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SMALL DEFORMATIONS OF CERTAIN COMPACT MANIFOLDS OF CLASS L

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The notion of a manifold of Class L was introduced by Kato [6]. A manifold of Class L is a complex 3-fold into which there exists an open embedding of a certain domain of $\boldsymbol{P}^3$. The most significant property of Class L is that we can connect any two Class L manifolds complex analytically to obtain another Class L manifold.

We define a complex 3-fold $M = M(1)$ as follows. Let $[\zeta_0: \zeta_1: \zeta_2: \zeta_3]$ be the system of homogeneous coordinates of $\boldsymbol{P}^3$. Put

$$l_0 = \{\zeta_0 = \zeta_1 = 0\},\ l_\infty = \{\zeta_2 = \zeta_3 = 0\} .$$

We denote $\boldsymbol{P}^3 - l_0 - l_\infty$ by W. Let g be a holomorphic automorphism of W sending $[\zeta_0: \zeta_1: \zeta_2: \zeta_3]$ to $[\zeta_0: \zeta_1: \alpha\zeta_2. \alpha\zeta_3]$, where α is a complex number with $0 < |\alpha| < 1$. We define M to be the quotient space of W by $\langle g\rangle$, where $\langle g\rangle$ indicates the infinite cyclic group generated by g. Then M is shown to be a compact manifold of Class L. So we can construct $M(2)$, a new compact manifold of Class L by connecting two copies of M. We construct $M(n)$, $n \in \boldsymbol{N}$, inductively with n copies of M.

The main purpose of this paper is to determine all the small deformations of $M(n)$ for all $n \in \boldsymbol{N}$. The result for M is that any small deformation of M is biholomorphic to $W/\langle g_t\rangle$ where g_t is a holomorphic automorphism of W defined by $g_t([\zeta_0: \zeta_1: \zeta_2: \zeta_3]) = [\zeta_0 + t_1\zeta_1: t_2\zeta_0 + (1 + t_3)\zeta_1: (\alpha + t_4)\zeta_2 + t_5\zeta_3: t_6\zeta_2 + (\alpha + t_7)\zeta_3]$, where t_i $(i = 1, \cdots, 7)$ are complex numbers with $|t_i|$ small enough (Theorem 1). The result for $M(n)$, $n \geqq 2$, is more complicated than that for M. The complete and effectively parametrized complex analytic family of the small deformations of $M(n)$ has $15n - 12$ parameters. The details are stated in Theorems 2 and 3.

This paper consists of three sections.

In § 1, we give some definitions, for instance, the definitions of Class L, that of $M(n)$.

In §2, we investigate small deformations of M.

In §3, we study small deformations of $M(n)$, $n \geqq 2$.

We have the following conjecture;

CONJECTURE. *Let* X_1 and X_2 *be compact manifolds of Class L. Let* $X_1 \# X_2$ *denote any manifold we obtain by connecting* X_1 *and* X_2 *complex analytically. Then we have*

$$\dim H^2(X_1 \# X_2, \Theta) = \dim H^2(X_1, \Theta) + \dim H^2(X_2, \Theta) .$$

The author wrote this statement as Proposition in [10] but the proof contained a gap. The conjecture is ture if X_1 is $M(n)$, X_2 is M, and $X_1 \# X_2$ is $M(n+1)$ for any $n \geqq 1$.

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§ 1. Definitions.

1. The definition of Class L. For any positive real number r, we define a domain U_r in $\boldsymbol{P}^3$ as

$$U_r = \{[\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3] \in \boldsymbol{P}^3 ; |\zeta_0|^2 + |\zeta_1|^2 < r(|\zeta_2|^2 + |\zeta_3|^2)\} .$$

DEFINITION 1.1 ([6, p.1, Definition 1.1]). Let X be a complex manifold of dimension 3. X is said to be of *Class L* if X contains a domain biholomorphic to U_1, in other words, if there exists a holomorphic open embedding of U_1 into X.

