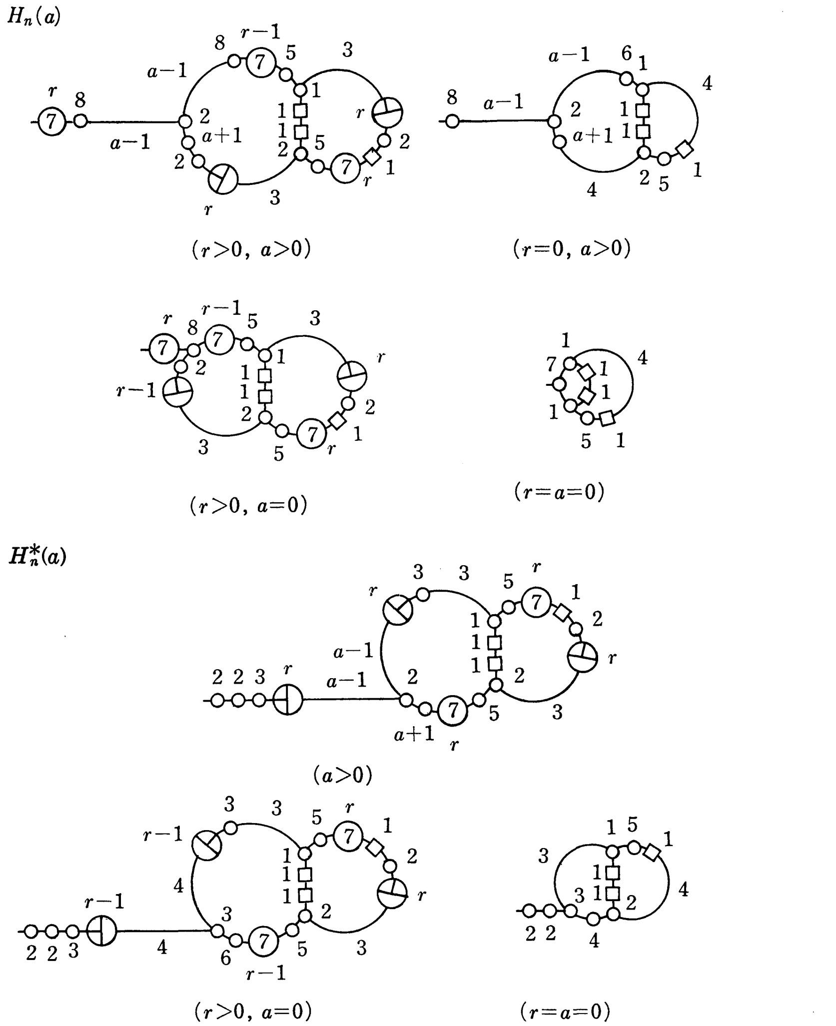
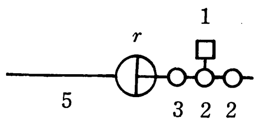


FIGURE 12.

$R = M/\sim$ provided with the quotient topology. The space R is connected and there exists a unique mapping φ of R onto $\boldsymbol{P}$ such that $\sigma^*f = \varphi \circ \pi$. By Stein's theorem on a proper holomorphic mapping, R has an analytic structure such that φ and π are holomorphic. (cf. Nishino [4])

Set $\Sigma(f) = \sigma^{-1}(I_f)$. The algebraic curve $\Sigma(f)$ consists of a finite number of non-singular rational curves. Let B_l $(l = 1, 2, \cdots, m)$ be the basic sections of σ^*f. The mapping $\pi|_{B_l}$ of B_l into R is non-constant and holomorphic. Hence R is the Riemann sphere. Setting $f_0 = (\sigma^{-1})^*\pi$, we see that f_0 is a primitive rational function on X and that $f = \varphi \circ f_0$. We

T_n^*



T_n

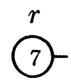
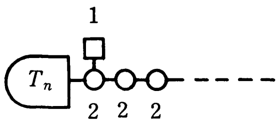
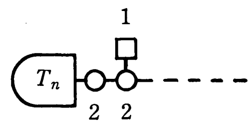


FIGURE 13.



$K_n^*(a_1)$

(n: odd)



$K_n^*(a_1)$

(n: even)

FIGURE 14.

easily get the remainder of the statement.

Suppose that f is primitive. If there exists a neighbourhood U_c of a point c on $\boldsymbol{P}$ such that the triple $\langle N, f|_N, U_c\rangle$ defines a trivial topological fibre bundle where $N = f^{-1}(U_c)$, then c is called a *regular value* of f. The fibre $L_c = f^{-1}(c)$, that is, the level curve of f with regular value c, is called a *regular level curve* of f. A regular level curve is irreducible, non-singular and of order one. A point c on $\boldsymbol{P}$ which is not a regular value of f is called a *critical value* of f. A level curve of f with critical value is called *critical*.

Let E be the set of critical points of the mapping $\sigma^*f\colon M \to \boldsymbol{P}$. The image $\sigma^*f(E)$ of E consists of a finite number of points on $\boldsymbol{P}$. Set $E' = (\sigma^*f)^{-1}(\sigma^*f(E))$ and $R' = \sigma^*f(M \setminus E')$. For each point p on R', there exists a neighbourhood V of p such that the triple $\langle N, \pi|_N, V\rangle$ is a trivial topological fibre bundle with the projection $\pi|_N$ where $N = \pi^{-1}(V)$. For each point p on R' except for a finite number of points, the number of intersections of the fibre $F = \pi^{-1}(p)$ with the sum $B_1 \cup B_2 \cup \cdots \cup B_m$ of the basic sections of σ^*f equals a constant number independent of p. Hence the set of critical values of f is finite and prime curves of a rational function on X are homeomorphic to each other except for a finite number of them.

Let n be the number of boundary components of general prime curves of f. Denote by $\#(I_f)$ the number of indetermination points of f. We see easily that $\#(I_f) \leqq m \leqq n$. If f is of $\boldsymbol{C}^*$-type, then $n = 2$. If f is of direct $\boldsymbol{C}^*$-type, that is, if $m = 2$, then $\#(I_f)$ is 1 or 2. If f is of torsional $\boldsymbol{C}^*$-type, that is, if $m = 1$, then $\#(I_f) = 1$.

2. The following lemma will be applied again and again in this paper.

NOETHER'S LEMMA. *Let C be a smooth irreducible rational compact curve on a compact smooth rational surface M. Suppose that the self-intersection number (C^2) of C is zero. Then there exists a rational function h of $\boldsymbol{P}$-type on M such that C is a regular level curve of h.*

The rational function h of $\boldsymbol{P}$-type in the above lemma is primitive because the level curve C is of order one.

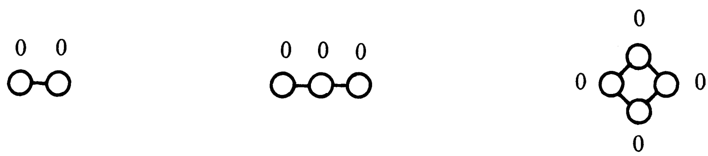
If a compact algebraic curve C on a smooth rational surface M is a level curve of a primitive rational function f of $\boldsymbol{P}$-type on M, then we say that C has the *property* (P). Suppose that C is also a level curve of a primitive rational function g on M. For each compact prime curve S of g not intersecting C, the restriction $f|_S$ does not take the value $f(C)$ so that $f|_S$ must be constant. Hence there exists an analytic automorphism T of $\boldsymbol{P}$ such that $g = T \circ f$. This means that the order of an irreducible component C_l of C is independent of the choice of the function f. We call this order of C_l as a component of a curve C with the property (P) the *component order* of C_l with respect to C. Suppose that C is irreducible. The curve C must be non-singular and rational. The self-intersection number (C^2) of C must be zero. By Noether's lemma, there exists a primitive rational function g of $\boldsymbol{P}$-type on M such that C is a level curve of order one of g. Hence the (component) order of C as a curve with the property (P) is one.

Suppose that C is reducible. Denote by $[C]$ the divisor defined by the equation $f - f(C) = 0$. (When $f(C) = \infty$, $[C]$ denotes the divisor defined by the equation $1/f = 0$.) The self-intersection number $([C]^2)$ of the divisor $[C]$ is zero. The virtual genus of $[C]$ is zero because f is of $\boldsymbol{P}$-type. As is well-known, at least one component D of C is an exceptional curve of the first kind. Let $\sigma\colon M \to \sigma(M)$ be the σ-process which contracts D. The image curve $\sigma(C)$ is the level curve of $(\sigma^{-1})^* f$ with the value $f(C)$. Hence the curve $\sigma(C)$ has the property (P). Let D' be any component of C different from D. Clearly, the component order of $\sigma(D')$ with respect to $\sigma(C)$ equals the component order of D' with respect to C. Contracting exceptional components of the first kind of C successively, we get an irreducible curve with the property (P). Hence we obtain the following lemma.

LEMMA 1. *An algebraic curve with the property (P) on a smooth rational algebraic surface contains at least one irreducible component of order one.*

The following proposition is fundamental in our later discussion.

PROPOSITION 2. *Suppose there exists an algebraic curve C on a compact smooth rational surface M such that the complement $M \setminus C$ contains no compact algebraic curve. Then there exists a pair of rational functions (h_1, h_2) such that the mapping $\theta: M \to \boldsymbol{P} \times \boldsymbol{P}$, defined by $\theta(p) = (h_1(p), h_2(p))$ $(p \in M)$, is a biregular isomorphism of M onto $\boldsymbol{P} \times \boldsymbol{P}$ if C satisfies one of the following three conditions* (i), (ii), (iii): (i) *C consists of two irreducible components C_1, C_2 with the property* (*P*) *such that* $(C_1 \cdot C_2) = 1$, (ii) *C consists of three irreducible components C_1, C_2, C_3 with the property* (*P*) *such that* $(C_1 \cdot C_2) = (C_2 \cdot C_3) = 1$ *and* $(C_1 \cdot C_3) = 0$, *and* (iii) *C consists of four irreducible components C_1, C_2, C_3, C_4 with the property* (*P*) *such that* $(C_1 \cdot C_2) = (C_2 \cdot C_3) = (C_3 \cdot C_4) = (C_4 \cdot C_1) = 1$ *and* $(C_1 \cdot C_3) = (C_2 \cdot C_4) = 0$. (*See Figure* 15)



(i) (ii) (iii)

FIGURE 15.

PROOF. Suppose first that (i) holds. There exist primitive rational functions h_1, h_2 of $\boldsymbol{P}$-type on M such that C_l $(l = 1, 2)$ is a regular level curve of h_l. We prove that each level curve of h_l is irreducible. Assume that a level curve F_1 of h_1 is reducible. The restriction $h_1|_{C_2}$ of h_1 to C_2 is a rational function of degree one on C_2. Hence F_1 intersects C_2 at a regular point of F_1 transversally. Let D be an irreducible component of F_1 which does not intersect C_2. We get $D \cap C_1 = D \cap C_2 = \varnothing$. Hence D is a compact curve in $M \setminus C$, a contradiction to the assumption. Therefore each level curve of h_1 is irreducible. Similarly, each level curve of h_2 is irreducible. For each level curve F_1' of h_1, we have $(F_1' \cdot C_2) = 1$. Hence the restriction $h_2|_{F_1'}$ of h_2 to F_1' is a rational function of degree one on F_1. This means that θ is a biregular isomorphism of M onto $\boldsymbol{P} \times \boldsymbol{P}$.

Next suppose that (ii) holds. Since C_1 and C_2 satisfy (i), it is sufficient to prove that there exists no compact curve in $M \setminus (C_1 \cup C_2)$. Let h_1, h_2 be the same functions as in the first case. There exists a primitive rational function h_3 on M such that C_3 is a level curve of h_3. Since $C_1 \cap C_3 = \varnothing$, there exists an analytic automorphism T_1 of $\boldsymbol{P}$ such that $h_3 = T_1 \circ h_1$. Suppose that a compact algebraic curve D is contained in

$M\setminus(C_1\cup C_2)$. By assumption, D must intersect C_3. Hence the restriction $h_3|_D$ is not constant. On the other hand, since $C_1\cap D=\varnothing$, the restriction $h_1|_D$ must be constant, a contradiction to the fact $h_3=T_1\circ h_1$ Hence there is no compact algebraic curve in $M\setminus(C_1\cup C_2)$.

Finally suppose (iii) holds. Since C_1, C_2 and C_3 satisfy (ii), it is sufficient to prove that there exists no compact algebraic curve in $M\setminus(C_1\cup C_2\cup C_3)$. There exists a primitive rational function h_4 on M such that C_4 is a level curve of h_4. Since $C_2\cap C_4=\varnothing$, there exists an analytic automorphism T_2 of $\boldsymbol{P}$ such that $h_4=T_2\circ h_2$. Suppose that a compact algebraic curve D is contained in $M\setminus(C_1\cup C_2\cup C_3)$. By assumption, D must intersect C_4. Hence $h_4|_D$ is not constant, a contradiction to the fact $h_4=T_2\circ h_2$. Thus we have our proposition.

§ 2. Reducible curves with the property (P).

1. A compact connected algebraic curve E on a smooth algebraic surface V is called an (reducible) exceptional curve of the first kind if there exists a sequence of regular mappings of smooth surfaces

$$V=V_n\xrightarrow{\tau_n}V_{n-1}\xrightarrow{\tau_{n-1}}\cdots\xrightarrow{\tau_2}V_1\xrightarrow{\tau_1}V_0=V'$$

such that each τ_l is a σ-process which contracts an irreducible component of the curve $(\tau_{l+1}\circ\tau_{l+2}\circ\cdots\circ\tau_n)(E)$ on V_l and that $\tau_1\circ\tau_2\circ\cdots\circ\tau_n(E)$ is a one point set. Hence E is a tree of rational curves on V and a singular point of E is an ordinary double point where two components of E intersect each other.

An irreducible curve of the first kind is non-singular, rational and with the self-intersection number -1. Conversely, by Castelnuovo's theorem, we know that a compact non-singular irreducible rational curve E with the self-intersection number -1 on a smooth algebraic surface V is an exceptional curve of the first kind. If an algebraic curve C on V intersects E, then $(\tau(C)^2)=(C^2)+k^2$ where k is the multiplicity of the point $\tau(E)$ on the curve $\tau(C)$.

LEMMA 2. *Suppose that E is reducible exceptional curve of the first kind on V which contains only one irreducible component with the self-intersection number -1. If E is a linear tree, then the graph of E is given by Figure* 16.

Here the numbers a_l and b_l are non-negative integers.

PROOF. We prove this lemma by induction on the number of the irreducible components of E. Let $\tau_1\colon V\to\tau_1(V)$ be the σ-process which contracts the component D_0 of E whose self-intersection number is -1.

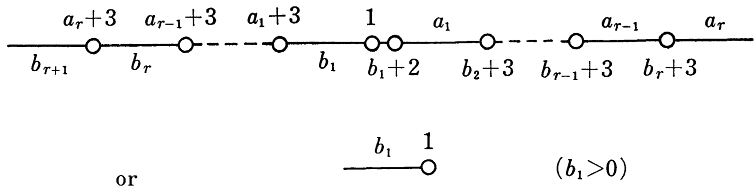


FIGURE 16.

The image $\tau_1(E)$ of E is an exceptional curve of the first kind on $\tau_1(V)$. Since each singular point of $\tau_1(E)$ is an ordinary double point, D_0 intersects at most two other components of E. The self-intersection number of a component D of E differs from the self-intersection number of $\tau_1(D)$ if and only if D_0 intersects D. Suppose that D_0 intersects two other components D_1, D_2 of E and that $(\tau_1(D_1)^2) = (\tau_1(D_2)^2) = -1$. Let $\tau_2 \colon \tau_1(V) \to \tau_2(\tau_1(V))$ be the σ-process which contracts $\tau_1(D_2)$. The self-intersection number of $\tau_2(\tau_1(D_1))$ is 0. Noether's lemma leads us to a contradiction because $\tau_2(\tau_1(E))$ must be an exceptional curve of the first kind on $\tau_2(\tau_1(V))$ or one point. Hence we see that $\tau_1(E)$ contains only one component with the self-intersection number -1. Therefore we have our lemma.

Now, we can study the case where E is non-linear. Let E be an exceptional curve of the first kind which contains only one component D_0 with the self-intersection number -1. A component D of E is called a *diverging* component of E if D intersects at least three other components of E. Let A_1' be the maximal connected linear tree of components of E containing D_0 which does not contain a diverging component of E. Then A_1' must intersect a diverging component D_1 of E at one of the edge of A_1'. Set $A_1 = A_1' \cup D_1$. The curve A_1 is an exceptional curve of the first kind. Denote by σ_1 the composite of σ-processes which contracts all the components of A_1'. The self-intersection number $(\sigma_1(D_1)^2)$ of the image $\sigma_1(D_1)$ must be -1. Hence we obtain the following lemma.

LEMMA 2$'$. *If the number of diverging components of E is $k-1$, then the outline of the graph of E is as in Figure* 17.

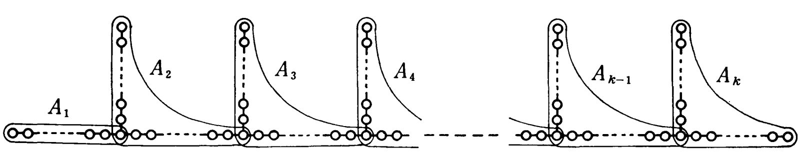


FIGURE 17.

Especially, each component D of E intersects at most three other components of E. In the above figure, if we substitute -1 for the self-intersection number of the diverging component common to A_{l-1} and A_l, then the part A_l of the graph becomes the graph of an exceptional curve of the first kind satisfying the condition in Lemma 2. The part of E represented by A_l is called the *l-th branch* of E. By the sequence $\tau_n, \tau_{n-1}, \cdots, \tau_2, \tau_1$, the branches of E are contracted successively.

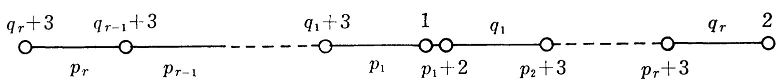
2. An algebraic curve C with the property (P) on a rational smooth surface consists of non-singular rational curves. Each singular point of C is an ordinary double point. If C is reducible, then the self-intersection number (C_l^2) of each component C_l of C is negative. This fact is easily proved by Noether's lemma. The first Betti number of C is zero.

LEMMA 3. *Suppose that C is a reducible algebraic curve with the property (P) on a smooth rational surface M.*

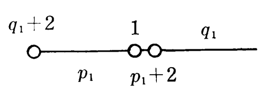
(i) *If the component order of a component C_0 of C (with respect to C) is one, then each connected component of the closure of $C \setminus C_0$ is an exceptional curve of the first kind.*

(ii) *If two irreducible components C_1, C_2 of C satisfy $(C_1 \cdot C_2) = 1$ and $(C_1^2) = (C_2^2) = -1$, then the component orders of the curves C_1, C_2 with respect to C are both one and C consists of only these curves, that is, $C = C_1 \cup C_2$.*

(iii) *Suppose that C_1 is the unique exceptional component of the first kind of C and suppose that the closure of $C \setminus C_1$ consists of two connected components, both of which contain components of component order one with respect to C. Then the graph of C is given by Figure* 18.



$(r \geqq 2)$



$(r=0, 1)$

FIGURE 18.

Here, the numbers p_l, q_l are non-negative integers. When C has $n-1$ components with the self-intersection number smaller than -2, the number r in the graph of C means the integer $[n/2]$.