To define the connecting operation of two Class L manifolds, we need a holomorphic automorphism σ of $\boldsymbol{P}^3$ defined by

$$\sigma([\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3]) = [\zeta_2 : \zeta_3 : \zeta_0 : \zeta_1] .$$

For any real number ε greater than 1, we define a domain $N(\varepsilon)$ in $\boldsymbol{P}^3$ by

$$N(\varepsilon) = U_\varepsilon - \overline{U_{1/\varepsilon}}$$

where $\overline{}$ indicates the closure. Then the following is clear.

LEMMA 1.2. (i) *For any positive real number r, U_r is biholomorphic to U_1.*

(ii) $\sigma(N(\varepsilon)) = N(\varepsilon)$.

Let X be a manifold of Class L. Then from Definition 1.1 and Lemma 1.2 there exists a holomorphic open embedding of U_ε into X.

DEFINITION 1.3 ([6, p. 3]). Let X_1 and X_2 be manifolds of Class L and

$$i_\nu : U_\varepsilon \to X_\nu, \ \nu = 1, 2$$

be holomorphic open embeddings. Writing $X_\nu - \overline{i_\nu(U_{1/\varepsilon})}$ as $X_\nu^\#(\nu = 1, 2)$, we define a complex manifold $Z(X_1, X_2, i_1, i_2) = X_1^\# \cup X_2^\#$ by identifying a point $x_1 \in i_1(N(\varepsilon)) \subset X_1^\#$ with the point $i_2 \circ \sigma \circ i_1^{-1}(x_1) \in X_2^\#$.

REMARK 1. The complex structure of $Z(X_1, X_2, i_1, i_2)$ depends on the open embeddings i_1 and i_2. We shall see this fact later in §3.2.

REMARK 2. If X_1 and X_2 are compact, then $X_1 \# X_2$ is also compact.

LEMMA 1.4. *$N(\varepsilon)$ is of Class L.*

PROOF. For a real number λ, we define a holomorphic open embedding τ of U_ε into $\boldsymbol{P}^3$ by

$$\tau([\zeta_0\colon \zeta_1\colon \zeta_2\colon \zeta_3]) = [\zeta_0 + \lambda\zeta_2\colon \zeta_1 + \lambda\zeta_3\colon \lambda\zeta_2 - \zeta_0\colon \lambda\zeta_3 - \zeta_1] .$$

Since $\zeta_0 \neq 0$ or $\zeta_1 \neq 0$ in U_ε, we have

$$U_\varepsilon = (U_\varepsilon \cap \{\zeta_0 \neq 0\}) \cup (U_\varepsilon \cap \{\zeta_1 \neq 0\}) .$$

Taking a system of local coordinates

$$(x_0, y_0, z_0) = (\zeta_1/\zeta_0, \zeta_2/\zeta_0, \zeta_3/\zeta_0) \quad \text{in} \quad U_\varepsilon \cap \{\zeta_0 \neq 0\}$$

and

$$(x_1, y_1, z_1) = (\zeta_0/\zeta_1, \zeta_2/\zeta_1, \zeta_3/\zeta_1) \quad \text{in} \quad U_\varepsilon \cap \{\zeta_1 \neq 0\} ,$$

we let

$$V_i = \{(x_i, y_i, z_i) \in U_\varepsilon \cap \{\zeta_i \neq 0\};\ |x_i| < 2\} \quad \text{for} \quad i = 0, 1 .$$

Then it is clear that $\{V_0, V_1\}$ is also an open covering of U_ε. Since $|x_i| < 2$ $(i = 0, 1)$, we have

$$\begin{aligned}(|y_0 - 1/\lambda|^2 + |z_0 - x_0/\lambda|^2)/\varepsilon &< |1/\lambda + y_0|^2 + |x_0/\lambda + z_0|^2 \\ &< \varepsilon(|y_0 - 1/\lambda|^2 + |z_0 - x_0/\lambda|^2) , \\ (|y_1 - x_1/\lambda|^2 + |z_1 - 1/\lambda|^2)/\varepsilon &< |x_1/\lambda + y_1|^2 + |1/\lambda + z_1|^2 \\ &< \varepsilon(|y_1 - x_1/\lambda|^2 + |z_1 - 1/\lambda|^2)\end{aligned}$$

when we take λ large enough. Thus we get $\tau(U_\varepsilon) \subset N(\varepsilon)$. □