PROOF. (i) Let C_l $(l = 0, 1, 2, \cdots, n)$ be the irreducible components of C and v_l be the component order of C_l with respect to C. By assumption, the divisor $[C]=\sum_{l=0}^{n} v_l[C_l]$ satisfies $([C]^2)=\pi([C])=0$, where $\pi([C])$ is the virtual genus of $[C]$. Let K be the canonical bundle of M. Since $\pi([C])=\{([C]^2)+(K\cdot[C])\}/2+1$, we have $(K\cdot[C])=(K\cdot[C_0])+\sum_{l=1}^{n} v_l(K\cdot[C_l])=-2$. By $\pi([C_l])=0$, we get $(K\cdot[C_l])=-2-(C_l^2)$ $(l=0,1,\cdots,n)$. The fact $(C_0^2)\leqq -1$ shows $(K\cdot[C_0])\geqq -1$. Hence $\sum_{l=1}^{n} v_l(K\cdot[C_l])=\sum_{l=1}^{n} v_l\{-2-(C_l^2)\}\leqq -1$. Therefore, there exists a component C_m such that $(C_m^2)\geqq -1$. By $(C_m^2)\leqq -1$, we get $(C_m^2)=-1$. By Castelnuovo's theorem, we see that C_m is an exceptional curve of the first kind. Let $\tau\colon M\to\tau(M)$ be the σ-process which contracts C_m. The image $\tau(C)$ of C is a curve with the property (P) on $\tau(M)$. The component order of $\tau(C_0)$ with respect to $\tau(C)$ is also one. By induction, we thus have (i).

(ii) Let $\tau\colon M\to\tau(M)$ be the σ-process which contracts C_2. The self-intersection number $(\tau(C_1)^2)$ of the image $\tau(C_1)$ is zero. By Noether's lemma, $\tau(C_1)$ has the property (P). On the other hand, the image $\tau(C)$ has the property (P). Hence $\tau(C)=\tau(C_1)$, $C=C_1\cup C_2$ and the curves C_1, C_2 are of order one.

(iii) Let C_2, C_3 be the components of C intersecting C_1 and $\tau\colon M\to\tau(M)$ be the σ-process which contracts C_1. We may suppose that the self-intersection number $(\tau(C_2)^2)$ of $\tau(C_2)$ is -1. Assume that the self-intersection number $(\tau(C_3)^2)$ is -1. By (ii) of this lemma, we get $C=C_1\cup C_2\cup C_3$. This is the case where $r=p_1=q_1=0$ in Figure 18. Assume $(\tau(C_3)^2)<-1$. The curve $\tau(C_2)$ is the unique exceptional component of the first kind of $\tau(C)$. By (i) of this lemma, the component order of $\tau(C_2)$ with respect to $\tau(C)$ is not one. Since $\tau(C_2)$ intersects at most two other components of $\tau(C)$, the closure of $\tau(C)\setminus\tau(C_2)$ consists of two connected components, each of which contains a component of order one. By induction, we thus have (iii).

By the following lemma, we can calculate orders of components of C in Lemma 3. The proof is easy and may be omitted.

LEMMA 4. *Let Γ be the bicylinder $\{(x,y)\in \boldsymbol{C}^2 \mid |x|<1, |y|<1\}$. Denote by $\sigma\colon\sigma^{-1}(\Gamma)\to\Gamma$ the σ-process which gives the blowing-up at the origin $(0,0)\in\Gamma$. Set $f(x,y)=x^p y^q$ for a pair of non-negative integers p, q. Then the curve $\sigma^{-1}((0,0))$ is a prime curve of order $p+q$ of $\sigma^* f$.*

Chapter II. Functions belonging to the class (D_0).

§ 1. Outlines.

1. Sketch of the graph of $\Sigma(f)$. We say that a primitive rational function f of $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2$ belongs to the class (D_0) if f satisfies the following conditions: (i) f has only one indetermination point p_0, (ii) the level curve of f with value 0 consists of two prime curves S_{01}, S_{02} of C-type and of order one, and the curve S_{01} intersects S_{02} transversally at a point in $P^2 \setminus \{p_0\}$, (iii) the level curve of f with value ∞ consists of only one prime curve S_∞ of C-type, (iv) the level curve of f with value 1 consists of only one multiple prime curve S_1 of C^*-type, and (v) the other level curves are irreducible and of C^*-type. By the classification in Chapter III, § 1.4, f is of direct C^*-type. In this section, we suppose that f belongs to the class (D_0) and give a sketch of the graph of $\Sigma(f) = \sigma^{-1}(p_0)$ and level curves of σ^*f where $\sigma: M \to P^2$ is the minimal resolution of the indetermination point of f.

There exist two basic sections B_1, B_2 of σ^*f. Since σ is minimal, we may suppose $(B_1^2) = -1$. Each restriction $\sigma^*f|_{B_i}$ $(i = 1, 2)$ is a rational function of degree one on B_i. Hence each level curve F_c of σ^*f with value c intersects B_i transversally at a regular point of F_c. The prime curve of F_c intersecting B_i must be of order one. Let S be a prime curve of f with value c. We denote by $\widetilde{S}$ the proper transform by σ^{-1} of the closure of S in P^2. If a component C of F_c is not a proper transform of the closure of a prime curve of f, C is a component of $\Sigma(f)$, and, since σ is minimal, $(C^2) < -1$. Suppose that F_c is reducible. Since F_c has the property (P), at least one component of F_c is an exceptional curve of the first kind. Hence at least one proper transform of the closure of a prime curve with value c is an exceptional curve of the first kind. Suppose $c \neq 0$. The level curve of f with value c consists of only one prime curve S_c. Hence the proper transform $\widetilde{S}_c$ of the closure of S_c must be an exceptional curve of the first kind.

Suppose that $c \neq 0, \infty$. If S_c is of order one, Lemma 3 (i) shows that F_c is irreducible, that is, F_c is the proper transform of the closure of S_c. Since S_c is of C^*-type and since F_c is simply connected, F_c intersects each B_j $(j = 1, 2)$ at one point and $(F_c \cap B_1) \neq (F_c \cap B_2)$. If S_c is multiple, then F_c must be reducible. Hence $\widetilde{S}_c$ is an exceptional curve of the first kind. Since $\widetilde{S}_c \cap (B_1 \cup B_2) = \varnothing$ and S_c is of C^*-type, F_c and $\widetilde{S}_c$ satisfy the condition of Lemma 3(iii). Especially F_c must be a linear tree of rational curves. Since a component of F_c intersecting $B_1 \cup B_2$ must be of order one and since $\Sigma(f)$ must be connected, two components represented at the edge of the graph in Figure 18 intersect $B_1 \cup B_2$. We denote by K_i $(i = 1, 2)$ the component of the level curve F_1 with value 1 which intersects B_i. Let $\tau: M \to \tau(M)$ be the composite of σ-processes which contracts $F_1 \setminus K_i$. Then K_j $(i \neq j)$ is the last component contracted by those σ-processes.

By assumption, S_{01} and S_{02} are of order one. By Lemma 3(i), both of $\tilde{S}_{01}$ and $\tilde{S}_{02}$ are exceptional curves of the first kind. By Lemma 3(ii), $F_0 = \tilde{S}_{01} \cup \tilde{S}_{02}$. We may suppose that $\tilde{S}_{01}$ intersects B_1 and that $\tilde{S}_{02}$ intersects B_2. Since S_∞ is of $\boldsymbol{C}$-type, $\tilde{S}_\infty$ intersects $\Sigma(f)$ at only one point.

Here we prove that B_1 and B_2 have no common point. Suppose that $B_1 \cap B_2 \neq \emptyset$. Since $\Sigma(f)$ is simply connected, B_1 intersects B_2 at only one point q. The level curve of σ^*f with value $\sigma^*f(q)$ intersects $B_1 \cup B_2$ only at q. Hence $\sigma^*f(q) = \infty$. Since a singular point of $\Sigma(f)$ is necessarily an ordinary double point, $F_\infty = \tilde{S}_\infty$. Since $(B_1^2) = -1$ and since $\Sigma(f)$ is an exceptional curve of the first kind, B_1 intersects at most two other components of $\Sigma(f)$. Hence S_1 is the unique *multiple* prime curve of $\boldsymbol{C}^*$-type. The shape of $\Sigma(f)$ is given by solid lines in Figure 19, while interrupted lines in the figure represent proper transforms of the closure of prime curves of f.

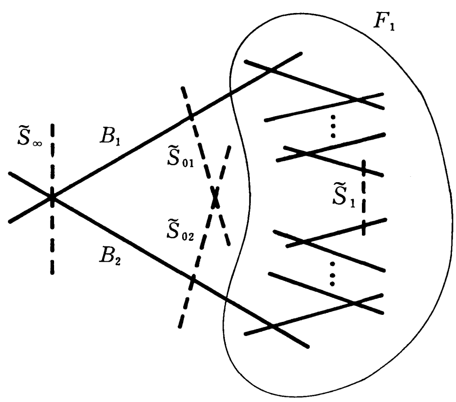


FIGURE 19.

Let $\tau: M \to \tau(M)$ be the composite of σ-processes which contracts the closure of $F_1 \setminus K_1$ and $\tilde{S}_{01}$. The curve $\tau(B_1) \cup \tau(K_1)$ satisfies the condition of Proposition 2(i). Hence $\tau(M)$ is biregularly isomorphic to $\boldsymbol{P} \times \boldsymbol{P}$. On the other hand, since $\Sigma(f)$ is an exceptional curve of the first kind, $(B_2^2) \leqq -2$ (see Lemma 2). Hence $(\tau(B_2)^2) \leqq -1$, a contradiction to the fact that there is no algebraic curve with the negative self-intersection number in $\boldsymbol{P} \times \boldsymbol{P}$. Hence $B_1 \cap B_2 = \emptyset$.

Since $\Sigma(f)$ is connected, $\tilde{S}_\infty \cap (B_1 \cup B_2) = \emptyset$ and the closure of $F_\infty \setminus \tilde{S}_\infty$ intersects both B_1 and B_2. Since B_1 intersects at most two other components of $\Sigma(f)$, S_1 must be the unique *multiple* prime curve of $\boldsymbol{C}^*$-type. Since $\tilde{S}_\infty$ is the unique exceptional component of the first kind of F_∞, we have by Lemma 3(i) that a component of F_∞ intersecting $B_1 \cup B_2$

interrects only one more component of F_∞. One or two components of F_∞ intersect $B_1 \cup B_2$. Denote the component of F_∞ intersecting B_1 by T. By Lemma 3(i), the closure of $F_\infty \setminus T$ is an exceptional curve of the first kind.

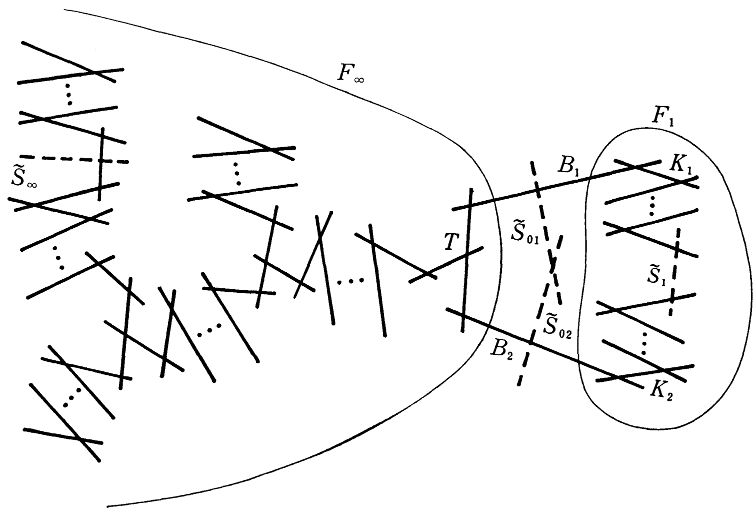


FIGURE 20.

Suppose that T also intersects B_2. In this case, the shape of $\Sigma(f)$ is as in Figure 20. Let $\tau: M \to \tau(M)$ be the composite of σ-processes

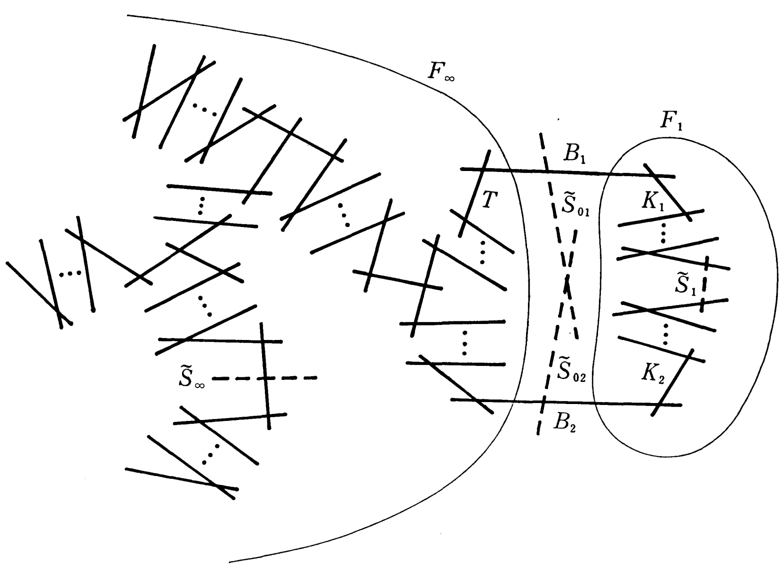


FIGURE 21.

which contracts $\tilde{S}_{01}$, the closure of $F_1 \setminus K_1$ and the closure of $F_\infty \setminus T$. The curves $\tau(T)$, $\tau(K_1)$, $\tau(B_1)$ satisfy the condition of Proposition 2(ii). Hence $\tau(M)$ is biregularly isomorphic to $\boldsymbol{P}\times\boldsymbol{P}$. Hence $(\tau(B_2)^2) = 0$, and $(B_2^2) = -1$.

Suppose that T does not intersect B_2. In this case, the shape of $\Sigma(f)$ is as in Figure 21. Let $\rho: M \to \rho(M)$ be the composite of σ-processes which contracts the branches of the closure of $F_\infty \setminus T$ except the last branch. Then $\rho(F_\infty)$ satisfies the condition of Lemma 3(iii). Hence the component of F_∞ intersecting B_2 is the last component of $F_\infty \setminus T$ contracted by this sequence of σ-processes.

Let $\tau: M \to \tau(M)$ be the composite of σ-processes which contracts $\tilde{S}_{01}$, the closure of $F_1 \setminus K_1$ and the closure of $F_\infty \setminus T$. The curves $\tau(T)$, $\tau(K_1)$, $\tau(B_1)$ satisfy the condition of Proposition 2(ii). Hence $\tau(M)$ is biregularly isomorphic to $\boldsymbol{P}\times\boldsymbol{P}$. Thus $(\tau(B_2)^2) = 0$ and $(B_2^2) = -2$.

2. Basic examples.

[A] In the first place, we introduce the simplest rational function f_0 which satisfies the conditions (i), (ii), (iii) and (v) in the former subsection. Suppose that the graph of $\Sigma(f_0)$ is as given by solid lines in Figure 22. The mapping $\sigma: M \to \boldsymbol{P}^2$ is the composite of σ-processes which contracts exceptional components of $\Sigma(f_0)$ successively. Hence we can calculate the self-intersection numbers of the curves $\bar{S}_{01}$, $\bar{S}_{02}$, $\bar{S}_\infty$ on $\boldsymbol{P}^2$ by this graph. It shows that the degrees of the curves S_{01}, S_{02}, S_∞ are 2, 1, 1, respectively. There exists a homogeneous coordinate $(X:Y:Z)$ of $\boldsymbol{P}^2$ such that $\bar{S}_{01} = \{YZ - X^2 = 0\}$, $\bar{S}_{02} = \{X = 0\}$ and $\bar{S}_\infty = \{Y = 0\}$. By Lemma 4, $\tilde{S}_\infty$ is a prime curve of order 3 of $\sigma^* f_0$ with value ∞. Hence, by using a suitable constant a, we can write f_0 as $(*)$ $f_0 = aX(YZ - X^2)/Y^3$.

Conversely, for a homogeneous coordinate $(X:Y:Z)$, a rational func-

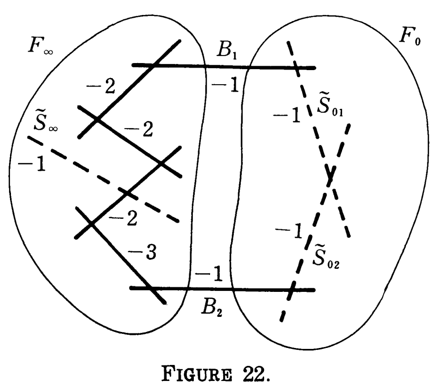


FIGURE 22.

tion f_0 given by $(*)$ has only one indetermination point $(0:0:1)$ whose level curve L_c with value c $(c \neq 0, \infty)$ is given by $\{aX(YZ - X^2) - cY^3 = 0\}$. Hence L_c is irreducible, of C^*-type and of order one. The graph of $\Sigma(f_0)$ is the same as that in Figure 22.

Consider the level curve L_1 of f_0 with value 1. By the projective transformation Φ: $X' = \alpha X$, $Y' = \alpha^2 Y$, $Z' = Z$ where $\alpha^3 = a^{-1}$, f_0 can be written as $f_0 = X'(Y'Z' - X'^2)/Y'^3$. Hence $L_1 = \{X'Y'Z' - X'^3 - Y'^3 = 0\}$. This fact is used in the next example.

[B] Here, we introduce the simplest rational function $f_{1,0}$ belonging to the class (D_0). The function $f_{1,0}$ belongs to D_1^0. Suppose that the graph of $\Sigma(f_{1,0})$ is as in Figure 23. (See also Figure 3.) By Lemma 4, the prime curve S_1 of $f_{1,0}$ is of order 2 and the prime curve S_∞ of $f_{1,0}$ is

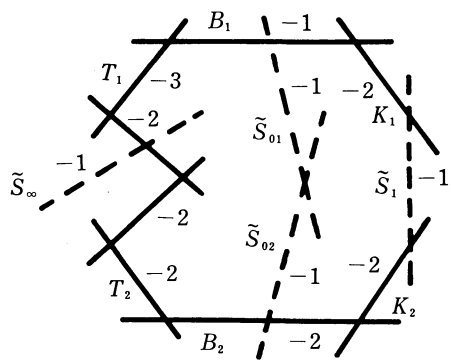


FIGURE 23.

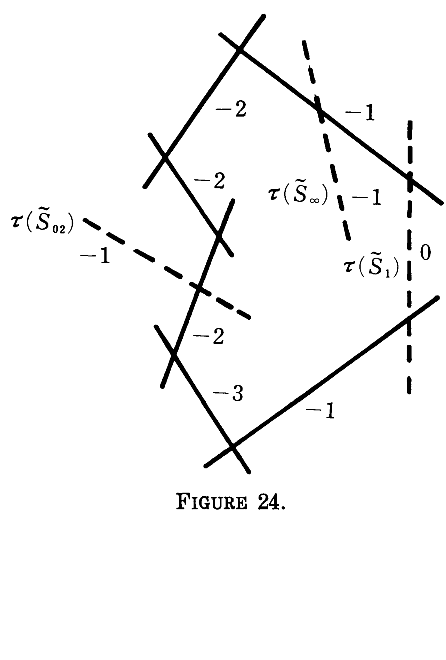


FIGURE 24.

of order 3. Denote by T_i the component of F_∞ intersecting B_i, and by $\tau: M \to \tau(M)$ the composite of σ-processes with contracts $T_1 \cup B_1 \cup K_1$ and gives the blowing-up at the intersection of $\tilde{S}_1$ and K_2. The graph of the total image of $\Sigma(f_{1,0})$ under τ is as in Figure 24, where the image $\tau(\tilde{S}_{01})$ of $\tilde{S}_{01}$ is omitted. Denote by $\tau(\tilde{S}_1)$ the proper image of $\tilde{S}_1$ under τ. Removing $\tau(\tilde{S}_1)$ from this graph, we get the same graph as that of $\Sigma(f_0)$, with $\tilde{S}_{02}$ removed, of f_0 in Example [A]. Denote by $\omega: \tau(M) \to \omega(\tau(M))$ the composite of σ-processes which contracts $\tau(\tilde{S}_\infty)$ and $\tau(T_2) \cup \tau(B_2) \cup \tau(K_2) \cup \tau(\tilde{S}_{02})$. The graph of the total image Σ^* of $\Sigma(f_{1,0})$ under $\omega \circ \tau$ is as in Figure 25, where the cross represents the point $\omega(\tau(\tilde{S}_\infty))$. Since

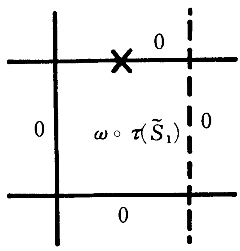


FIGURE 25.