LEMMA 1.5. *$X_1 \# X_2$ is of Class L.*

PROOF. $X_1 \# X_2$ contains a domain biholomorphic to $N(\varepsilon)$ which is of Class L. Hence $X_1 \# X_2$ is of Class L. □

2. Definition of the manifolds $M(n)$. Here we are going to define compact complex manifolds $M(n)$ of which we shall study the small deformations later. We have already defined l_0, l_∞, W and g in Introduction. Then we have:

PROPOSITION 1.6. *$\langle g \rangle$ acts on W properly discontinuously without fixed points.*

PROOF. It is easy to see that $\langle g\rangle$ acts on W without fixed points. We show that $\langle g\rangle$ acts properly discontinuouly on W. Let μ be a real number larger than 1 and ν a natural number such that $|\alpha|^{\nu} < 1/\mu$. Then for any integer n with $n \geqq \nu$, we have

$$g^n(N(\mu)) \cap N(\mu) = \emptyset .$$

Since any compact subsets K_1 and K_2 of W are contained in $N(\mu)$ for a suitable real number μ and since we can take a natural number ν for μ so that the above equality holds, we have

$$\sharp\{n \in \boldsymbol{Z};\ g^n(K_1) \cap K_2 \neq \emptyset\} < 2\nu . \qquad \square$$

DEFINITION 1.7. Let W and $\langle g\rangle$ be as above. We define a complex manifold $M = M(1)$ as the quotient space of W by $\langle g\rangle$, i.e.

$$M = W/\langle g\rangle .$$

REMARK 1. M is compact because M is the image of compact $N(\mu)$ for μ sufficiently large.

REMARK 2. M is diffeomorphic to $S^1 \times S^2 \times S^3$ where S^n is the standard n-sphere.

Taking real numbers β, γ, δ such that $|\alpha| < \beta < \gamma < \delta < 1$, we define domains U_0, U_W, U_∞ in W as follows:

$$U_0 = \{\zeta \in W;\ |\alpha|(|\zeta_2|^2 + |\zeta_3|^2) < |\zeta_0|^2 + |\zeta_1|^2 < \delta(|\zeta_2|^2 + |\zeta_3|^2)\} ,$$
$$U_W = \{\zeta \in W;\ \gamma(|\zeta_2|^2 + |\zeta_3|^2) < |\zeta_0|^2 + |\zeta_1|^2 < (|\zeta_2|^2 + |\zeta_3|^2)/\gamma\} ,$$
$$U_\infty = \{\zeta \in W;\ (|\zeta_2|^2 + |\zeta_3|^2)/\delta < |\zeta_0|^2 + |\zeta_1|^2 < \beta(|\zeta_2|^2 + |\zeta_3|^2)/|\alpha|^2\} .$$

By the definition of U_0, U_W, and U_∞, we have

$$gU_0 \cap U_\infty \neq \emptyset , \quad gU_W \cap U_\infty = \emptyset .$$

This shows that M is a manifold we obtain by identifying $\zeta \in gU_0 \cap U_\infty$ with $g^{-1}(\zeta) \in U_0 \cap g^{-1}U_\infty$ in $U_0 \cup U_W \cup U_\infty$.

PROPOSITION 1.8. *M is of Class L.*

PROOF. Let π be the natural projection of W to M. Since M contains a domain $\pi(U_W)$ which is biholomorphic to $U_W = N(1/\gamma)$, the proposition is clear. $\square$

We construct $M(n)$ with n copies of M. We denote by M^j the j-th copy of M. By Lemma 1.4 and Proposition 1.8, we have a holomorphic open embedding $\iota = \pi \circ \tau$ of U_ε into M, where τ is a map defined in the proof of Lemma 1.4. We denote by ι^j the holomorphic open embedding ι of U_ε into M^j. We define $M(2)$ by $Z(M^1, M^2, \iota^1, \iota^2)$. We define Class L manifolds

$M(n)$ for $n \geqq 3$ inductively. Suppose that $M(n)$ is defined to be $Z(M(n-1), M^n, \iota_{n-1}, \iota^n)$ with a holomorphic open embedding ι_{n-1} of U_ε into $M(n-1)$, then we define

$$M(n+1) = Z(M(n), M^{n+1}, \iota^n|_{N(\varepsilon)}\circ\tau, \iota^{n+1}) = Z(M(n), M^{n+1}, \iota_{n-1}|_{N(\varepsilon)}\circ\sigma\circ\tau, \iota^{n+1})$$

where $\iota^n|_{N(\varepsilon)}$ (resp. $\iota_{n-1}|_{N(\varepsilon)}$) is the restriction of ι^n (resp. ι_{n-1}) to $N(\varepsilon)$.

§ 2. Small deformations of M.

1. Cohomology groups of M. Let $W_{\eta\eta'}$ be a domain in W defined by

$$W_{\eta\eta'} = \{\eta(|\zeta_2|^2 + |\zeta_3|^2) < |\zeta_0|^2 + |\zeta_1|^2 < \eta'(|\zeta_2|^2 + |\zeta_3|^2)\}\ ,$$

where η and η' are real number such that $\eta' > \eta > 0$. Since the line $l_0 = \{\zeta_0 = \zeta_1 = 0\}$ does not intersect W, we can cover $W_{\eta\eta'}$ by the two domains $W_{\eta\eta'} \cap \{\zeta_0 \neq 0\}$ and $W_{\eta\eta'} \cap \{\zeta_1 \neq 0\}$ whose system of local coordinates are

$$x_0 = \zeta_1/\zeta_0,\ y_0 = \zeta_2/\zeta_0,\ z_0 = \zeta_3/\zeta_0 \quad \text{in} \quad W_{\eta\eta'} \cap \{\zeta_0 \neq 0\}$$

and

$$x_1 = \zeta_0/\zeta_1,\ y_1 = \zeta_2/\zeta_1,\ z_0 = \zeta_3/\zeta_1 \quad \text{in} \quad W_{\eta\eta'} \cap \{\zeta_1 \neq 0\}$$

We remark that these two domains are Reinhardt domains on which every holomorphic function can be expanded as a unique Laurent series with respect to the system of local coordinates (x_i, y_i, z_i), $i = 0, 1$. Moreover the two domains intersect hyperplanes $\{x_0 = 0\}$, $\{y_0 = 0\}$, $\{z_0 = 0\}$ and $\{x_1 = 0\}$, $\{y_1 = 0\}$, $\{z_1 = 0\}$ respectively, so every holomorphic function on each of the two domains admits a unique Taylor series expansion.

LEMMA 2.1. *Let Θ be the tangent sheaf. An element of $H^0(W_{\eta\eta'}, \Theta)$ (resp. $H^0(W, \Theta)$, resp. $H^0(\boldsymbol{P}^3, \Theta)$) is expressed on $W_{\eta\eta'} \cap \{\zeta_0 \neq 0\}$ (resp. $W \cap \{\zeta_0 \neq 0\}$, resp. $\boldsymbol{P}^3 \cap \{\zeta_0 \neq 0\}$) as follows:*

$$\begin{aligned}(a_1 + b_1x_0 &+ c_1y_0 + d_1z_0 + ex_0^2 + fx_0y_0 + gx_0z_0)\partial/\partial x_0 \\ &+ (a_2 + b_2x_0 + c_2y_0 + d_2z_0 + ex_0y_0 + fy_0^2 + gy_0z_0)\partial/\partial y_0 \\ &+ (a_3 + b_3x_0 + c_3y_0 + d_3z_0 + ex_0z_0 + fy_0z_0 + gz_0^2)\partial/\partial z_0\ ,\end{aligned}$$

where a_i, b_i, c_i, d_i, e, f, g are complex numbers for $i = 1, 2, 3$. Conversely, a vector field on $W_{\eta\eta'} \cap \{\zeta_0 \neq 0\}$ (resp. $W \cap \{\zeta_0 \neq 0\}$, resp. $\boldsymbol{P}^3 \cap \{\zeta_0 \neq 0\}$) of the above type is extended to an element of $H^0(W_{\eta\eta'}, \Theta)$ (resp. $H^0(W, \Theta)$, resp. $H^0(\boldsymbol{P}^3, \Theta)$)