$\omega \circ \tau(M) \setminus (\Sigma^* \cup \omega \circ \tau(\tilde{S}_1))$ is analytically isomorphic to the complement of an algebraic curve on $\boldsymbol{P}^2$, it contains no compact curve. By Proposition 2(iii), $\omega(\tau(M))$ is biregularly isomorphic to $\boldsymbol{P} \times \boldsymbol{P}$. There exists a rational function h on $\omega(\tau(M))$ such that $\omega(\tau(\tilde{S}_1))$ is a level curve of order one of h. Set $f = (\omega \circ \tau \circ \sigma^{-1})^* h$. We may suppose that $h(\omega(\tau(\tilde{S}_{02}))) = \infty$, $h(\omega(\tau(\tilde{S}_\infty))) = 0$ and $h(\omega(\tau(\tilde{S}_1))) = 1$. Then $f(S_{02}) = \infty$, $f(S_\infty) = 0$, $f(S_1) = 1$. Denote by S' the level curve of h with value 0 and denote by S the proper transform of S' under $(\omega \circ \tau \circ \sigma^{-1})^{-1}$. Then $f(S \setminus \{p_0\}) = 0$ and $\sigma \circ \tau^{-1}: \tau(M) \to \boldsymbol{P}^2$ is the minimal resolution of the indetrmination point of f. By the graph of $\Sigma(f)$, we see that f if the function f_0 in Example [A]. Hence, in a homogeneous coordinate $(X:Y:Z)$ of $\boldsymbol{P}^2$, $\bar{S}_1 = \{XYZ - X^3 - Y^3 = 0\}$, $\bar{S}_\infty = \{YZ - X^2 = 0\}$, $\bar{S}_{02} = \{Y = 0\}$ and $S = \{X = 0\}$. Set $v_{-1} = X$, $v_0 = Y$, $v_1 = YZ - X^2$, $u = XYZ - X^3 - Y^3$. Consider the rational function $g = v_1^3/u^2$ on $\boldsymbol{P}^2$. Since S_1 is a level curve of order 2 of $f_{1,0}$ and since $\bar{S}_\infty = \{v_1 = 0\}$, there must exist an analytic automorphism Φ of $\boldsymbol{P}$ such that $f_{1,0} = \Phi \circ g$. The level curve of g with value -1 is $\{v_1^3 + u^2 = 0\} \setminus \{p_0\}$. Since $v_1^3 + u^2 = v_1^3 + (v_{-1}v_1 - v_0)^2 \equiv v_1^2(v_1 + v_{-1}^2) \equiv 0 \pmod{v_0}$, $v_2 = (v_1^3 + u^2)/v_0$ is a homogeneous polynomial of (X, Y, Z). Hence the level curve of g with value -1 is $S_{01} \cup S_{02}$. Thus $\bar{S}_{01} = \{v_2 = 0\}$ and $f_{1,0} = (g+1)/g = v_0 v_2/v_1^3$.

Conversely, starting from the graph of $\Sigma(f_0)$, we can construct $f_{1,0}$. Consider the rational function $f_0 = X(YZ - X^2)/Y^3$ for a homogeneous coordinate $(X:Y:Z)$ of $\boldsymbol{P}^2$. Removing $\tilde{S}_{02}$ from the graph of $\Sigma(f_0)$ and applying to the graph the operation inverse to τ, we get Figure 26(a). Contracting the encircled components, we get Figure 26(b). By Proposition 2(iii), there exists a rational function $\tilde{h}$ such that the curves corresponding to the vertical solid lines are level curves of $\tilde{h}$. We may suppose that $\tilde{h}$ takes the values 0, 1, and ∞ on the proper images of the curves $\bar{S}_\infty$, $\bar{S}_1$ and $\bar{S}_{01}$ of f_0. The graph of $\Sigma(f)$ of the transform f of $\tilde{h}$ on $\boldsymbol{P}^2$ is as in Figure 23, which assures the existence of the function belonging to D_1^0.

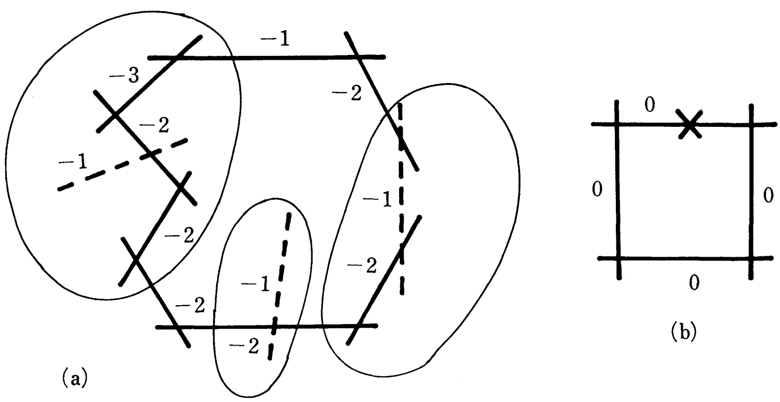


FIGURE 26.

Another proof of this fact is as follows. A smooth rational surface M and a rational function f on M for which the graph of $\Sigma(f)$ is as in Figure 23 are constructed by a finite sequence of blowing-ups on $\boldsymbol{P}\times\boldsymbol{P}$. Let $\sigma'\colon M \to N$ be the composite of σ-processes which contracts $\Sigma(f)$. By the formula on the Euler characteristic in Chapter III, § 1.4, the Euler characteristic of N is 3. Hence $N \cong \boldsymbol{P}^2$.

[C] In the last place, we introduce a rational function $f_{1,1}$ belonging to D_1^{1+}. Suppose that the graph of $\Sigma(f_{1,1})$ is as in Figure 27. (See Figures 4 and 5.) The order of S_1 is 2. Denote by $\rho_1\colon M \to \rho_1(M)$ the composite of σ-processes which contracts the encircled components of $\Sigma(f_{1,1})$ in Figure 27. The graph of $\rho_1(\Sigma(f_{1,1}))$ is given in Figure 28. Denote by C the component of $\Sigma(f_{1,1})$ with the self-intersection number $-(a+1)$. The self-intersection number $(\rho_1(C)^2)$ of the image $\rho_1(C)$ is -1. Denote by $\omega\colon \rho_1(M) \to \omega(\rho_1(M))$ the composite of σ-processes which contracts the components of $\rho_1(\Sigma(f_{1,1}) \cup \tilde{S}_\infty \cup \tilde{S}_{02})$ encircled by fine interrupted lines. The

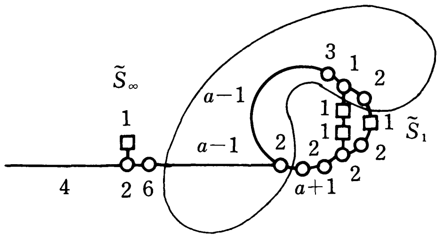


FIGURE 27.

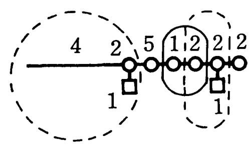


FIGURE 28.

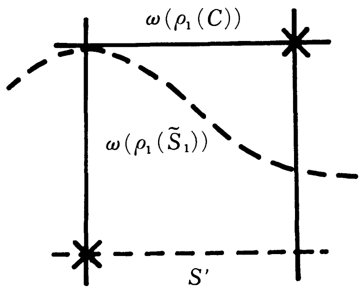


FIGURE 29.

graph of $\omega(\rho_1(\Sigma(f_{1,1})))$ is as in Figure 29. The interrupted curved line represents the image $\omega(\rho_1(\tilde{S}_1))$. The curve $\omega(\rho_1(\tilde{S}_1))$ is tangent to $\omega(\rho_1(C))$ with order $a-1$, that is, $(\omega(\rho_1(\tilde{S}_1))\cdot\omega(\rho_1(C)))=a$. By Proposition 2(ii), there exist rational functions h_1, h_2 on $\omega(\rho_1(M))$ such that the curves corresponding to the vertical lines in the graph of $\omega(\rho_1(\Sigma(f_{1,1})))$ are level curves of h_1 and such that $\omega(\rho_1(C))$ is a level curve of h_2. The mapping θ defined by $\theta(p)=(h_1(p), h_2(p))$ for $p\in\omega(\rho_1(M))$ is a biregular isomorphism of $\omega(\rho_1(M))$ onto $\boldsymbol{P}\times\boldsymbol{P}$. Set $R=(\omega\circ\rho_1\circ\sigma^{-1})^*h_1$. The rational function R is of $\boldsymbol{C}$-type on $\boldsymbol{P}^2$. The graph of $\rho_1(\Sigma(f_{1,1}))$ is that of the minimal resolution of indetermination points of R. In the graph of $\omega(\rho_1(\Sigma(f_{1,1})))$, the crosses represent the images $\omega(\rho_1(\tilde{S}_\infty))$, $\omega(\rho_1(\tilde{S}_{02}))$. Hence the curves S_∞, S_{02} are level curve of R. Set $\psi=(\omega\circ\rho_1\circ\sigma^{-1})^*h_2$. The rational function ψ is of $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2$. Denote by S' the level curve of h_2 which contains the image $\omega(\rho_1(\tilde{S}_\infty))$. Denote by $\tilde{S}$ the proper transform of S' under the mapping $(\omega\circ\rho_1)^{-1}$. The curve $(\sigma(\tilde{S})\setminus\{p_0\})\cup S_\infty$ is a level curve of ψ and the curve S_{02} is another level curve of ψ. Denotes by μ: $\rho_1(M)\to\mu(\rho_1(M))$ the composite of σ-processes which contracts the component of $\rho_1(\Sigma(f_{1,1}))$ encircled by a fine curved line in the graph of $\rho_1(\Sigma(f_{1,1}))$ in Figure 28. The graph of $\mu(\rho_1(\Sigma(f_{1,1})))$ is as in Figure 30. In this graph, S'' is the proper transform of $S=\sigma(\tilde{S})$ under the mapping $\mu\circ\rho_1\circ\sigma^{-1}$. This graph is the same as that of $\Sigma(f_{1,0})$, with S_{02} removed, of $f_{1,0}$ in

Example [B]. By Proposition 2(iii), we can show that there exists a rational function k on $\mu(\rho_1(M))$ having the same family of level curves as that of $\sigma^* f_{1,0}$ in Example [B]. Especially, the level curve of k with value $k(\mu \circ \rho_1(\tilde{S}_{02}))$ consists of two prime curves one of which is $\mu \circ \rho_1(\tilde{S}_{02})$ and another of which is an algebraic curve relevant to $\tilde{S}_{02}$ of $f_{1,0}$ in Example [B]. Hence, in a homogeneous coordinate $(X:Y:Z)$ of $\boldsymbol{P}^2$, $\bar{S}_\infty = \{v_1 = 0\}$, $\bar{S}_{02} = \{v_2 = 0\}$ and $S = \{u = 0\}$. Since R is primitive and since $\deg v_1$ and $\deg v_2$ are coprime, we may suppose that $R = R_{1,1} = v_2^{\deg(v_1)}/v_1^{\deg(v_2)} = v_2^2/v_1^5$. From the graph of $\rho_1(\Sigma(f_{1,1}))$, we see that the curve $\rho_1(\tilde{S}_{02})$ is a prime curve of order one of $\omega^* h_2$ and that $\rho_1(\tilde{S})$ is a prime curve of order one of $\omega^* h_2$. Hence we may suppose that $\psi = \psi_{1,1} = v_1 u/v_2$. Then $\omega(\rho_1(\tilde{S}_1))$ is defined by the equation $h_2 = P(h_1)$ on $\omega(\rho_1(M))$, where $P(z)$ is a polynomial in z of degree a. Hence S_1 is the level curve of order one of the rational function $\psi_{1,1} - P(R_{1,1})$ on $\boldsymbol{P}^2$ with value 0. Hence, for a homogeneous polynomial $P_a(z_1, z_2)$ in (z_1, z_2) of degree a with $P_a(1, 0) \neq 0$, we have $S_1 = \{u v_1^{5a+1} - P_a(v_2^2, v_1^5) v_2 = 0\}$.

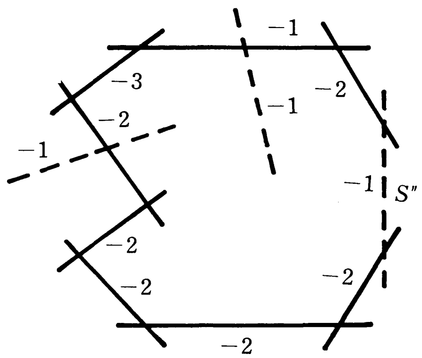


FIGURE 30.

Set $u_1 = u v_1^{5a+1} - P_a(v_2^2, v_1^5) v_2$. Then $\bar{S}_1 = \{u_1 = 0\}$. Since $\psi_{1,1} - P(R_{1,1})$ is a rational function of degree $\deg v_2 + a \deg v_1^5 = 10a + 5$ on $\boldsymbol{P}^2$, $\bar{S}_1$ is of degree $10a + 5$. Consider the rational function $g = v_1^{\deg(u_1)}/u_1^2$ on $\boldsymbol{P}^2$. Since S_1 is a level curve of order 2 of $f_{1,1}$ and since $\bar{S}_\infty = \{v_1 = 0\}$, there must exist an analytic automorphism Φ of $\boldsymbol{P}$ such that $f_{1,1} = \Phi(g)$. The level curve of g with the value -1 is $\{v_1^{\deg(u_1)} + u_1^2 = 0\} \setminus \{p_0\}$. Since $v_1^{\deg(u_1)} + u_1^2 \equiv v_1^{\deg(u_1)} + u^2 v_1^{2(5a+1)} = v_1^{2(5a+1)}\{v_1^3 + u^2\} = v_1^{2(5a+1)} v_0 v_2 \equiv 0 \pmod{v_2}$, we see that $w_1 = (v_1^{\deg(u_1)} + u_1^2)/v_2$ is a homogeneous polynomial of (X, Y, Z). Hence the level curve of g with the value -1 is $S_{01} \cup S_{02}$. We obtain $\bar{S}_{01} = \{w_1 = 0\}$ and $f_{1,1} = (g + 1)/g = v_2 w_1 / v_1^{10a+5}$.

Conversely, starting from the graph of $\Sigma(f_{1,0})$, we can construct the graph of $\Sigma(f_{1,1})$, using Proposition 2 (ii). It assures the existence of the functions $R_{1,1}$, $\psi_{1,1}$, $f_{1,1}$, for a homogeneous polynomial $P_a(z_1, z_2)$ in (z_1, z_2) satisfying $P_a(1, 0) \neq 0$.

§ 2. Determination.

1.1. Now we prove that, if f belongs to the class (D_0) and if the graph of $\Sigma(f)$ is *linear*, the graph of $\Sigma(f)$ is given by Figure 3 in Chapter 0. Suppose that T intersects B_2. Then, as was seen in § 1.1, the shape of $\Sigma(f)$ must be as in Figure 31. Let ρ: $M \to \rho(M)$ be the σ-process which contracts $\tilde{S}_\infty$. Since σ is minimal, $(T^2) \leqq -2$. So $(\rho(T)^2) \leqq -1$, a contradiction to the fact that $\rho(T)$ must have the property (P). We thus obtain $T \cap B_2 = \varnothing$ and $(B_2^2) = -2$.

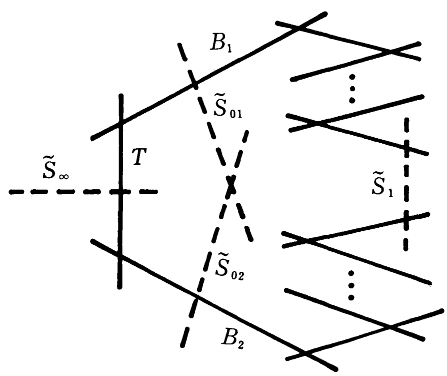


FIGURE 31.

For simplicity, we suppose that F_1 has at least one component with the self-intersection number smaller than -2. Let K_l ($l = 1$ or 2) be the component of F_1 corresponding to the left edge of the graph in Lemma 3 (iii) and K_m ($m = 2$ or 1) be the component corresponding to the right edge of the graph.

(i) The case $(l, m) = (1, 2)$. By Lemma 2, the graph of $\Sigma(f)$ must be as in Figure 32. In Figure 32, the portion in the parenthesis may not exist. First we prove $p_1 = 0$. Suppose that $p_1 > 0$. Then the number of components of F_1 with the self-intersection number smaller than -2

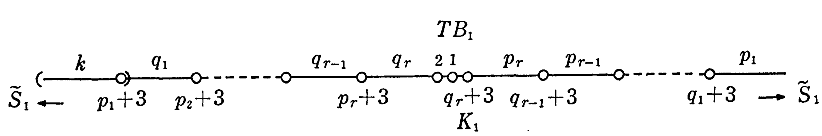
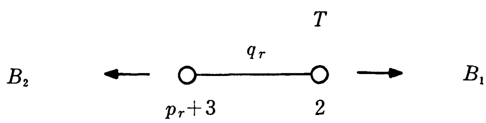


FIGURE 32.

is $2r$. Hence the portion in the parenthesis must exist. By Lemma 3(iii), we have $k=0$ and $p_1+3=p_1+2$, a contradiction. So we obtain $p_1=0$.

If the portion in the parenthesis does not exist, we have $q_1+3=q_1+2$ by Lemma 3(iii), a contradiction. Hence the portion in the parenthesis must exist. Let $\rho\colon M\to\rho(M)$ be the σ-process which contracts $\tilde{S}_\infty$. As was seen in § 1.1, $\rho(F_\infty)$ satisfies the condition of Lemma 3(iii). We thus see that the graph of $F_\infty\cap\Sigma(f)$ must be as in Figure 33.



$(p_r+3=q_r+1)$

FIGURE 33.

Therefore, the graph of F_1 must be as in Figure 34.

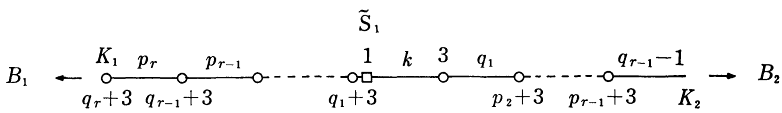


FIGURE 34.

By Lemma 3(iii), we see $k+2=q_1+3$, $q_1+3=q_2+3,\cdots,q_{r-2}+3=q_{r-1}+3$, $(q_{r-1}-2)+3=q_r+3$; $3=p_2+3$, $p_2+3=p_3+3,\cdots,p_{r-1}+3=p_r+3$. Hence $p_1=p_2=\cdots=p_r=0$; $q_r=2$, $q_{r-1}=q_{r-2}=\cdots=q_1=4$ and $k=5$. Thus f belongs to D^0_{2r} $(r=1,2,\cdots)$.

(ii) The case $(l,m)=(2,1)$. By Lemma 2, the graph of $\Sigma(f)$ must be as in Figure 35.