PROOF. From the above remark, an element θ of $H^0(W_{\eta\eta'}, \Theta)$ is expressed in $W_{\eta\eta'} \cap \{\zeta_i \neq 0\}$ as

$$\theta = \sum_{l,m,n\geq 0} a^i_{lmn} x^l_i y^m_i z^n_i \partial/\partial x_i + \sum_{l,m,n\geq 0} b^i_{lmn} x^l_i y^m_i z^n_i \partial/\partial y_i + \sum_{l,m,n\geq 0} c^i_{lmn} x^l_i y^m_i z^n_i \partial/\partial z_i$$

where a^i_{lmn}, b^i_{lmn}, c^i_{lmn} are complex numbers for $i = 0, 1$. Since the two expressions above must coincide in $W_{\eta\eta'} \cap \{\zeta_0 \neq 0, \zeta_1 \neq 0\}$, we have the lemma for $W_{\eta\eta'}$ by the uniqueness of the Laurent expansion. The calculations for W and $\boldsymbol{P}^3$ are completely the same as those for $W_{\eta\eta'}$. The converse is clear. □

PROPOSITION 2.2. *An element of $H^0(M, \Theta)$ is identified with an element of $H^0(W \cap \{\zeta_0 \neq 0\}, \Theta)$ of the form*

$$\begin{aligned}&(a_1 + b_1 x_0 + c x_0^2)\partial/\partial x_0 + (a_2 y_0 + b_2 z_0 + c x_0 y_0)\partial/\partial y_0\\ &\quad + (a_3 y_0 + b_3 z_0 + c x_0 z_0)\partial/\partial z_0,\ a_i,\ b_i,\ c \in \boldsymbol{C} \quad (i = 1, 2, 3)\ .\end{aligned}$$

In particular, $\dim H^0(M, \Theta) = 7$.

PROOF. It is easy to see that an element of $H^0(M, \Theta)$ is identified with an element of $H^0(W, \Theta)$ which is invariant under the action of $\langle g\rangle$, i.e., with $\theta \in H^0(W, \Theta)$ such that

$$(g^n)_{*p}\theta_p = \theta_{g^n(p)} \quad \text{for any} \quad n \in \boldsymbol{Z} \quad \text{and any} \quad p \in W\ .$$

Assume that θ is the one mentioned in Lemma 2.1. Then

$$\begin{aligned}&g_*\theta|_{W\cap\{\zeta_0\neq 0\}}\\ &= \left(a_1 + b_1 x_0 + \frac{1}{\alpha}c_1 y_0 + \frac{1}{\alpha}d_1 z_0 + e x_0^2 + \frac{1}{\alpha}f x_0 y_0 + \frac{1}{\alpha}g x_0 z_0\right)\frac{\partial}{\partial x_0}\\ &\quad + \left(a_2 + b_2 x_0 + \frac{1}{\alpha}c_2 y_0 + \frac{1}{\alpha}d_2 z_0 + e x_0 y_0 + \frac{1}{\alpha^2}f y_0^2 + \frac{1}{\alpha^2}g y_0 z_0\right)\alpha\frac{\partial}{\partial y_0}\\ &\quad + \left(a_3 + b_3 x_0 + \frac{1}{\alpha}c_3 y_0 + \frac{1}{\alpha}d_3 z_0 + e x_0 z_0 + \frac{1}{\alpha^2}f y_0 z_0 + \frac{1}{\alpha^2}g z_0^2\right)\alpha\frac{\partial}{\partial z_0}\ .\end{aligned}$$

From the above equation it is obvious that the condition $(g^n)_{*p}\theta_p = \theta_{g^n(p)}$ is equivalent to

$$c_1 = d_1 = a_2 = b_2 = a_3 = b_3 = f = g = 0\ . \qquad □$$

PROPOSITION 2.3. $H^3(M, \Theta) = 0$.