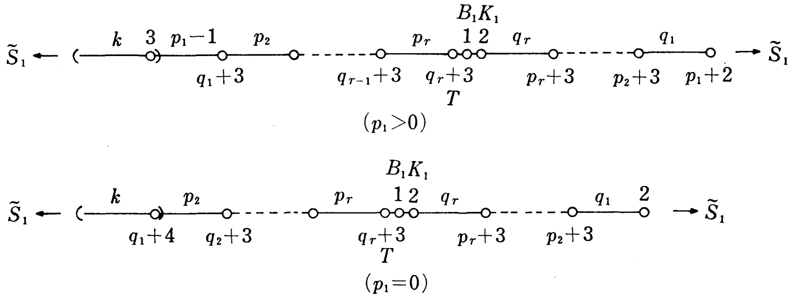
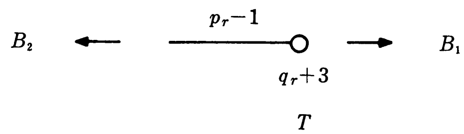


FIGURE 35.

The portion in the parenthesis may not exist. If $p_1 = 0$, the number of components of F_1 with the self-intersection number smaller than -2 is $2r-1$ and a component of F_∞ cannot exist in the graph, a contradiction. Hence p_1 must be positive.

By Lemma 3(iii), the graph of $F_\infty \cap \Sigma(f)$ must be as in Figure 36.



$(q_r+3=p_r-1)$

FIGURE 36.

Therefore, the graph of F_1 must be as in Figure 37.

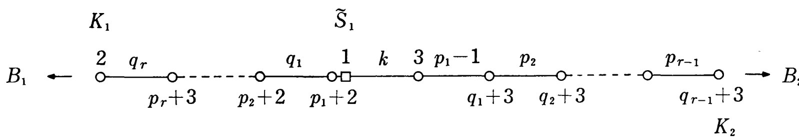


FIGURE 37.

By Lemma 3(iii), we see $p_1 = k$, $p_2 = p_1 - 1$, $p_3 = p_2, \cdots, p_r = p_{r-1}$; $q_1 = 0$, $q_2 = q_1, \cdots, q_r = q_{r-1}$. Hence $q_1 = q_2 = \cdots = q_r = 0$, $p_2 = p_3 = \cdots = p_r = 4$, $p_1 = 5$, $k = 5$. So we see that f belongs to D_{2r+1} $(r = 0, 1, \cdots)$. By (i) and (ii), we obtain the graph of $\Sigma(f)$ in Figure 3 in Chapter 0, § 2.

1.2. Let $f_{n,0}$ be the rational function belonging to D_n^0. Let $\sigma\colon M_n \to \boldsymbol{P}^2$ be the minimal resolution of indetermination points of $f_{n,0}$. We define a *τ-transformation* $\tau_n\colon M_n \to \tau_n(M_n)$ of $f_{n,0}$ as follows. If n is odd, τ_n is the composite of σ-processes which contracts the components of $\Sigma(f_{n,0})$ represented by the diagram in Figure 38(i). If n is even, τ_n is the composite of σ-processes which contracts the components of $\Sigma(f_{n,0})$ represented by the diagram in Figure 38(ii).

FIGURE 38.

The graph of $\tau_n(\Sigma(f_{n,0}) \cup \tilde{S}_{02} \cup \tilde{S}_1 \cup \tilde{S}_\infty)$ is the graph of D_{n-1}^0 with $\tilde{S}_{02}$ removed. By Proposition 2(iii), there exists a curve relevant to $\tilde{S}_{02}$ in the graph of D_{n-1}^0. Hence there must exist a unique rational function $f_{n-1,0}$ on $\boldsymbol{P}^2$ belonging to D_{n-1}^0 such that $\Sigma(f_{n-1,0}) = \tau_n(\Sigma(f_{n,0}))$, whose minimal resolution of indetermination points is $\sigma \circ \tau_n^{-1}: \tau_n(M_n) \to \boldsymbol{P}^2$. Set $M_{n-1} = \tau_n(M_n)$. Let $\tau_{n-1}: M_{n-1} \to \tau_{n-1}(M_{n-1})$ be the τ-transformation on M_{n-1} of $f_{n-1,0}$. As stated before, there exists a curve relevant to $\tilde{S}_{02}$ in the graph of D_{n-2}^0 and a unique rational function $f_{n-2,0}$ on $\boldsymbol{P}^2$ belonging to D_{n-2}^0 such that $\sigma \circ \tau_n^{-1} \circ \tau_{n-1}^{-1}: \tau_{n-1}(M_{n-1}) \to \boldsymbol{P}^2$ is the minimal resolution of the indetermination points of $f_{n-2,0}$. Set $M_{n-2} = \tau_{n-1}(M_{n-1})$. Repeating these processes, we get a sequence of τ-transformations

$$M_n \xrightarrow{\tau_n} M_{n-1} \xrightarrow{\tau_{n-1}} M_{n-2} \xrightarrow{\tau_{n-2}} \cdots \xrightarrow{\tau_3} M_2 \xrightarrow{\tau_2} M_1$$

and rational functions $f_{j,0}$ $(j = 1, 2, \cdots, n)$ on $\boldsymbol{P}^2$ belonging to D_j^0 such that $\Sigma(f_{j,0}) = \tilde{\tau}_j(\Sigma(f_{n,0}))$, where $\tilde{\tau}_j = \tau_{j+1} \circ \tau_{j+2} \circ \cdots \circ \tau_n$.

By Example [B], $f_{1,0}$ is written as $f_{1,0} = v_2 v_0 / v_1^3$, where $v_0 = Y$, $v_1 = YZ - X^2$, $u = XYZ - X^3 - Y^3$ and $v_2 = (v_1^3 + u^2)/v_0$ in a homogeneous coordinate $(X:Y:Z)$ of $\boldsymbol{P}^2$. Denote by S_{01}^j, S_{02}^j, S_1^j, S_∞^j the prime curves S_{01}, S_{02}, S_1, S_∞ of $f_{j,0}$, respectively. We get the recurrence relation $S_{01}^j = S_\infty^{j+1}$, $S_\infty^j = S_{02}^{j+1}$ $(j = 1, 2, \cdots, n-1)$. Suppose that v_i $(i = j-2, j-1, j, j+1)$ are irreducible homogeneous polynomials defining S_∞^i, respectively. Suppose furthermore $v_{j+1} = (v_j^3 + u^{m_j})/v_{j-1}$ and $v_j = (v_{j-1}^3 + u^{m_{j-1}})/v_{j-2}$, where $m_i = \deg(v_i)$. Since $\bar{S}_1^j = \bar{S}_1^1 = \{u = 0\}$, the level curve of a rational function $v_{j+1}v_{j-1}/v_j^3$ with value 1 is S_1^j. Hence $f_{j,0}$ is written as $v_{j+1}v_{j-1}/v_j^3$ and $3m_j = m_{j+1} + m_{j-1}$. So we have

$$\begin{aligned} v_{j-1}^3(v_{j+1}^3 + u^{m_{j+1}}) &= (v_j^3 + u^{m_j})^3 + v_{j-1}^3 u^{m_{j+1}} \equiv u^{3m_j} + v_{j-1}^3 u^{m_{j+1}} \pmod{v_j} \\ &= u^{m_{j+1}}(u^{m_{j-1}} + v_{j-1}^3) \equiv u^{m_{j+1}} v_j v_{j-2} \equiv 0 \pmod{v_j} . \end{aligned}$$

By our assumption, v_j and v_{j-1} are coprime. Hence $v_{j+2} = (v_{j+1}^3 + u^{m_{j+1}})/v_j$ is a homogeneous polynomial.

Consider the rational function $g = v_{j+1}^3/u^{m_{j+1}}$. Since $\{v_{j+1} = 0\} \setminus \{p_0\}$ is the level curve of $f_{j+1,0}$ of order 3 and since $\bar{S}_1^{j+1} = \{u = 0\}$, there must exist an analytic automorphism Φ of $\boldsymbol{P}$ such that $f_{j+1,0} = \Phi \circ g$. Since $\overline{\{g = -1\}} = \{v_{j+1}^3 + u^{m_{j+1}} = 0\} = \{v_j v_{j+2} = 0\}$, we have $\bar{S}_{01}^{j+1} = \{v_{j+2} = 0\}$, so that $f_{j+1,0} = (g+1)/g = v_{j+2}v_j/v_{j+1}^3$. By induction, we get the recurrence formula in Chapter 0, § 2.1.

Conversely, starting from the graph of $\Sigma(f_{1,0})$, we can construct the graph of $\Sigma(f_{n,0})$ by a method similar to that in Example [B]. It assures that the function gotten by the recurrence formula for a homogeneous coordinate $(X:Y:Z)$ belongs to $D_{n,0}$.

The restriction $f_{n,0}|_V$ of $f_{n,0}$ to $V = \boldsymbol{P}^2 \setminus (\bar{S}_{01} \cup \bar{S}_1 \cup \bar{S}_\infty)$ is of proper $\boldsymbol{C}^*$-type. Since m_n and m_{n+1} are coprime, the rational function $R = v_{n+1}^{m_n}/v_n^{m_{n+1}}$ is primitive. By Theorem 1, the restriction $R|_V$ is of proper direct $\boldsymbol{C}^*$-type. By the classification in Chapter III, § 1.4, R must be of $\boldsymbol{C}$-type.

Let $\xi\colon N \to \boldsymbol{P}^2$ be the minimal resolution of the indetermination points of R. The graph of $\Sigma(R) = \xi^{-1}(p_0)$ is easily determined, which appears in 2.2 in this section.

2.1. In this subsection, we suppose that the graph of $\Sigma(f)$ is not linear and $\Sigma(f)$ has k (> 0) diverging components. For simplicity, suppose furthermore that F_1 has at least three components with the self-intersection number smaller than -2. Suppose that T intersects B_2. Then, using Lemma 3(iii) for F_1, we see that $\Sigma(f)$ cannot be an exceptional curve of the first kind, a contradiction. We thus obtain $T \cap B_2 = \varnothing$.

Define K_1 and K_2 as in § 1 and define K_l and K_m as in § 2.1.1. Denote by A_j ($j = 1, 2, \cdots, k+1$) the j-th branch of $\Sigma(f)$ (see Chapter I, § 2.1). By Lemma 3(i), the curve $\overline{F_\infty \setminus T}$ is an exceptional curve of the first kind which contains only one irreducible exceptional curve $\tilde{S}_\infty$ of the first kind. Denote by A'_j the j-th branch of $\overline{F_\infty \setminus T}$. By assumption, $\overline{F_\infty \setminus T}$ has k diverging components. There occur four cases, that is, (i) $(l, m) = (1, 2)$ and $p_1 > 0$, (ii) $(l, m) = (2, 1)$ and $p_1 > 0$, (iii) $(l, m) = (1, 2)$ and $p_1 = 0$, (iv) $(l, m) = (2, 1)$ and $p_1 = 0$. In the following, we determine $\Sigma(f)$ in the case (i). In the remaining cases, $\Sigma(f)$ is determined similarly.

By Lemma 3(iii), we know that the graph of F_1 is as in Figure 39.

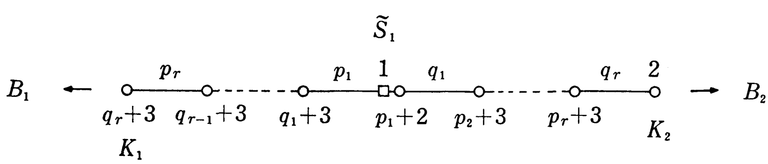


FIGURE 39.

By Lemma 2, the graph of A_1 is as in Figure 40.

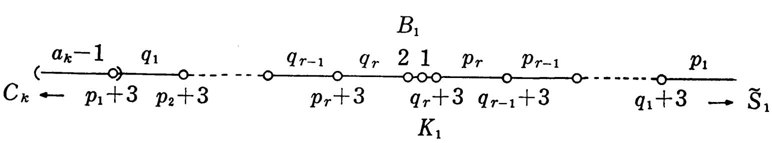


FIGURE 40.

The portion in the parenthesis may not exist. In such a case, we put $a_k = 0$. We denote by C_j the diverging component of $\overline{F_\infty \setminus T}$ common to A'_j and A'_{j+1}. By Lemma 3, the graph of A'_{k+1} is as in Figure 41.

Hence, by Lemma 2, the graph of A_2 is as in Figure 42.

In Figure 42, the symbol in Figure 43(i) is the abbreviation for the diagram in Figure 43(ii) and the symbol in Figure 43(iii) is the abbrevi-

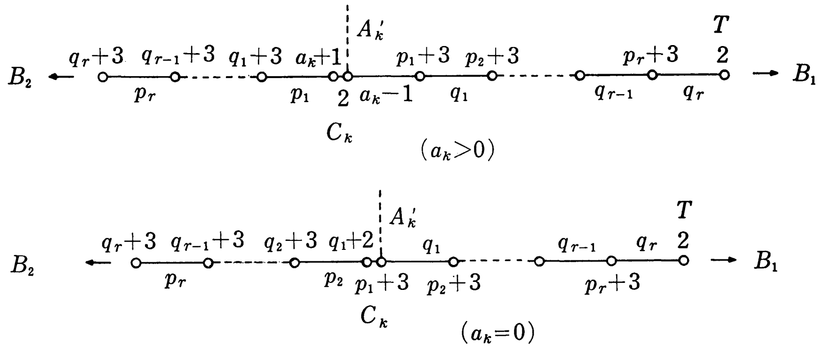


FIGURE 41.

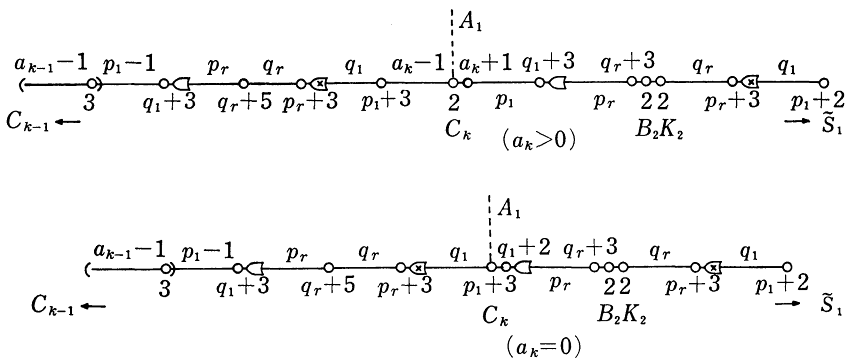


FIGURE 42.

ation for the diagram in Figure 43(iv).

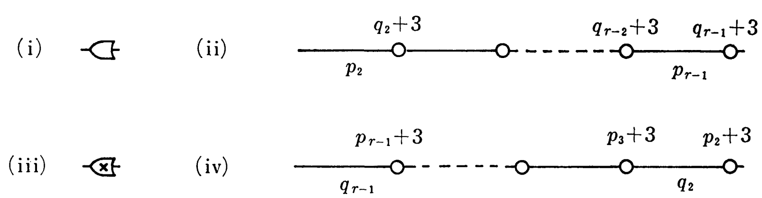


FIGURE 43.

By Lemma 3, the graph of A_k' is as in Figure 44. When $a_k = 0$, the portions in the braket must vanish. By Lemma 2, the graph of A_3 is as in Figure 45.

Repeating these processes, we obtain the graph of A_k. If k is odd, the graph of A_k is the same as the graph of A_3. If k is even, the graph of A_k is the same as the graph of A_2. So we see that the graph of $A_0' \setminus \tilde{S}_\infty$ must be as in Figure 46.

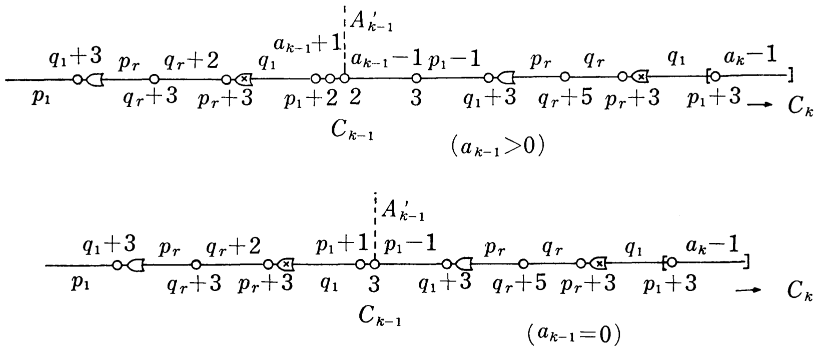


FIGURE 44.

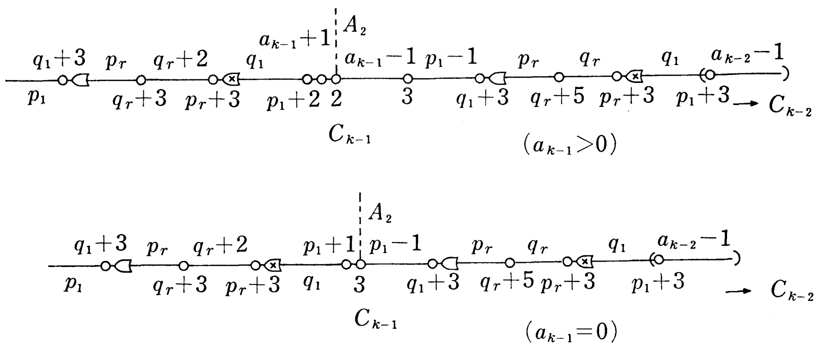


FIGURE 45.

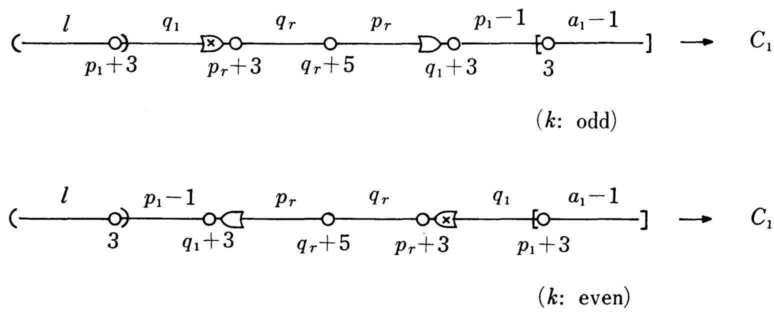


FIGURE 46.

The portions in the braket must vanish if $a_1 = 0$. Since A'_0 is an exceptional curve of the first kind, we can determine p_j, q_j ($j = 1, 2, \cdots, r$) by Lemma 2. If k is odd, the portion in the parenthesis does not exist. We see $q_1 + 3 = 3$, $q_2 + 3 = q_1 + 3, \cdots, q_r + 3 = q_{r-1} + 3$, $q_r + 5 = p_r + 1$, $p_r + 3 = p_{r-1} + 3, \cdots, p_3 + 3 = p_2 + 3$ and $p_2 + 3 = (p_1 - 1) + 3$. Hence $q_1 = q_2 = \cdots = q_r = 0$, $p_2 = p_3 = \cdots = p_r = 4$ and $p_1 = 5$. So, if k is odd, we see that f belongs to D_{2r+1}^{k+} $(r = 2, 3, \cdots)$. If k is even, the portion in the parenthesis must exist. We see $l + 3 = p_1 + 3$, $(p_1 - 1) + 3 = p_2 + 3$, $p_2 + 3 = p_3 + 3, \cdots, p_{r-1} + 3 = p_r + 3$, $p_r + 1 = q_r + 5$, $q_{r-1} + 3 = q_r + 3, \cdots, q_1 + 3 = q_2 + 3$ and $3 = q_1 + 3$. Hence $q_1 = q_2 = \cdots = q_r = 0$, $p_2 = p_3 = \cdots, p_r = 4$ and $p_1 = l = 5$. Thus, if k is even, we see that f belongs to D_{2r+1}^{k-} $(r = 2, 3, \cdots)$. In the remaining cases, we can determine the graph of $\Sigma(f)$ similarly. We get the graph of $\Sigma(f)$ in Chapter 0, § 2.3. Thus we obtain Proposition 0.

2.2. Let $f = f_{n,k}$ be a rational function belonging to $D_n^{k\pm}$. Let

$$M_k = M'_m \xrightarrow{\sigma_m} M'_{m-1} \xrightarrow{\sigma_{m-1}} \cdots \xrightarrow{\sigma_2} M'_1 \xrightarrow{\sigma_1} M'_0 = \boldsymbol{P}^2$$

be the minimal resolution of the indetermination points of $f_{n,k}$ by σ-processes. We define the birational mapping $\rho_k\colon M_k \to N_k$ as follows. Let v, w be homogeneous polynomials which define S_∞, S_{02}. The restriction $f|_V$ of f to $V = \boldsymbol{P}^2 \setminus (\bar{S}_{02} \cup \bar{S}_1 \cup \bar{S}_\infty)$ is of proper $\boldsymbol{C}^*$-type on V. Consider the rational function $R = w^{\alpha_1}/v^{\alpha_2}$ on $\boldsymbol{P}^2$ for coprime positive integers α_1 and α_2 satisfying $\alpha_1 \deg(w) = \alpha_2 \deg(v)$. Since α_1 and α_2 are coprime, R is primitive. Hence, by Theorem 1, the restriction $R|_V$ is of proper direct $\boldsymbol{C}^*$-type on V. Hence R is of special type on $\boldsymbol{P}^2$ and is not of torsional $\boldsymbol{C}^*$-type. By the classification in Chapter III, § 1.4, R must be of $\boldsymbol{C}$-type. Let $\xi_k\colon N_k \to \boldsymbol{P}^2$ be the minimal resolution of the indetermination point p_0 of R by σ-processes. The graph of $\Sigma(R) = \xi_k^{-1}(p_0)$ is easily obtained from the graph of $\Sigma(f_{n,k}) = \tilde{\sigma}_k^{-1}(p_0)$ where $\tilde{\sigma}_k = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_m$. Set $\rho_k = (\xi_k^{-1}) \circ \tilde{\sigma}_k$. The definition of ρ_k is independent of the choice of v, w. Another definition of ρ_k is as follows. Suppose $a_k > 0$. If $f \in D_n^{k+}$ and k is odd, or, if $f \in D_n^{k-}$ and k is even, then ρ_k is the composite of σ-processes which contracts the components of $\Sigma(f_{n,k})$ represented by the graph in Figure 47.

If $f \in D_n^{k+}$ and k is even, or, if $f \in D_n^{k-}$ and k is odd, then ρ_k is the composite of σ-processes which contracts the components of $\Sigma(f_{n,k})$ repre-

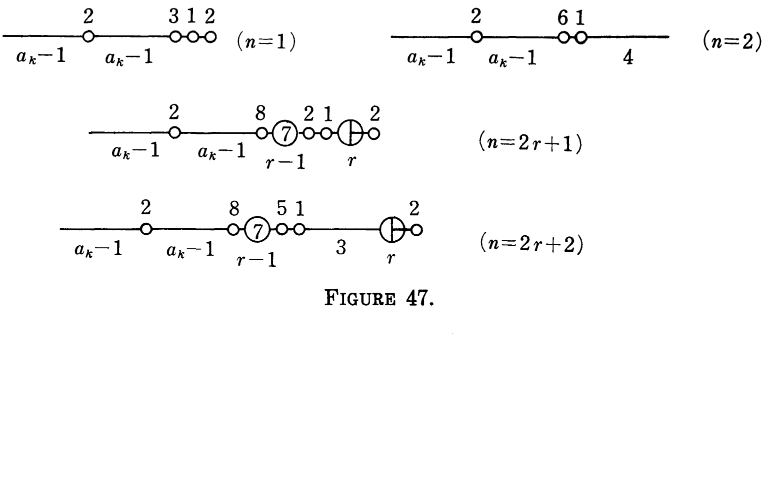


FIGURE 47.

sented by the graph in Figure 48.

If $a_k = 0$, the birational mapping ρ_k is determined similarly. The graph of $\Sigma(R)$ when $k = 1$ and $f \in D_n^{1+}$ is as in Figure 49.

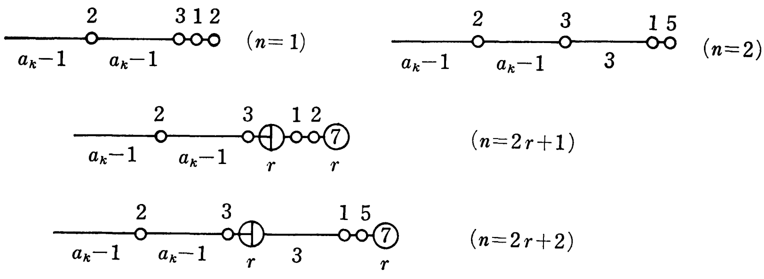


FIGURE 48.

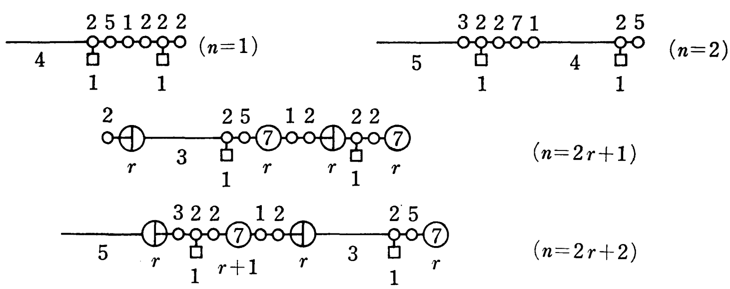


FIGURE 49.

If $k = 1$, by the shape of the graph of $\Sigma(R)$, we can determine S_∞ and S_{02} by a method similar to that in Example [C]. If $f \in D_n^{1+}$, then $\bar{S}_\infty = \{v_n = 0\}$ and $\bar{S}_{02} = \{v_{n+1} = 0\}$ in a homogeneous coordinate $(X : Y : Z)$ of $\boldsymbol{P}^2$. If $f \in D_n^{1-}$, then $\bar{S}_\infty = \{v_n = 0\}$ and $\bar{S}_{02} = \{v_{n-1} = 0\}$.

In the following, we suppose $f \notin D_1^{1-}$. Let H_0, H_∞ be the level curves of $\xi_k^* R$ with values 0, ∞, respectively. The graph of H_0 is linear. Let E be the unique exceptional irreducible component of the first kind of $\Sigma(R)$. The restriction $\xi_k^* R|_E$ of $\xi_k^* R$ to E is non-constant. Let C_∞ be the irreducible component of H_∞ intersecting E and let C_0 be the irreducible component of H_0 located at the edge of the graph of H_0 which does not intersect E. Let the mapping $\omega\colon N_k \to \omega(N_k)$ be the composite of σ-processes which contracts $\overline{(H_0 \setminus C_0)} \cup \overline{(H_\infty \setminus C_\infty)}$. The image $\omega(\Sigma(R))$ consists of three smooth rational curves $\omega(H_\infty)$, $\omega(E)$, $\omega(H_0)$ with the property (P) such that $(\omega(H_\infty) \cdot \omega(E)) = (\omega(E) \cdot \omega(H_0)) = 1$, $(\omega(H_\infty) \cdot \omega(H_0)) = 0$. By Proposition 2(ii), there exist rational functions h_1, h_2 of $\boldsymbol{P}$-type on $\omega(N_k)$ with no critical value such that $\omega(H_\infty)$ and $\omega(H_0)$ are level curves of h_1 and such that $\omega(E)$ is a level curve of h_2. Since R is primitive, we may suppose that $\xi_k^* R = \omega^* h_1$. Let S' be the level curve of h_2 passing through the point $\omega(H_\infty \setminus C_\infty)$. The curve S' satisfies $(S' \cdot \omega(H_\infty)) = (S' \cdot \omega(H_0)) = 1$ and $(S' \cdot \omega(E)) = 0$.

For $k \geqq 2$, we define the birational mapping $\tau_k \colon M_k \to \tau_k(M_k)$ in the following way. If $n = 1$ and $a_k = 0$, then τ_k is the composite of σ-processes which contracts one of the parts of $\Sigma(f_{n,k})$ represented by the diagram in Figure 50(i). If $n = 2$ and $a_k = 0$ and if the right hand side of the graph of $\Sigma(f_{n,k})$ is $H_n(a_k)$, then τ_k is the composite of σ-processes which contracts the components of $\Sigma(f_{n,k})$ represented by the diagram in Figure 50(ii). If $n = 2$ and $a_{k-1} = 0$ and if the right-hand side of the graph of $\Sigma(f_{n,k})$ is $H_n^*(a_k)$, then τ_k is the composite of σ-processes which contracts the encircled components of $\Sigma(f_{n,k})$ in the graphs in Figure 51

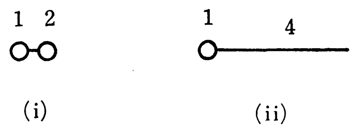


FIGURE 50.

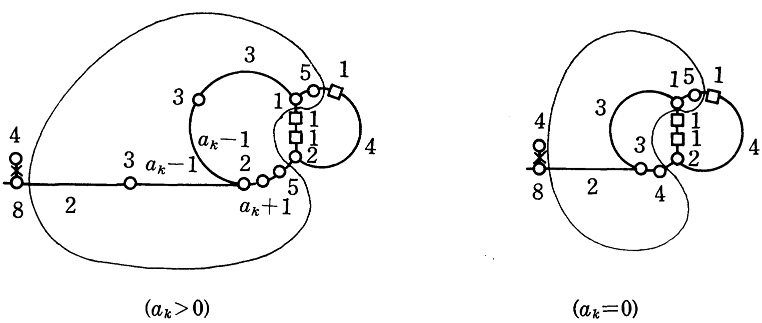


FIGURE 51.

and blows-up a point marked by a cross in these graphs. In the other cases, we define τ_k by $\tau_k = \sigma_{i+1} \circ \sigma_{i+2} \circ \cdots \circ \sigma_m$, where i is the integer such that $\sigma_i \colon M_i' \to M_{i-1}'$ is the σ-process which contracts the image of $\sigma_{i+1} \circ \cdots \circ \sigma_m(B_2)$ on M_i'. We call the mapping τ_k the *τ-transformation* with respect to $f_{n,k}$ and set $M_{k-1} = \tau_k(M_k)$.

Let $\tilde{S}'$ be the proper image of S' under the mapping $\rho_k^{-1} \circ \omega^{-1}$. The graph of $\tau_k(\Sigma_f \cup \tilde{S}_{02} \cup \tilde{S}_\infty \cup \tilde{S}')$ is the graph of $D_n^{k-1\pm}$ with $\tilde{S}_{02}$ removed and $\tau_k(\tilde{S}')$ corresponds to $\tilde{S}_1$ in this graph. By Proposition 2(iii), there exists a curve relevant to $\tilde{S}_{02}$ in the graph of $D_n^{k-1\pm}$ (see Example [C]). Hence there exists a unique rational function $f_{n,k-1}$ on $\boldsymbol{P}^2$ belonging to $D_n^{k-1\pm}$

whose minimal resolution of the indetermination point is $\tilde{\sigma}_k \circ \tau_k^{-1}: M_{k-1} \to \boldsymbol{P}^2$ and such that $\Sigma(f_{n,k-1}) = \tau_k(\Sigma(f_{n,k}))$. If $f \in D_n^{k+}$, then $f_{n,k-1} \in D_n^{k-1+}$. If $f \in D_n^{k-}$, then $f_{n,k-1} \in D_n^{k-1-}$. Let $\tau_{k-1}: M_{k-1} \to \tau_{k-1}(M_{k-1})$ be the τ-transformation with respect to $f_{n,k-1}$ and set $\tau_{k-1}(M_{k-1}) = M_{k-2}$. As stated before, we know that there exist curves relevant to $\tilde{S}_{02}$ and $\tilde{S}_1$ in the graph of $D_n^{k-2\pm}$. There exists a unique rational function $f_{n,k-2}$ on $\boldsymbol{P}^2$ belonging to $D_n^{k-2\pm}$ such that $\Sigma(f_{n,k-2}) = \tau_{k-1}(\Sigma(f_{n,k-1}))$. Repeating these processes, we get a sequence of τ-transformations $\tau_j: M_j \to M_{j-1}$ and rational functions $f_{n,j}$ on $\boldsymbol{P}^2$ belonging to $D_n^{j\pm}$ $(j = 1, 2, \cdots, k)$, such that $\Sigma(f_{n,j}) = \tilde{\tau}_j(\Sigma(f_{n,k}))$ where $\tilde{\tau}_j = \tau_{j+1} \circ \tau_{j+2} \circ \cdots \circ \tau_k$. The level curve of $f_{n,k}$ with value ∞ is also the level curve of $f_{n,j}$ with value ∞ $(j = 1, 2, \cdots, k-1)$. Hence we may suppose that $v = v_n$. Denote by $S_{01,j}$, $S_{02,j}$, $S_{1,j}$ the prime curves S_{01}, S_{02}, S_1 of $f_{n,j}$, respectively. We get the recurrence relation $S_{01,j} = S_{02,j+1}$.

Let u_{k-1} be a homogeneous polynomial defining $\bar{S}_{1,k-1} = \tilde{\sigma}_k(\tilde{S}')$. The proper image of $\bar{S}_\infty$ under the mapping $\omega \circ \xi_k^{-1}$ is the point $\omega(\overline{H_\infty \setminus C_\infty})$. The proper image of $S_{02,k}$ under $\omega \circ \xi_k^{-1}$ is the point $\omega(\overline{H_0 \setminus C_0})$. Hence we may suppose $(\omega \circ \xi_k^{-1})^* h_2 = u_{k-1} v_n^{\mu_k} / w^{s_k}$, where μ_k and s_k are positive integers detemined as follows. Let $\eta_k: N_k' \to \boldsymbol{P}^2$ be a minimal resolution of the indetermination point p_0 of $\psi = (\omega \circ \xi_k^{-1})^* h_2$. The graph of $\Sigma(\psi) = \eta_k^{-1}(p_0)$ is easily obtained from the graph of $\Sigma(R)$. The graph of the level curve of $\eta_k^* \psi$ with value ∞ is given in Figure 52(a) if $n = 1$, given in Figure 52(b) if $f \in D_n^{k+}$ and k is odd, or, if $f \in D_n^{k-}$ and k is even, and given in Figure 52(c) if $f \in D_n^{k+}$ and k is even, or, if $f \in D_n^{k-}$ and k is odd.

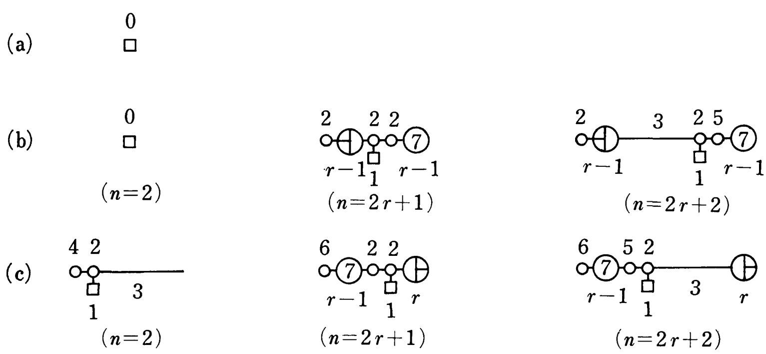


FIGURE 52.

By Lemma 4, the order s_k of the level curve $S_{02,k}$ of ψ with the value ∞ is determined. If $f \in D_n^{k+}$, then $s_k = (b_{2n} + 3(-1)^k b_{2n-3})/2$. If $f \in D_n^{k-}$, then $s_k = (b_{2n} + 3(-1)^{k-1} b_{2n-3})/2$. The integer μ_k satisfies $\mu_k = (s_k \deg w - \deg u_{k-1})/m_n$ for $m_n = \deg v_n$.

The proper image S of $\bar{S}_{1,k}$ under the mapping $\omega\circ\xi_k^{-1}$ is a smooth rational curve satisfying $(S\cdot\omega(E))=a_k$ and $(S\cdot\omega(H_0))=(S\cdot\omega(H_\infty))=1$. If $a_k>0$, then S is tangent to $\omega(E)$ with order a_k-1 at the point $\omega(E)\cap\omega(H_\infty)$. Since $\omega(H_\infty)$ is the level curve of h_1 with value ∞ and $\omega(E)$ is the level curve of h_2 with value ∞, the equation $h_2=P_{a_k}(h_1)$ on $\omega(N_k)$ defines S, where $P_{a_k}(z)$ is a polynomial of degree a_k. Hence $S_{1,k}$ is the level curve of the rational function $\psi-P_{a_k}(R)$ of order one with the value 0. Set $w_{k-1}=w$. The polynomial $u_k=u_{k-1}v_n^{\mu_k+a_k\deg(w_{k-1})}-P_{a_k}(w_{k-1}^{m_n},v_n^{\deg(w_{k-1})})w_{k-1}^{s_k}$ defines $S_{1,k}$ where $P_{a_k}(z_1,z_2)$ is a homogeneous polynomial in (z_1,z_2) of degree a_k and $P_{a_k}(1,0)\neq 0$. The degree of u_k is $a_km_n\deg(w_{k-1})+s_k\deg(w_{k-1})=m_n(\mu_k+a_k\deg(w_{k-1}))+\deg(u_{k-1})$. Suppose furthermore $v_n^{\deg(u_{k-1})}+u_{k-1}^{m_n}=w_{k-1}w_{k-2}$, where w_{k-2} is a homogeneous polynomial defining $\bar{S}_{02,k-1}=\bar{S}_{01,k-2}$.

Consider the rational function $g=v_n^{\deg(u_k)}/u_k^{m_n}$ on $\boldsymbol{P}^2$. By Proposition 0, $S_{1,k}$ is the level curve of $f_{n,k}$ of order m_n. Since $\bar{S}_\infty=\{v_n=0\}$, there must exist an analytic automorphism Φ of $\boldsymbol{P}$ with $f_{n,k}=\Phi(g)$. The level curve of g with the value -1 is defined by $v_n^{\deg(u_k)}+u_k^{m_n}$. Since

$$\begin{aligned}v_n^{\deg(u_k)}+u_k^{m_n}&\equiv v_n^{\deg(u_k)}+(u_{k-1}v_n^{\mu_k+a_k\deg(w_{k-1})})^{m_n}\pmod{w_{k-1}}\\&=\{v_n^{\deg(u_{k-1})}+u_{k-1}^{m_n}\}v_n^{m_n(\mu_k+a_k\deg(w_{k-1}))}\\&\equiv w_{k-1}w_{k-2}v_n^{m_n(\mu_k+a_k\deg(w_{k-1}))}\equiv 0\pmod{w_{k-1}}\,,\end{aligned}$$

$w_k=(v_n^{\deg(u_k)}+u_k^{m_n})/w_{k-1}$ is a homogeneous polynomial. Hence the level curve of g with the value -1 is $S_{01,k}\cup S_{02,k}$ and $\bar{S}_{01,k}$ is defined by w_k. Hence $f_{n,k}=(g+1)/g=w_kw_{k-1}/v_n^{\deg(u_k)}$. We obtain the recurrence formula in Chapter 0, § 2.1 by induction.

Conversely, starting from the graph of $\Sigma(f_{n,0})$, we can construct the graph of $\Sigma(f_{n,k})$ using Proposition 2(ii). So the function obtained by the recurrence formula belongs $D_n^{k\pm}$.

Since $\deg(v_j)$ and $\deg(v_{j+1})$ are coprime and since $\deg(w_k)=m_n\deg(u_k)-\deg(w_{k-1})$, we can prove inductively that $\deg(w_k)$ and m_n are coprime. Hence the rational function $R_{n,k}=w_{k-1}^{m_n}/v_n^{\deg(w_{k-1})}$ is primitive. Kashiwara has proved that a rational function R of $\boldsymbol{C}$-type belonging to $\mathscr{F}_{\mathrm{II}}$ on $\boldsymbol{P}^2$ is written as $R=\Lambda(R_{n,k})$ in a homogeneous coordinate $(X:Y:Z)$, where $\Lambda(z)$ is a rational function of z. By a method similar to those in Chapter II and Chapter III, § 1, we can prove the result of Kashiwara. Set $\psi_{n,k}=u_{k-1}v_n^{\mu_k}/w_{k-1}^{s_k}$. As was seen in this section, the mapping θ defined by $\theta(p)=(R_{n,k}(p),\psi_{n,k}(p))$ is a birational biregular isomorphism of $\boldsymbol{P}^2\setminus(\bar{S}_\infty\cup\bar{S}_{02,k})$ onto $\boldsymbol{C}^*\times\boldsymbol{C}$.