PROOF. By the Kodaira-Serre duality, we have

$$H^3(M, \Theta) = H^0(M, \Omega^1 \otimes \Omega^3)$$

where Ω^p is the sheaf of germs of holomorphic p-forms. By [6, p. 7, Proposition 2.3]: we have

$$H^0(X, (\Omega^1)^{\otimes m_1} \otimes (\Omega^2)^{\otimes m_2} \otimes (\Omega^3)^{\otimes m_3}) = 0$$

for a Class L manifold X if m_1, m_2, m_3 are non-negative integers such that

$m_1 + m_2 + m_3 > 0$. We conclude that $H^3(M, \Theta) = 0$. □

PROPOSITION 2.4. $H^2(M, \Theta) = 0$.

Again by the Kodaira-Serre duality,

$$H^2(M, \Theta) = H^1(M, \Omega^1 \otimes \Omega^3) .$$

We shall show here that $H^1(M, \Omega^1 \otimes \Omega^3) = 0$. For that purpose, we first have:

PROPOSITION 2.5. *M is a holomorphic fibre bundle over $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ with elliptic curves as fibres.*

PROOF. Let $\tilde{p}: W \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be the holomorphic map sending $[\zeta_0: \zeta_1: \zeta_2: \zeta_3]$ to $([\zeta_0: \zeta_1], [\zeta_2: \zeta_3])$. Then it is easy to see that $(W, \boldsymbol{P}^1 \times \boldsymbol{P}^1, \tilde{p})$ becomes a holomorphic fibre bundle with $\boldsymbol{C}^* = \boldsymbol{C} - \{0\}$ as the fibres. Since $\tilde{p}(g(\zeta)) = \tilde{p}(\zeta)$ for any $\zeta \in W$, $\tilde{p}$ induces a map $p: M \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$. The action of $\langle g \rangle$ on W induces the action of $\langle \alpha \rangle$ on $\boldsymbol{C}^*$, the fibre of $(W, \boldsymbol{P}^1 \times \boldsymbol{P}^1, \tilde{p})$. This means that the fibre of $(M, \boldsymbol{P}^1 \times \boldsymbol{P}^1, p)$ is $\boldsymbol{C}^*/\langle \alpha \rangle$, which is an elliptic curve. □

From now on, we write S instead of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ for simplicity and sometimes write Ω^1_M, Ω^1_S and so on to avoid confusion. Now we begin to calculate $H^1(M, \Omega^1_M \otimes \Omega^3_M)$. By the Leray spectral sequence, we have

$$0 \to E_2^{1,0} \to H^1(M, \Omega^1_M \otimes \Omega^3_M) \to E_3^{0,1} \to 0$$

where $E_2^{q,r} = H^q(S, R^r p_*(\Omega^1_M \otimes \Omega^3_M))$ and $E_3^{0,1} = \mathrm{Ker}(E_2^{0,1} \to E_2^{2,0}) \subset E^{0,1}$. Hence we have an exact sequence

$$0 \to E_2^{1,0} \to H^1(M, \Omega^1_M \otimes \Omega^3_M) \to E_2^{0,1} .$$

We need now to calculate $E_2^{1,0}$ and $E_2^{0,1}$.

LEMMA 2.6.

$$R^i p_* \mathcal{O}_M = \begin{cases} \mathcal{O}_s, & i = 0, 1 , \\ 0, & i \geqq 2 . \end{cases}$$

PROOF. (i) $i = 0$.

$$(R^0 p_* \mathcal{O}_M)_{(x,y)} = H^0(p^{-1}(x, y), \mathcal{O}_M) \quad \text{for any} \quad (x, y) \in S .$$

Since $p^{-1}(x, y)$ is an elliptic curve, which is compact, we see $(R^0 p_* \mathcal{O}_M)_{(x,y)} = \mathcal{O}_{S,(x,y)}$. From this we get $R^0 p_* \mathcal{O}_M = \mathcal{O}_S$.