When f belongs to D_1^{1-}, we can also prove the recurrence formula in Chapter 0, § 2.1 similarly. The rational function $R_{1,0}$ of $\boldsymbol{C}$-type belongs to $\mathscr{F}_{\mathrm{I}}$.

Chapter III. Rational functions of C^*-type on P^2.

§ 1. Critical level curves.

1. Rational functions of proper C^*-type. Here we prove the following.

PROPOSITION 3. *Let f be a primitive rational function of direct C^*-type on P^2. Thre exist a triple S_1, S_2, S_3 of prime curves of f and an analytic automorphism T of P such that the restriction $T \circ f|_V$ of $T \circ f$ to $V = P^2 \setminus (\bar{S}_1 \cup \bar{S}_2 \cup \bar{S}_3)$ is of proper direct C^*-type, where $\bar{S}_l$ is the closure of S_l in P^2.*

We use the following lemma to prove this. (See M. Oka [5, p.233].)

LEMMA 5. *Let C be an algebraic curve on P^2. If C has l irreducible components, then the first Betti number $b_1(P^2 \setminus C)$ of $P^2 \setminus C$ equals $l-1$.*

PROOF OF PROPOSITION 3. Let $\sigma: M \to P^2$ be the minimal resolution of the indetermination points of f by σ-processes. Let B_1 and B_2 be basic sections of $\sigma^* f$. We may suppose $(B_1^2) = -1$. The curve B_1 intersects the other components of $\Sigma(f) = \sigma^{-1}(I_f)$ at most two points. Hence, for a suitable analytic automorphism T of P, each point p of B_1 satisfying $T \circ f(p) \neq 0, \infty$ is a regular point of $\Sigma(f)$. Clearly, σ is the minimal resolution of the indetermination points of $g = T \circ f$. Since $\sigma^* g|_{B_i}$ is a rational function degree one on B_i, each level curve F_c of $\sigma^* g$ with value c intersects B_i at one ordinary point of F_c transversally for each i. Since a componet of F_c intersecting B_1 is of order one, Lemma 3(i) shows that the union E_c of all prime curves of the level curve F_c which do not intersect B_1 is an exceptional curve of the first kind. Let the mapping $\tau: M \to \tau(M)$ be the composite of σ-processes which contracts the curve $\cup E_c$, where c varies over C^*. The restriction $h|_{V'}$ of $h = (\sigma \circ \tau^{-1})^* g$ to $V' = \tau(M) \setminus (\{h = 0\} \cup \{h = \infty\} \cup \tau(B_1) \cup \tau(B_2))$ is of proper direct C^*-type. The restriction $\sigma \circ \tau^{-1}|_{V'}$ of the birational mapping $\sigma \circ \tau^{-1}$ is a biregular mapping of V' onto $V = \sigma \circ \tau^{-1}(V')$. Since the first Betti number of V' equals 2, Lemma 5 implies that $P^2 \setminus V$ is an algebraic curve with three irreducible components C_1, C_2, C_3 which are the closures of prime curves S_1, S_2, S_3 of f, respectively. Since $h|_{V'}$ is of proper direct C^*-type, $T \circ f|_V$ is of proper direct C^*-type.

2. The first Betti numbers and the Euler characteristics of level curves.

LEMMA (Nishino). *Let f be a primitive rational function on P^2.*

Denote by L the topological model of regular level curves of f and let L_0 be a critical level curve of f. Then $b_1(L_0) \leqq b_1(L)$ and $\chi(L_0) \geqq \chi(L)$, where $b_1(*)$ and $\chi(*)$ are the first Betti-number and the Euler characteristic of $*$, respectively.

PROOF. Let $(X:Y:Z)$ be a homogeneous coordinate of $\boldsymbol{P}^2$. Suppose that f is a rational function of degree n on $\boldsymbol{P}^2$. The function f is represented as $f = P/Q$, where P, Q are homogeneous polynomials in $(X:Y:Z)$ of degree n. Set $N = {}_{n+2}C_2 - 1$. We denote homogeneous coordinates in $\boldsymbol{P}^N$ be $W_{k_0k_1k_2}$ where k_0, k_1, k_2 are arbitrary non-negative integers such that $k_0 + k_1 + k_2 = n$. The Veronese mapping $v: \boldsymbol{P}^2 \to \boldsymbol{P}^N$ defined by $W_{k_0k_1k_2} = X^{k_0}Y^{k_1}Z^{k_2}$ is an analytic imbedding of $\boldsymbol{P}^2$ into $\boldsymbol{P}^N$. The so-called Veronese variety $v(\boldsymbol{P}^2)$ is smooth in $\boldsymbol{P}^N$. Suppose $P = \sum A_{k_0k_1k_2}X^{k_0}Y^{k_1}Z^{k_2}$ and $Q = \sum B_{k_0k_1k_2}X^{k_0}Y^{k_1}Z^{k_2}$. Set $\tilde{P} = \sum A_{k_0k_1k_2}W_{k_0k_1k_2}$ and $\tilde{Q} = \sum B_{k_0k_1k_2}W_{k_0k_1k_2}$. Then, the rational function $\tilde{P}/\tilde{Q}$ of degree one on $\boldsymbol{P}^N$ satisfies $f = v^*(\tilde{P}/\tilde{Q})$.

Let L_0 be a critical level curve of f with value c_0 and let c_1 $(\neq c_0)$ be a complex number. We regard $H = \{\tilde{P} - c_1\tilde{Q} = 0\}$ as the hyperplane at infinity and denote by $(w_1, w_2, \cdots, w_N)$ an inhomogeneous coordinate of $\boldsymbol{C}^N = \boldsymbol{P}^N \setminus H$. The manifold $\tilde{V} = v(\boldsymbol{P}^2) \setminus H$ is biregularly isomorphic to the domain $V = \{p \in \boldsymbol{P}^2 \setminus I_f \mid f(p) \neq c_1\}$. Set $\tilde{\Omega}^\alpha = \{\sum_{k=1}^N |w_k|^2 < \alpha^2\} \cap \tilde{V}$ for each real positive number α. Denote by Ω^α the inverse image $v^{-1}(\tilde{\Omega}^\alpha)$. Then $\Omega^\alpha \subset\subset V$. Set $\tilde{\varphi} = \sum_{k=1}^N |w_k|^2$ and $\varphi = v^*\tilde{\varphi}$. The function φ is strongly pluri-subharmonic on V. Let α_0 be a number such that $L_0 \cap \Omega^{\alpha_0} \neq \emptyset$. Then there is an open neighbourhood U of c_0 such that $L_c^{\alpha_0} = L_c \cap \Omega^{\alpha_0} \neq \emptyset$ for any $c \in U$, where L_c is the level curve of f with value c. For each real number $\alpha > \alpha_0$, we denote by L_c^α the analytic continuation of $L_c^{\alpha_0}$ in Ω^α.

If $\alpha_0 < \alpha < \beta$, then $b_1(L_c^\alpha) \leqq b_1(L_c^\beta)$ and $\chi(L_c^\alpha) \geqq \chi(L_c^\beta)$. To see this, suppose that $l = b_1(L_c^\alpha) > b_1(L_c^\beta)$. Let $\mathfrak{c}_1, \mathfrak{c}_2, \cdots, \mathfrak{c}_l$ be l cycles on L_c^α whose homology classes $[\mathfrak{c}_1]_\alpha, [\mathfrak{c}_2]_\alpha, \cdots, [\mathfrak{c}_l]_\alpha$ generate $H_1(L_c^\alpha, \boldsymbol{Z})$. There must exist a set of integers $(m_1, m_2, \cdots, m_l)$ such that at least one m_j is not zero and such that the homology class $[\mathfrak{c}]_\beta$ of $\mathfrak{c} = m_1\mathfrak{c}_1 + m_2\mathfrak{c}_2 + \cdots + m_l\mathfrak{c}_l$ is the zero element of $H_1(L_c^\beta, \boldsymbol{Z})$. Hence there exists a subdomain S of L_c^β such that $\partial S \subset \operatorname{supp} \mathfrak{c}$ and such that $S \not\subset L_c^\alpha$. Since $\partial S \subset \Omega^\alpha$, $\varphi|_S$ takes a maximal value at an interior point of S, which contradicts the fact that φ is strongly pluri-subharmonic. Hence $b_1(L_c^\alpha) \leqq b_1(L_c^\beta)$. Suppose that $\chi(L_c^\alpha) < \chi(L_c^\beta)$. By assumption, there exists a simply connected component of $L_c^\beta \setminus L_c^\alpha$ whose boundary is contained in $\partial\Omega^\alpha$. It leads us to a contradiction. Hence $\chi(L_c^\alpha) \geqq \chi(L_c^\beta)$.

From the above fact, we see that $b_1(L_c^\alpha) \leqq b_1(L_c)$ and $\chi(L_c^\alpha) \geqq \chi(L_c)$.

Suppose that α_0 is so large that $L_{c_0}^{\alpha_0}$ intersects all the irreducible components of L_{c_0}. Let $p_1, p_2, \cdots, p_k$ be the singular points of L_{c_0}. For a sufficiently large α, we have $b_1(L_{c_0}^{\alpha}) = b_1(L_{c_0})$, $\chi(L_{c_0}^{\alpha}) = \chi(L_{c_0})$ and $\{p_j\} \subset L_{c_0}^{\alpha}$. Suppose that L_c is a regular level curve of f. There exist bicylinders γ_j with the center p_j $(j = 1, 2, \cdots, k)$ in coordinate neighbourhoods such that $L_{c_0}^{\alpha} \cap \gamma_j$ is simply connected and for c sufficiently near to c_0, $L_c^{\alpha} \setminus \cup \gamma_j$ is a topological covering surface of $L_{c_0}^{\alpha} \setminus \cup \gamma_j$ (with no branch point and with no relative boundary). Hence $b_1(L_c^{\alpha}) \geqq b_1(L_{c_0}^{\alpha})$, $\chi(L_c^{\alpha}) \leqq \chi(L_{c_0}^{\alpha})$. Therefore $b_1(L_c) \geqq b_1(L_{c_0})$ and $\chi(L_c) \leqq \chi(L_{c_0})$.

COROLLARY. *Each prime curve of a rational function of special type on $\boldsymbol{P}^2$ is of $\boldsymbol{C}$-type or of $\boldsymbol{C}^*$-type.*

REMARK. Let S be a prime curve of f with value c and $\{p_j\}$ be the set of intersections of S with the other prime curves of f with value c. Then $S' = S \setminus \{p_j\}$ satisfies $b_1(S') \leqq b_1(L)$ and $\chi(S') \geqq \chi(L)$. The proof is almost the same as that of Lemma 6.

3. Proof of Theorem 1. Let C be an algebraic curve on $\boldsymbol{P}^2$ such that the restriction $f|_V$ of a rational function f on $\boldsymbol{P}^2$ to $V = \boldsymbol{P}^2 \setminus C$ is of proper $\boldsymbol{C}^*$-type. Let $\sigma\colon M \to \boldsymbol{P}^2$ be a resolution of the indetermination points of f by a finite sequence of σ-processes. Set $\tilde{V} = \sigma^{-1}(V)$. The mapping $\sigma|_{\tilde{V}}\colon \tilde{V} \to V$ is a biholomorphic mapping of $\tilde{V}$ onto V. Set $\tilde{h} = \sigma^* f|_{\tilde{V}}$ and $M' = M \setminus (\{\sigma^* f = 0\} \cup \{\sigma^* f = \infty\})$. Let L_c denote the level curve of $\tilde{h}$ with value c and F_c be the level curve of $\tilde{h}_1 = \sigma^* f|_{M'}$ with value c. Assume that F_c is reducible. Since L_c is irreducible and of order one, we see by Lemma 3(i) that the closure of $F_c \setminus \bar{L}_c$ is an exceptional curve of the first kind. Let $\tau\colon M' \to \tau(M')$ be the composite of σ-processes which contracts $\cup (F_c \setminus L_c)$. Set $h = (\tau^{-1})^* \tilde{h}$ and set $h_1 = (\tau^{-1})^* \tilde{h}_1$. The function h_1 is of $\boldsymbol{P}$-type and each level curve of h_1 is irreducible. The image of the union of basic sections of $\sigma^* f$ under τ is $H = \tau(M') \setminus \tau(V)$. So H intersects each $\tau(F_c)$ at two points transversally. Hence H is a smooth algebraic curve in $\tau(M')$.

Let g be a primitive rational function on $\boldsymbol{P}^2$ whose restriction $g|_V$ to V does not take the values 0, ∞ on V. Set $k = (\sigma \circ \tau^{-1})^* g|_{\tau(V)}$ and $k_1 = (\sigma \circ \tau^{-1})^* g|_{\tau(M')}$. Since k does not take the values 0, ∞ on $\tau(V)$, k has no point of indetrmination on H. Therefore, if the restriction $k|_{\tau(L_c)}$ of k to some level curve $\tau(L_c)$ of h is constant, then the restriction of k to any $\tau(L_c)$ must be constant. Hence g is of proper $\boldsymbol{C}^*$-type.

Suppose that the restriction $k|_{\tau(L_c)}$ is non-constant for each $c \in \boldsymbol{C}^*$. For a fixed number c, let $\mu_c\colon \tau(L_c) \to \boldsymbol{C}^*$ be an analytic isomorphism of $\tau(L_c)$ onto $\boldsymbol{C}^*$. The variable $\zeta = \mu_c(p)$ $(p \in \tau(L_c))$ is a global coordinate of

$\tau(L_c)$. The mapping $k \circ \mu_c^{-1}$ is a non-constant regular mapping of C^* onto C^*. Hence we have $k \circ \mu_c^{-1} = \alpha\zeta^m$ for a non-zero constant α and a non-zero integer m. So each prime curve of k is non-singular and of order one and the mapping $k: \tau(V) \to C^*$ is surjective. An irreducible component of H is a prime curve of k_1 with value 0 or ∞. Hence the closure of a prime curve of k does not intersect H. A prime curve S of k is a covering surface over C^* with the projection $h|_S: S \to C^*$ which has no branch point and no relative boundary. Hence each prime curve of k is of C^*-type. By Lemma 6, each level curve of k is irreducible. Therefore k is of proper C^*-type on $\tau(V)$. This means that the restriction $g|_V$ of g to V is of proper C^*-type. If f is of direct C^*-type, the first Betti number of $\tau(V)$ is 2. Hence g must be of direct C^*-type, if f is of direct C^*-type. This Theorem 1 is established.

4. Classification. Let f be a primitive rational function of C^*-type on P^2 and B be the union of basic sections of σ^*f. For a suitable set $e^* = \{a_1, a_2, \cdots, a_m\}$ of values of σ^*f the triple $F = \langle M^*, \sigma^*f|_{M^*}, P\setminus e^*\rangle$ is a locally trivial analytic family of curves with the fibre C^*, where $M^* = M\setminus(B\cup(\cup_k(\sigma^*f)^{-1}(a_k)))$. If f is of direct C^*-type, then $b_1(M^*) = m$. If f is of torsional C^*-type, then $b_1(M^*) = m - 1$.

Set $C = \cup_k L_{a_k}$ where L_{a_k} is the level curve of f with value a_k. Since $P^2\setminus C$ is homeomorphic to M^*, we get the following proposition from Lemma 5.

PROPOSITION 4. *If f is of direct C^*-type on P^2, then f has one level curve with two irreducible components and the other level curves of f are irreducible. If f is of torsional C^*-type on P^2, then each level curve of f is irreducible.*

Since each level curve of σ^*f is simply connected, a singular point of each level curve of f is an ordinary double point. Since each level curve of σ^*f intersects B at most two points, a connected component of each level curve of f has at most two boundary points. Each prime curve of f is smooth and non-compact. If a level curve of f is irreducible, of C^*-type and of order one, then it is regular. By Lemma 6 and Remark after it, we see the following facts for a critical level curve L_0 of f. (a) If $\chi(L_0) = 0$, then L_0 is of C^*-type, irreducible and multiple. (b) If $\chi(L_0) = 1$, then one of the following three cases occurs. (i) L_0 consists of two prime curves both of which are of C-type. They intersect each other at one point in $P^2\setminus I_f$ transversally. (ii) L_0 consists of two prime curves disjoint in $P^2\setminus I_f$ and one of them is of C-type and the other is of C^*-type. (iii) L_0 is irreducible and of C-type. (c) If $\chi(L_0) = 2$, then

L_0 consists of two prime curves disjoint in $\boldsymbol{P}^2 \setminus I_f$ both of which are of $\boldsymbol{C}$-type.

Let $e = \{c_1, c_2, \cdots, c_m\}$ be the set of critical values of f and L_i be the level curve of f with value c_i. Set $X = \boldsymbol{P}^2 \setminus (I_f \cup (\cup_{i=1}^m L_i))$. The triple $\langle X, f|_X, \boldsymbol{P} \setminus e \rangle$ is a locally trivial analytic family of curves with the fibre $\boldsymbol{C}^*$. Hence $\chi(X) = \chi(\boldsymbol{C}^*)\chi(\boldsymbol{P} \setminus e)$. Therefore, $\chi(\boldsymbol{P}^2 \setminus I_f) = \chi(\boldsymbol{C}^*)\chi(\boldsymbol{P}) + \sum_{i=1}^m \{(\chi(L_i) - \chi(\boldsymbol{C}^*)\}$. Since $\chi(\boldsymbol{C}^*) = 0$, we see $\chi(\boldsymbol{P}^2 \setminus I_f) = \sum_{i=1}^m \chi(L_i)$. Suppose that f has two points of indetermination. Since $\chi(\boldsymbol{P}^2 \setminus I_f) = 1$, we have $\sum_{i=1}^m \chi(L_i) = 1$. By Lemma 6, $\chi(L_i) \geqq 0$ for each i. Hence one critical level curve L_{i_0} satisfies $\chi(L_{i_0}) = 1$. If $i \neq i_0$, then $\chi(L_i) = 0$ and all the level curves of f, except L_{i_0}, are irreducible. Since f must be of direct $\boldsymbol{C}^*$-type, Proposition 4 shows that L_{i_0} must be reducible. Therefore, f belongs to one of the following two classes.

Class (A): A level curve L_1 satisfies the condition in (b), (i) and the other level curves are irreducible and of $\boldsymbol{C}^*$-type.

Class (B): A level curve L_1 satisfies the condition in (b), (ii) and the other level curves are irreducible and of $\boldsymbol{C}^*$-type.

Suppose that I_f consist of only one point. Since $\chi(\boldsymbol{P}^2 \setminus I_f) = 2$, we have $\sum_{i=1}^m \chi(L_i) = 2$. We may suppose that $\chi(L_1) = \chi(L_2) = 1$ and $\chi(L_i) = 0$ for $i \neq 1, 2$, or that $\chi(L_1) = 2$ and $\chi(L_i) = 0$ for $i \neq 1$. Assume that $\chi(L_1) = \chi(L_2) = 1$. If f is of direct $\boldsymbol{C}^*$-type, then, by Proposition 4, we may suppose that L_1 satisfies the condition in (b), (iii) and L_2 satisfies the condition in (b), (i) or (b), (ii). Therefore, f belongs to one of the following two classes.

Class (C): A level curve L_1 satisfies the condition in (b), (iii) and another level curve L_2 satisfies the condition (b), (ii) and, furthermore, the other level curves are irreducible and of $\boldsymbol{C}^*$-type.

Class (D): A level curve L_1 satisfies the condition in (b), (iii) and another level curve L_2 satisfies the condition (b), (i) and, furthermore, the other level curves are irreducible and of $\boldsymbol{C}^*$-type.

If f is of tosional $\boldsymbol{C}^*$-type, then, by Proposition 4, each level curve of f is irreducible. Hence we have the following class.

Class (T): Two level curves L_1, L_2 satisfy the condition (b), (iii) and the other level curves are irreducible and of $\boldsymbol{C}^*$-type.

Assume that $\chi(L_1) = 2$. Then we have the following class.

Class (E): A level curve L_1 satisfies the condition in (c) and the other level curves are irreducible and of $\boldsymbol{C}^*$-type.

By Proposition 4, f is of torsional $\boldsymbol{C}^*$-type if and only if f belongs to Class (T). This fact is used in § 3. Suppose that R is a primitive rational function of $\boldsymbol{C}$-type on $\boldsymbol{P}^2$. By the same method as in the proof of

Proposition 4, we can prove that each level curve of R is irreducible. By Lemma 6, each level curve of R is of $\boldsymbol{C}$-type.

§ 2. Functions of direct C^*-type.

1. Functions of C^*-type with one point of indetermination. In this subsection, we determine the rational functions of direct $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2$ with one point of indetermination. By Proposition 3, there is an analytic automorphism T on $\boldsymbol{P}$ such that $T\circ f|_V$ is a rational function of proper $\boldsymbol{C}^*$-type on $V=\boldsymbol{P}^2\setminus(C_1\cup C_2\cup C_3)$, where each C_i ($i=1, 2, 3$) is the closure of the prime curve S_i of f. Let S_a be a prime curve of $\boldsymbol{C}$-type of f with value a different from S_1, S_2, S_3. Since $S_a\cap V$ is a prime curve of $\boldsymbol{C}^*$-type of $f|_V$, the level curve of f with the value a must satisfy the condition (b) (i) in § 1.4 of this chapter. By the classification of rational functions of $\boldsymbol{C}^*$-type in that subsection, at least two of the prime curves S_1, S_2, S_3 are of $\boldsymbol{C}$-type. We suppose that S_1 and S_2 are of $\boldsymbol{C}$-type.

Let t_i be an irreducible homogeneous polynomial defining C_i for each i. The function $T\circ f$ is written as $T\circ f=a_1t_1^{\alpha_1}t_2^{\alpha_2}t_3^{\alpha_3}$ for a non-zero constant a_1 and three integers α_1, α_2, α_3 satisfying $\sum_{i=1}^3\alpha_i\deg t_i=0$. We may suppose that α_1 is positive. Let β_1 and β_2 be coprime positive integers such that $\beta_1\deg t_1=\beta_2\deg t_2$ The rational function $R=t_1^{\beta_1}/t_2^{\beta_2}$ is primitive. By Theorem 1, the restriction $R|_V$ of R to V is of proper direct $\boldsymbol{C}^*$-type. Since R is not of torsional $\boldsymbol{C}^*$-type, the classification in § 1.4 shows that R must be of $\boldsymbol{C}$-type. Each level curve of R with the value different from 0, ∞ is irreducible and intersects C_3 at a point in $\boldsymbol{P}^2\setminus I_f$. Since the restriction $R|_{C_3}$ is a rational function of degree one on C_3, a level curve L of R is of order one and each L intersects C_3 transversally.

Suppose that f belongs to $\mathscr{F}_{\mathrm{II}}$. Then R belongs to $\mathscr{F}_{\mathrm{II}}$. Since R is primitive, R is written as $R=a_2R_{n,k}$ or as $R=a_2(R_{n,k})^{-1}$ in a homogeneous coordinate $(X:Y:Z)$ of $\boldsymbol{P}^2$, where a_2 is a non-zere constant. So $\deg t_1$ and $\deg t_2$ are coprime and $\beta_1=\deg t_2$, $\beta_2=\deg t_1$. Hence there exists a rational function $\psi_{n,k}$ of $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2$ such that the mapping θ defined by $\theta(p)=(R_{n,k}(p), \psi_{n,k}(p))$, $p\in\boldsymbol{P}^2\setminus(C_1\cup C_2)$ is a birational biregular isomorphism of $\boldsymbol{P}^2\setminus(C_1\cup C_2)$ onto $\boldsymbol{C}^*\times\boldsymbol{C}$. Then the equation $\psi_{n,k}=\Psi(R_{n,k})$ on $\boldsymbol{P}^2\setminus(C_1\cup C_2)$ defines C_3 for some rational function $\Psi(z)=P(z)/z^l$, where $P(z)$ is a polynomial in z and l is a non-negative integer. Hence C_3 is a prime curve of order one of the rational function $\psi_{n,k}-\Psi(R_{n,k})$. The locus of poles of $\psi_{n,k}-(R_{n,k})$ is contained in $C_1\cup C_2$. Since $\deg t_1$ and $\deg t_2$ are coprime, there exists an integer p such that S_1 is a prime curve of order α_1 with the value 0 of $f_0 = (R_{n,k})^p\{\psi_{n,k} - \Psi(R_{n,k})\}^{\alpha_3}$. Then the function f is written as $f = a_3 f_0$ for a non-zero constant a_3. Since f is of C^*-type, $a_3 \neq 0$. Since f is primitive, the integers p and α_3 are coprime.

Conversely, in a homogeneous coordinate $(X:Y:Z)$ of P^2, the equation $\psi_{n,k} = \Psi(R_{n,k})$ on $P^2\setminus(\{w_{k-1}=0\}\cup\{v_n=0\})$ defines a prime curve S of order one of the rational function $\psi_{n,k} - \Psi(R_{n,k})$ on P^2 where $\Psi(z) = P(z)/z^l$ and a polynomial $P(z)$ and a non-negative integer l are arbitrary. Since the level curves of $R_{n,k}$ with value 0, ∞ are multiple, the closure $\bar{S}$ does not intersect them. Hence S is of C^*-type. The restriction $R_{n,k}|_V$ of $R_{n,k}$ to $V = P^2\setminus(\{w_{k-1}=0\}\cup\{v_n=0\}\cup S)$ is of proper direct C^*-type. Consider the rational function $f_0 = (R_{n,k})^p\{\psi_{n,k} - \Psi(R_{n,k})\}^q$ for coprime integers p and q $(\neq 0)$. The curve S is a prime curve of C^*-type of f_0 with the value 0. Hence, by Theorem 1, f_0 is direct C^*-type on P^2. The curves $\{w_{k-1}=0\}$ and $\{v_n=0\}$ must be the closures of prime curves of f_0. Hence f_0 belongs to $\mathscr{F}_{\mathrm{II}}$. The restriction $\psi|_L$ of $\psi = \psi_{n,k} - \Psi(R_{n,k})$ to each level curve L of $R_{n,k}$ is a rational function of degree one on L. Hence the mapping θ_1 defined by $\theta_1(p) = (R_{n,k}(p), \psi(p))$, $p \in P^2\setminus(\{w_{k-1}=0\}\cup\{v_n=0\})$ is a birational biregular isomorphism of $P^2\setminus(\{w_{k-1}=0\}\cup\{v_n=0\})$ onto $C^*\times C$. Hence f_0 is primitive. As is seen in Proposition 5, a rational function of C^*-type with two points of indetermination belongs to $\mathscr{F}_{\mathrm{I}}$. Hence we obtain the following theorem announced in Chapter 0, § 2.2.

THEOREM 2. *A primitive rational function f on P^2 is of direct C^*-type and belongs to $\mathscr{F}_{\mathrm{II}}$ if and only if f is represented as $f = T\circ f_0$ where T is an analytic automorphism of P and $f_0 = R^p_{n,k}\{\psi_{n,k} - \Psi(R_{n,k})\}^q$. Here $(R_{n,k}, \psi_{n,k})$ is a pair of rational functions of special type belonging to $\mathscr{F}_{\mathrm{II}}$ given in Chapter 0, § 2, p and q $(\neq 0)$ are coprime integers and Ψ is a rational function $P(z)/z^l$ in one variable z for a polynomial P in z and a non-negative integer l.*

Suppose that f belongs to $\mathscr{F}_{\mathrm{I}}$. Then R belongs to $\mathscr{F}_{\mathrm{I}}$. Let C_2 be a prime curve of degree one. The rational function R defines a regular function T_1 on $C^2 = P^2\setminus C_2$. Let (x, y) be an inhomogeneous coordinate of P^2 with C_2 regarded as the complex line at infinity. Then T_1 is a polynomial function of (x, y). By Jung [1], we know that there exists a polynomial $T_2(x, y)$ such that the transformation $(x', y') = (T_1(x, y), T_2(x, y))$ is an algebraic automorphism of C^2. Hence f is written as $f = a_1 T_1^{\alpha_1}\{T_2 - \Psi(T_1)\}^{\alpha_3}$ for copime $\alpha_1 \in Z^+$ and $\alpha_3 \in Z\setminus\{0\}$, where $\Psi(z) = P(z)/z^l$, $P(z)$ is a polynomial in z and l is a non-negative integer. Since f is primitive, α_1 and α_3 are coprime. The converse is not true. There is a case where $T_1^p\{T_2 - \Psi(T_1)\}^q$ is of C-type.

2. Functions of C^*-type with two points of indetermination. In this subsection, we determine the rational functions of C^*-type with two points of indetermination. We use the following lemma.

LEMMA 7. *Let C be an algebraic curve on $\boldsymbol{P}^2$ whose complement $\boldsymbol{P}^2 \setminus C$ is simply connected. Then C is an irreducible curve of degree one, that is, a complex line.*

PROOF. Suppose that the degree ν of the curve C is not 1. Let L be a curve of degree one on $\boldsymbol{P}^2$. Let $(X:Y:Z)$ be a homogeneous coordinate of $\boldsymbol{P}^2$ such that $L=\{Z=0\}$. Let $P(X, Y, Z)$ be an irreducible homogeneous polynomial of (X, Y, Z) defining C. Consider an analytic function $\xi=\{Z^\nu/P(X, Y, Z)\}^{1/\nu}$. The Riemann domain of ξ over $\boldsymbol{P}^2 \setminus C$ is ν-sheeted and unramified with no relative boundary, which contradicts the assumption that $\boldsymbol{P}^2 \setminus C$ is simply connected. Thus we have our lemma.

Let f be a primitive rational function of C^*-type with two points of indetermination on $\boldsymbol{P}^2$. Let $\sigma: M \to \boldsymbol{P}^2$ be the minimal resolution of indetermination points of f by σ-processes. Let B_1, B_2 be two basic sections of σ^*f. Since σ is minimal, we have $(B_1^2)=(B_2^2)=-1$.

First, we prove that there is at most one irreducible multiple level curve of C^*-type of f. Suppose that f has two irreducible multiple level curves with the values c_1, c_2, respectively. The graphs of level curves σ^*f with the value c_1, c_2 are determined by Lemma 3 (iii) in a way similar to that of Chapter II, § 1.1. Each B_i $(i=1, 2)$ intersects at most two other components of $\Sigma(f)=\sigma^{-1}(I_f)$. Hence a level curve of σ^*f with the value different from c_1, c_2 consists only of proper transforms of prime curves of f under the mapping σ^{-1}. Since the restriction $\sigma^*f|_{B_i}$ of σ^*f to B_i is a rational function of degree one on B_i, f satisfies the condition of Class (A) in Chapter III, § 1.4. Only one level curve L of f consists of two prime curves S_1, S_2 of C-type which intersect at a point in $\boldsymbol{P}^2 \setminus I_f$. Denote by $\tilde{S}_1$, $\tilde{S}_2$ the proper transforms of $\bar{S}_1$, $\bar{S}_2$ under the mapping σ^{-1}, respectively, The curve $\tilde{S}_1 \cup \tilde{S}_2$ is a level curve of σ^*f. We may suppose that $\tilde{S}_i$ intersects B_i. Denote by F_{c_1}, F_{c_2} the level curves of σ^*f with the value c_1, c_2, respectively. Let K_i be the component of F_{c_i} intersecting B_i for each i. Let $\tau: M \to \tau(M)$ be the composite of σ-processes which contracts the curve $\overline{(F_{c_1} \setminus K_1)} \cup \overline{(F_{c_2} \setminus K_2)} \cup \tilde{S}_2$. Then $(\tau(B_1)^2)=(\tau(F_{c_1})^2)=(\tau(F_{c_2})^2)=0$. By Proposition 2 (ii), $\tau(M)$ is biregularly isomorphic to $\boldsymbol{P} \times \boldsymbol{P}$. On the other hand $(\tau(B_2)^2)=1$, which contradicts the fact that the self-intersection number of an algebraic curve on $\boldsymbol{P} \times \boldsymbol{P}$ is even. Hence we have proved our assertion.

The function f has only one reducible level curve L, which consists of two prime curves S_1, S_2. At least one of them is of $\boldsymbol{C}$-type. We suppose that S_1 is of $\boldsymbol{C}$-type. If f has an irreducible multiple level curve of $\boldsymbol{C}^*$-type, denote it by S_3. If f has no irreducible multiple level curve of $\boldsymbol{C}^*$-type, denote by S_3 an arbitrary regular level curve. Let t_i be an irreducible homogeneous polynomial which defines $\bar{S}_i$ ($i = 1, 2, 3$). Consider the rational functions $g_2 = t_2^{\alpha_2}/t_1^{\alpha_1}$, $g_3 = t_3^{\beta_3}/t_1^{\beta_1}$, where coprime positive integers α_1, α_2 satisfy $\alpha_2 \deg t_2 = \alpha_1 \deg t_1$ and coprime positive integers β_1, β_3 satisfy $\beta_3 \deg t_3 = \beta_1 \deg t_1$. Since the restriction $f|_V$ of f to $V = \boldsymbol{P}^2 \setminus (\bar{S}_1 \cup \bar{S}_2 \cup \bar{S}_3)$ is of proper direct $\boldsymbol{C}^*$-type, Theorem 1 shows that the restrictions $g_2|_V$, $g_3|_V$ are of proper direct $\boldsymbol{C}^*$-type. By the classification in § 1.4 in this chapter, g_2 and g_3 are of $\boldsymbol{C}$-type on $\boldsymbol{P}^2$. Since g_2 is of $\boldsymbol{C}$-type on $\boldsymbol{P}^2$, the restriction $g_2|_{S_3}$ of g_2 to S_3 is a rational function of degree one on S_3. Hence each level curve of g_2 intersects S_3 transversally at a point and is of order one. Since g_3 is of $\boldsymbol{C}$-type on $\boldsymbol{P}^2$, the restriction $g_3|_{S_2}$ of g_3 to S_2 is a rational function of degree one on S_2. Hence each level curve of g_3 intersects S_2 transversally at a point and is of order one. We obtain $\alpha_2 = \beta_3 = 1$. Hence the restriction of g_2 to each level curve L of g_3 with a finite value is a rational function of degree one on L. The restriction of g_3 to each level curve L' of g_2 with a finite value is a rational function of degree one on L'. Hence the mapping θ defined by $\theta(p) = (g_2(p), g_3(p))$, $p \in \boldsymbol{P}^2 \setminus S_1$, is a biregular isomorphism of $\boldsymbol{P}^2 \setminus \bar{S}_1$ onto $\boldsymbol{C}^2$. By Lemma 7, S_1 is an algebraic curve of degree one. By Proposition 1, we obtain the following.

PROPOSITION 5. *If a rational function f of $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2$ has two indetermination points, then f belongs to $\mathscr{F}_{\mathrm{I}}$. In an inhomogeneous coordinate (x, y) of $\boldsymbol{P}^2$, f is written as $f = \Lambda(T_1^m/T_2^n)$ for coprime integers m, n, and conversely. Here $T_1(x, y)$, $T_2(x, y)$ are polynomials of x, y such that $(x', y') = (T_1(x, y), T_2(x, y))$ is an algebraic automorphism of $\boldsymbol{C}^2$ and $\Lambda(z)$ is a rational function of z. If both T_1 and T_2 are of degree one, then $(m, n) \neq (1, 1)$.*

3. Transformation group defined by $\boldsymbol{C}^*$. Here we give an alternative proof of the first half of Proposition 5. We also obtain a result on an analytic transcendental automorphism of the complement of an algebraic curve on $\boldsymbol{P}^2$. Let S be a prime curve of $\boldsymbol{C}$-type of f. The restriction $f|_V$ of f to $V = \boldsymbol{P}^2 \setminus \bar{S}$ has only one point p_1 of indetermination. Each level curve of $f|_V$ is irreducible and of $\boldsymbol{C}^*$-type. Let S_a be a level curve of order ν of f with the value a. Let μ_a: $S_a \to \boldsymbol{C}^*$ be an analytic isomorphism of S_a onto $\boldsymbol{C}^* = \{w \mid 0 < |w| < 1\}$ which maps a neighbourhood of

p_1 into a neighbourhood of the origin $w = 0$. For a fixed non-zero complex number c, we consider an analytic automorphism ψ_c^a of C^* defined by $\psi_c^a(w) = c^\nu w$. The transformation $T_c^a = \mu_a^{-1} \circ \psi_c^a \circ \mu_a$ of S_a is independent of choice of μ_a. The mapping T_c of $V \setminus \{p_1\}$ into itself defined by $T_c(p) = T_c^{f(p)}(p)$, $p \in V \setminus \{p_1\}$, is bijective. We prove the following.

LEMMA 8. *The mapping T_c is an analytic automorphism of $V \setminus \{p_1\}$.*

PROOF. To see that T_c is holomorphic in a neighbourhood of S_a, we may suppose that $a = 0$. For a sufficiently small positive number r, the punctured disc $\Gamma_r = \{z \mid z \neq 0, |z| < r\}$ does not contain a critical value. Consider the tube $V_r = \{p \in V \setminus \{p_1\} \mid f(p) \in \Gamma_r\}$. Denote by $\tilde{V}_r$ the domain of existence of the function $(f|_{V_r})^{1/\nu}$. Denote by $\tilde{\omega}\colon \tilde{V}_r \to V_r$ the canonical projection. The set $\tilde{S}_0 = \tilde{\omega}^{-1}(S_0)$ is an irreducible curve on V_r which is a ν-sheeted covering surface of S_0 with no branch point and with no relative boundary. Hence the domain $\tilde{V}_r$ is analytically isomorphic to $\Gamma_r \times C^*$. Since $\tilde{\omega}^{-1} \circ T_c \circ \tilde{\omega}$ defines an analytic automorphism of $\tilde{V}_r \setminus \tilde{S}_0$ whose analytic continuation to $\tilde{V}_r$ is still holomorphic on $\tilde{V}_r$, $T_c|_{V_r}$ is holomorphic on V_r, which proves the lemma.

By putting $T_c(p_1) = p_1$, T_c defines an analytic automorphism of V. Suppose that $|c| > 1$. Let U be a solid sphere with the center p_1. Since $V = \lim_{n\to\infty} T_c^n(U)$ for n-times iteration T_c^n of T_c, V is simply connected. Hence, by Lemma 7, S is a curve of degree one. Thus we have proved the first half of Proposition 5.

We also obtain the following.

COROLLARY TO LEMMA 8. *Let C be an algebraic curve in P^2. Suppose that the complement $P^2 \setminus C$ has a regular rational function of C^*-type any level curve of which is irreducible and of C^*-type. Then $P^2 \setminus C$ has an analytic transcendental automorphism.*

4. Exposition of examples. In this section, we give simple examples of the graphs $\Sigma(f)$ of rational functions of C^*-type with one point of indetermination, which will help the reader to understand the proof of Theorem 2.

Class (C): Figure 53.
Class (D): Figure 54.
Class (E): Figure 55.