(ii) $i = 1$. $R^1 q_* \mathcal{O}_M$ is a line bundle because

$$\dim H^1(p^{-1}(x, y), \mathcal{O}_M) = \dim H^1(\boldsymbol{C}^*/\langle \alpha \rangle, \mathcal{O}) = 1$$

for any $(x, y) \in S$ [1, p. 151, Theoreme 4.12. (ii)]. From the cohomology exact sequence

$$0 = H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S^*) \to H^2(S, \boldsymbol{Z}) ,$$

it is sufficient to prove that the restriction of $R^1p_*\mathcal{O}_M$ to $\{0\}\times\boldsymbol{P}^1$ and $\boldsymbol{P}^1\times\{0\}$ are trivial because $R^1p_*\mathcal{O}_M$ is a line bundle and because $\{0\}\times\boldsymbol{P}^1$ and $\boldsymbol{P}^1\times\{0\}$ generate $H^2(S, \boldsymbol{Z})$. This follows from the fact that $p^{-1}(\{0\}\times\boldsymbol{P}^1)$ and $p^{-1}(\boldsymbol{P}^1\times\{0\})$ are elliptic bundles with vanishing Chern numbers, by Kodaira [9, p. 772, Theorem 12].

(iii) $i \geqq 2$. It is clear because the fibre is 1-dimensional. □

LEMMA 2.7.

$$R^ip_*(p_*\Omega_S^1) = \begin{cases} \Omega_S^1, & i = 0, 1 , \\ 0, & i \geqq 2 . \end{cases}$$

PROOF. Since $\mathcal{O}_M$ satisfies the condition (b) of [1, p. 149, Theoreme 4.10], we get

$$R^ip_*(p^*\Omega_S^1) = R^ip_*(p^*\Omega_S^1 \otimes \mathcal{O}_M) = \Omega_S^1 \otimes R^ip_*\mathcal{O}_M .$$

The lemma follows from Lemma 2.6. □

LEMMA 2.8. *We have an exact sequence*

$$0 \to p^*\Omega_S^1 \to \Omega_M^1 \to \mathcal{O}_M \to 0 .$$

PROOF. In the following, we denote by ξ the fibre coordinate, induced by the coordinate of $\boldsymbol{C}^*$. An element on the stalk of $P^*\Omega^1$ at (x, y, ξ) has the form

$$\sum_{\lambda=1}^{m} (f_\lambda dx + g_\lambda dy) \otimes h_\lambda \quad \text{for} \quad f_\lambda, g_\lambda \in \mathcal{O}_{S,(x,y)}, h_\lambda \in \mathcal{O}_{M,(x,y,\xi)} .$$

Let $\alpha_{(x,y,\xi)}: p^*\Omega^1_{S,(x,y,\xi)} \to \Omega^1_{M,(x,y,\xi)}$ be the module homomorphism sending $\sum_{\lambda=1}^m (f_\lambda dx + g_\lambda dy) \otimes h_\lambda$ to $\sum_{\lambda=1}^m (f_\lambda h_\lambda dx + g_\lambda h_\lambda dy)$ and $\beta_{(x,y,\xi)}: \Omega^1_{S,(x,y,\xi)} \to \mathcal{O}_{M,(x,y,\xi)}$ be the module homomorphism sending $fdx + gdy + hd\xi/\xi$ to h, where f, g and h are elements of $\mathcal{O}_{M,(x,y,\xi)}$. It is easy to see that $\alpha_{(x,y,\xi)}$ and $\beta_{(x,y,\xi)}$ are defined independently of the choice of the local coordinates. It is obvious that there exist sheaf homomorphisms $\alpha: p^*\Omega_M^1 \to \Omega_M^1$ (resp. $\beta: \Omega_M^1 \to \mathcal{O}_M$) whose restrictions to the stalk on (x, y, ξ) are $\alpha_{(x,y,\xi)}$ (resp. $\beta_{(x,y,\xi)}$). Thus we have the exact sequence

$$0 \to p^*\Omega_M^1 \xrightarrow{\alpha} \Omega_M^1 \xrightarrow{\beta} \mathcal{O}_M \to 0 . \qquad \square$$