5. Class (A). Let f be a primitive rational function of C^*-type on P^2 belonging to the Class A in § 1.4 in this chapter. Let $\sigma\colon M \to P^2$ be the minimal resolution of the indetermination points of f. We determine

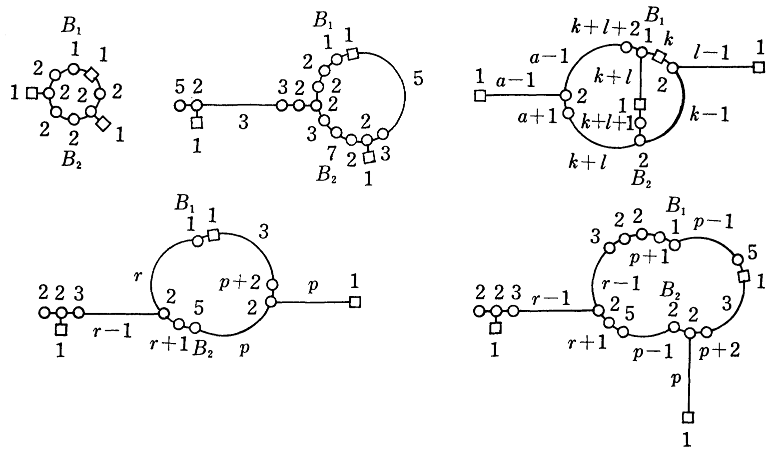


FIGURE 53.

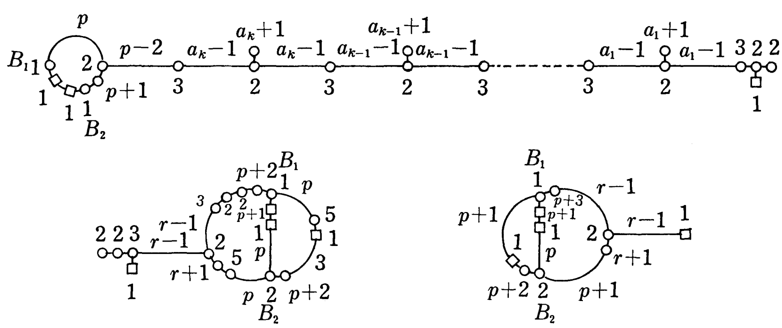


FIGURE 54.

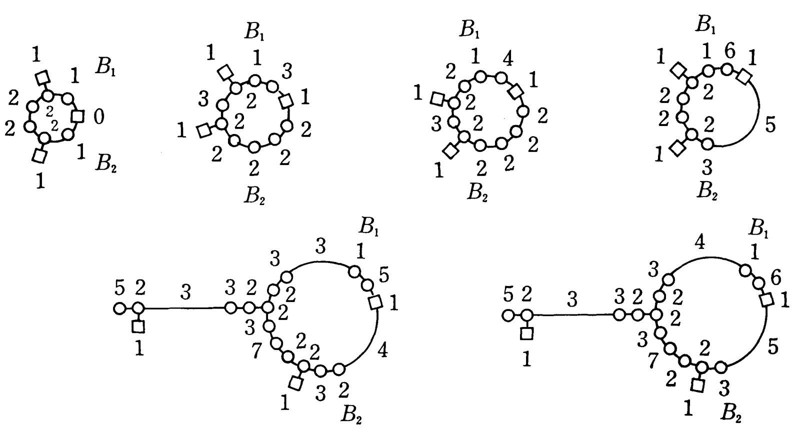


FIGURE 55.

the graph of $\Sigma(f)$ and f in another way, which shows a way to construct functions of $\boldsymbol{C}^*$-type with multiple prime curves of higher order.

We suppose that the level curve of f with the value 0 satisfies the condition (b)(i) in § 1.4. As was seen in § 2.2, we may suppose that each level curve of f with a finite non-zero value is of order one. Denote by S_1, S_2 the two prime curves of f with the value zero and by S_3 the level curve of f with the value ∞. Let $\tilde{S}_i$ be the proper image of $\bar{S}_i$ under the mapping σ^{-1} for each i. Let B_1, B_2 be the basic sections of σ^*f. Then $(B_1^2) = (B_2^2) = -1$. The level curve F_0 of σ^*f with the value 0 intersects each B_i $(i = 1, 2)$ at a simple point of F_0 transversally. The component of F_0 intersecting B_i is of order one. At least one of $\tilde{S}_1$ and $\tilde{S}_2$ is an exceptional curve of the first kind.

(1) The case where $(\tilde{S}_1^2) = (\tilde{S}_2^2) = -1$. Lemma 3(ii) shows $F_0 = \tilde{S}_1 \cup \tilde{S}_0$. We may suppose that $(\tilde{S}_1 \cdot B_1) = (\tilde{S}_2 \cdot B_2) = 1$. Suppose that the level curve F_∞ of σ^*f with the value ∞ is irreducible. Denote by τ the σ-process contracting S_1. Then $(\tau(\tilde{S}_2)^2) = (\tau(B_1)^2) = 0$. Hence, by Proposition 2(i), $\tau(M)$ is biregularly isomorphic to $\boldsymbol{P} \times \boldsymbol{P}$. On the other hand, $(\sigma(B_2)^2) = -1$, a contradiction. Hence F_∞ must te reducible. The graph of F_∞ is that in Lemma 3(iii). By Lemma 2, the graph of $\Sigma(f)$ must be as in Figure 56. From this, we see that S_1 and S_2 are of order one and

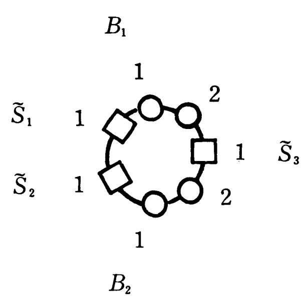


FIGURE 56.

that S_3 is of order two. Since $(\sigma(\tilde{S}_1)^2) = (\sigma(\tilde{S}_2)^2) = (\sigma(\tilde{S}_3)^2) = 1$, the curves S_1, S_2, S_3 are algebraic curves of degree one on $\boldsymbol{P}^2$. Let $(X_1 : X_2 : X_3)$ be a homogeneous coordinate of $\boldsymbol{P}^2$ such that $\bar{S}_i = \{X_i = 0\}$ $(i = 1, 2, 3)$. Under the inhomogeneous coordinate $(x, y) = (X_1/X_3, X_2/X_3)$ of $\boldsymbol{P}^2$, f is written as $f = axy$ for a non-zero constant a.

(2) The case where $(\tilde{S}_1^2) \neq (\tilde{S}_2^2)$. Suppose that $(\tilde{S}_1^2) = -1$. The closure of $F_0 \setminus \tilde{S}_2$ consists of two connected components each of which intersects a basic section of σ^*f. Hence F_0 satisfies the condition in Lemma 3(iii). Therefore the graph of $\Sigma(f)$ is as in Figure 57. By

Lemma 2, we have $q_r = q'_s$. Let τ be the composite of σ-processes which contracts B_1 and B_2. The graph of $\tau(\Sigma(f) \cup \tilde{S}_1 \cup \tilde{S}_2 \cup \tilde{S}_3)$ has the same property as the graph of $\Sigma(f)$ and is shorter than it, from which, using Proposition 2, we obtain another rational function belonging to the Class (A). The loci of zeros and poles of this new function are the same as those of f. Repeating these processes, our case (2) is reduced to the former case (1), Hence f is written as $f = ax^m y^n$ for positive integers m, n and for a non-zero constant a. Since f is primitive, m and n must be coprime.

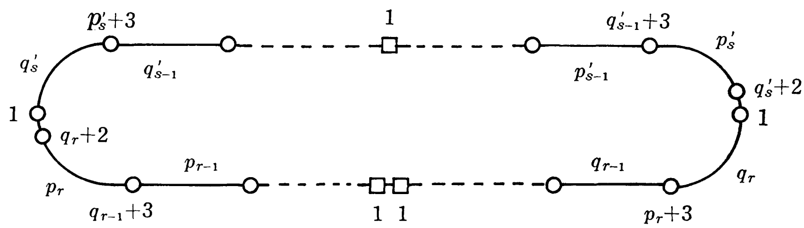


FIGURE 57.

§ 3. Non-existence of a rational function of torsional C^*-type on P^2.

1. Let f be a primitive rational function of torsional C^*-type on P^2. Let S_1, S_2 be two irreducible level curves of C-type of f. The other level curves of f are irreducible and of C^*-type. Let $\sigma: M \to P^2$ be the minimal resolution of the indetermination point p_0 of f by σ-processes. The basic section B of σ^*f satisfies $(B^2) = -1$. The restriction $\sigma^*f|_B$ of σ^*f to B is a rational function of degree two on B. Hence the curve B is a two-sheeted ramified covering surface over P with the projection σ^*f. By Lemma 3(iii), we see that the two ramification points of this covering surface are over the points $f(S_1)$ and $f(S_2)$. Denote by F_i the level curve of σ^*f with value $f(S_i)$ $(i = 1, 2)$ and by S_i the proper image of S_i under the mapping σ^{-1}.

Suppose that each level curve of σ^*f is irreducible. Then $B = \sigma^{-1}(\{p_0\})$ and σ is the blowing-up at p_0. The curve F_1 must be tangent to B with

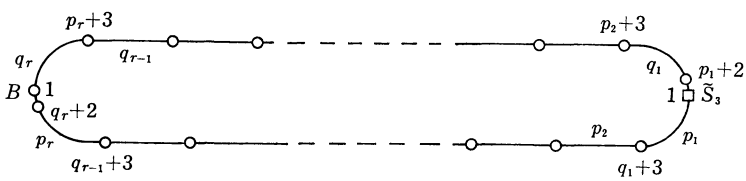


FIGURE 58.

order one. Hence $S_1 = \sigma(F_1)$ must be singular at p_0. Since the multiplicity of S_1 at p_0 is two and $(F_1^2) = 0$, we have $(\bar{S}_1^2) = 4$. This means that S_1 is an algebraic curve of degree two. Hence S_1 has no singular point, a contradiction. Hence at least one level curve of σ^*f must be reducible.

Suppose that a level curve F_3 of σ^*f with value a, different from $f(S_1)$ and $f(S_2)$, is reducible. Since the level curve S_3 of f with value a is of C^*-type, by Lemma 3(iii), it must be multiple. Since B intersects at most two other components of $\Sigma(f)$, each level curve of σ^*f with value different from $f(S_3)$ is irreducible. The graph of $\Sigma(f)$ is as in Figure 58, where $\tilde{S}_3$ denotes the proper image of $\bar{S}_3$ under the mapping σ^{-1}. It means that $\Sigma(f)$ has the property (P), which contradicts the fact that $\Sigma(f)$ is an exceptional curve. Hence each level curve of C^*-type of f must be of order one. The curve $\Sigma(f)$ consists of B, the closure Σ_1 of $F_1 \setminus \tilde{S}_1$ and the closure Σ_2 of $F_2 \setminus \tilde{S}_2$.

2. If $\Sigma_i \neq \emptyset$, then a component K_i of Σ_i intersects B at a point transversally. Suppose that K_i and $\tilde{S}_i$ are prime curves of order one of σ^*f and $\tilde{S}_i$ intersects B at $K_i \cap B$ transversally. By Lemma 3(i), the closure of $F_i \setminus \tilde{S}_i$ is an exceptional curve of the first kind, which contradicts the fact that σ is minimal. Hence K_i is a prime curve of order two of σ^*f and $\tilde{S}_i \cap B = \emptyset$.

Since F_i is reducible, $\tilde{S}_i$ is only one exceptional component of the first kind of F_i. Let $\tau_1\colon M \to \tau_1(M)$ be the σ-process which contracts $\tilde{S}_i$. Since S_i is of C-type, $\tau_1(F_i)$ has only one exceptional component of the first kind. Let

$$M \xrightarrow{\tau_1} M_1 \xrightarrow{\tau_2} \cdots \xrightarrow{\tau_{k-1}} M_{k-1} \xrightarrow{\tau_k} M_k$$

be a sequence of σ-processes τ_j $(j = 1, 2, \cdots, k)$ which contracts a component of $\tau_{j-1} \circ \tau_{j-2} \circ \cdots \circ \tau_2 \circ \tau_1(F_i)$. Set $\tilde{\tau}_j = \tau_j \circ \tau_{j-1} \circ \cdots \circ \tau_2 \circ \tau_1$. For a

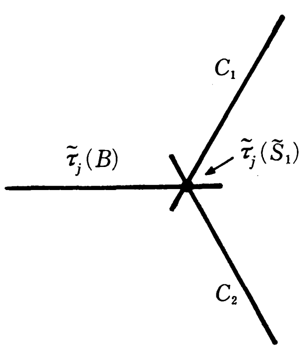


FIGURE 59.

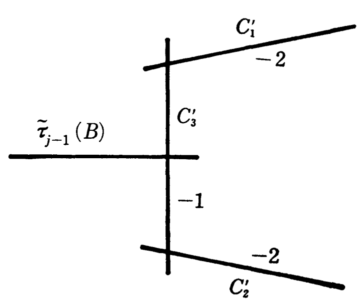


FIGURE 60.

sufficiently large j, the image $\tilde{\tau}_j(F_i)$ contains only components of order one. Since $(\tilde{\tau}_j(B)\cdot\tilde{\tau}_j(F_i))=2$ for each j, there exists j such that $\tilde{\tau}_j(B)$ is smooth and $\tilde{\tau}_j(B)$ intersects two components C_1, C_2 of order one of $\tilde{\tau}_j(F_i)$ transversally at the point $\tilde{\tau}_j(\tilde{S}_i)$. Figure 59 gives a sketch of $\tilde{\tau}_j(\Sigma(f))$. Hence the graph of $B\cup\Sigma_i$ is not linear. Since $\Sigma(f)$ is an exceptional curve of the first kind containing only one irreducible exceptional component of the first kind, at least one of the graphs of $B\cup\Sigma_1$ and $B\cup\Sigma_2$ is linear. Hence at least one of Σ_1 and Σ_2 must be empty. So we may suppose that $\Sigma_2=\varnothing$. By the fact stated in § 3.1, Σ_1 is not empty.

3. We suppose $i=1$ in the former subsection. By Lemma 3(i), we know that $(C_1^2)=(C_2^2)=-1$ and $\tilde{\tau}_j(F_1)=C_1\cup C_2$. Set $C_3'=\tau_j^{-1}(\tilde{\tau}_j(\tilde{S}_1))$. Denote by C_1', C_2' the proper transform of C_1, C_2 under the mapping τ_j^{-1}, respectively. The graph of $\tilde{\tau}_{j-1}(\Sigma(f))$ is as in Figure 60. The image $\tilde{\tau}_{j-1}(\tilde{S}_1)$ must be a point because S_1 is of C-type. Suppose that the point $\tilde{\tau}_{j-1}(\tilde{S}_1)$ is neither $C_1'\cap C_3'$ nor $C_2'\cap C_3'$. Denote by C_1'' the proper transform of C_1 under the mapping $\tilde{\tau}_j^{-1}$. Since C_1'' is a component of F_1 of order one, by Lemma 3(i), the closure of $F_1\setminus C_1''$ is an exceptional curve of the first kind. On the other hand, the proper transform of $C_1'\cup C_2'\cup C_3'$ under the mapping $\tilde{\tau}_{j-1}^{-1}$ is the last branch of $\Sigma(f)$. So F_1 must be exceptional, a contradiction. Hence the point $\tilde{\tau}_{j-1}(\tilde{S}_1)$ is $C_1'\cap C_3'$ or $C_2'\cap C_3'$. We may suppose that $\tilde{\tau}_{j-1}(\tilde{S}_1)=C_2'\cap C_3'$. The graph of $\tilde{\tau}_{j-2}(\Sigma(f))$ is as in Figure 61. Since $\Sigma(f)$ is an exceptional curve of the first kind, three components of $\Sigma(f)$ near B must be as in Figure 62(a). Suppose that the graph of Σ_1 is linear. By Lemma 2, the graph of $\Sigma(f)$ must be as in Figure 62(b). On the other hand, $\tau_1(F_1)=\tau_1(\Sigma_1)$ has the graph in Lemma 3(iii), which

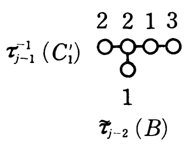
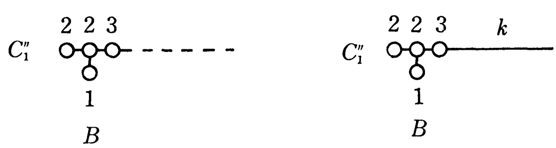


FIGURE 61.



(a) (b)

FIGURE 62.

is impossible. Hence Σ_1 has at least one diverging component. Since F_1 has the property (P), the graph of $\Sigma(f)$ is as in Figure 63(a) or (b).

Suppose that Σ_1 has only one diverging component. Since $\Sigma(f)$ is an exceptional curve of the first kind, the graph of $\Sigma(f)$ is as in Figure 64(a) or (b). The portion in the parenthesis may not exist. This contradicts the fact that F_1 has the property (P). Hence Σ_1 has at least two diverging components. Since F_1 has the property (P), the graph of $\Sigma(f)$ must be as in Figure 65(a) or (b). In Figure 65, the mark labeled $T(p)$ represents the diagram in Figure 66.

Suppose that Σ_1 has two diverging components. Since $\Sigma(f)$ is an exceptional curve of the first kind, the graph of $\Sigma(f)$ is as in Figure 67. This contradicts the fact that F_1 has the property (P). Repeating these

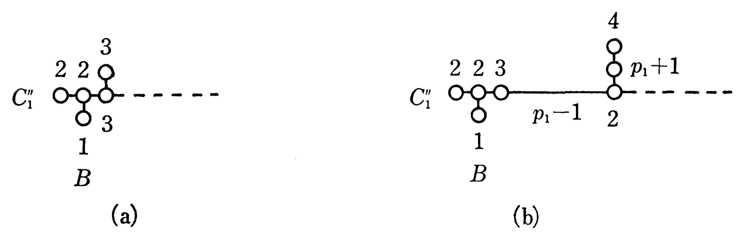


FIGURE 63.

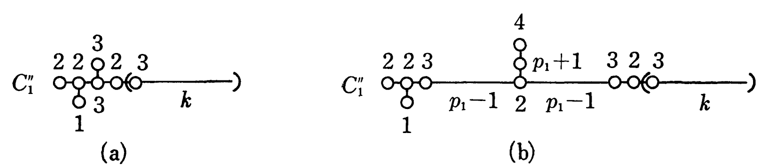


FIGURE 64.

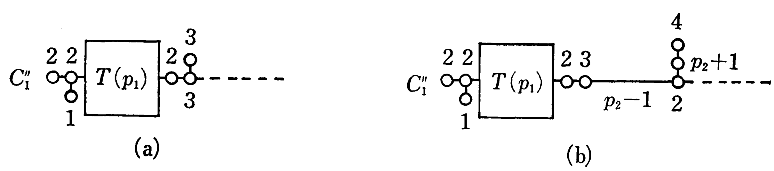


FIGURE 65.

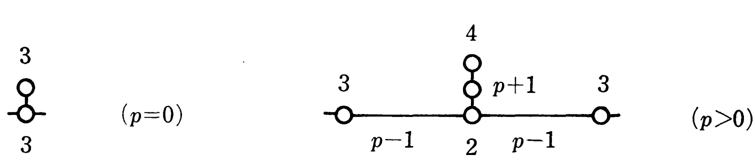


FIGURE 66.

processes, we see that $\Sigma(f)$ must have the graph with infinite length as in Figure 68.

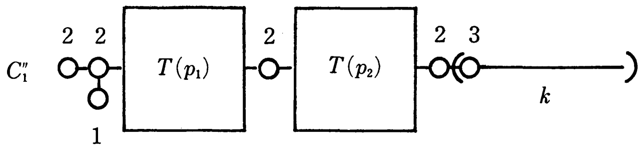


FIGURE 67.

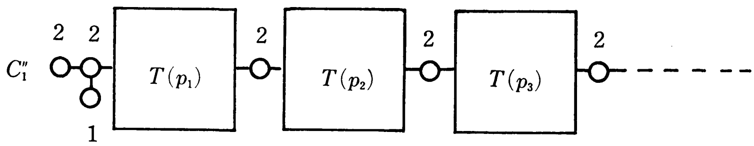


FIGURE 68.

It is a contradiction. Thus we have proved the following theorem.

THEOREM 3. *There exists no rational function of torsional C^*-type on the two-dimensional complex projective space.*

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, 980
JAPAN