LEMMA 2.9. *We have an exact sequence*

$$0 \to \Omega_S^1 \to R^0p_*\Omega_M^1 \to \mathcal{O}_S \to 0 .$$

PROOF. Since $(R^0p_*\Omega_M^1)_{(x,y)}$, the stalk of $R^0p_*\Omega_M^1$ at (x, y), is isomorphic to $H^0(p^{-1}(x, y), \Omega_M^1)$,

$$(R^0 p_* \Omega^1_M)_{(x,y)} = \left\{ \begin{matrix} \phi_1(x, y)dx + \phi_2(x, y)dy + \phi_3(x, y)d\xi/\xi; \\ \phi_i(x, y) \in \mathcal{O}_{S,(x,y)} \qquad (i = 1, 2, 3) \end{matrix} \right\} .$$

Let $\alpha'_{(x,y)}\colon \Omega^1_{S(x,y)} \to (R^0 p_* \Omega^1_M)_{(x,y)}$ be the map sending $\phi_1(x, y)dx + \phi_2(x, y)dy$ to $\phi_1(x, y)dx + \phi_2(x, y)dy$ and $\beta'_{(x,y)}\colon (R^0 p_* \Omega^1_M)_{(x,y)} \to \mathcal{O}_{S,(x,y)}$ be the map sending $\phi_1(x, y)dx + \phi_2(x, y)dy + \phi_3(x, y)d\xi/\xi$ to $\phi_3(x, y)$. It is easily checked that $\alpha'_{(x,y)}$ and $\beta'_{(x,y)}$ are well-defined. It is clear that there exist sheaf homomorphisms $\alpha'\colon \Omega^1_S \to R^0 p_* \Omega^1_M$ and $\beta'\colon R^0 p_* \Omega^1_M \to \mathcal{O}_S$ such that the restriction to the stalk at (x, y) of each homomorphism coincides with $\alpha'_{(x,y)}$, $\beta'_{(x,y)}$, respectively. Thus we have the exact sequence

$$0 \to \Omega^1_M \xrightarrow{\alpha'} R^0 p_* \Omega^1_M \xrightarrow{\beta'} \mathcal{O}_S \to 0 . \qquad \square$$

LEMMA 2.10. *We have an exact sequence*

$$0 \to \Omega^1_S \to R^1 p_* \Omega^1_M \to \mathcal{O}_S \to 0 .$$

PROOF. By Lemma 2.6 and Lemma 2.7, the long exact sequence arising from the short exact sequence in Lemma 2.8 reduces to

$$\begin{aligned} 0 \to \Omega^1_S \to R^0 p_* \Omega^1_M \to \mathcal{O}_S \\ \to \Omega^1_S \to R^1 p_* \Omega^1_M \to \mathcal{O}_S \to 0 . \end{aligned}$$

By Lemma 2.9, the lemma follows. $\square$

LEMMA 2.11. $p^* \Omega^2_S \cong \Omega^3_M$.

PROOF. $\Omega^3_{M,(x,y,\xi)}$, the stalk of Ω^3_M at (x, y, ξ), consists of elements

$$\phi(x, y, \xi)dx \wedge dy \wedge d\xi/\xi \quad \text{for} \quad \phi(x, y, \xi) \in \mathcal{O}_{M,(x,y,\xi)} .$$

On the other hand,

$$p^* \Omega^2_S = p^{-1} \Omega^2_S \underset{p^{-1}\mathcal{O}_S}{\otimes} \mathcal{O}_M$$

by definition, so $p^* \Omega^2_{S,(x,y,\xi)}$ consists of the elements

$$\sum_{\lambda=1}^{m} \psi_\lambda(x, y)dx \wedge dy \otimes f_\lambda(x, y, \xi)$$

with $\psi_\lambda(x, y) \in \mathcal{O}_{S,(x,y)}$, $f_\lambda(x, y, \xi) \in \mathcal{O}_{M,(x,y,\xi)}$. There exists a sheaf homomorphism α'' (resp. β'') of Ω^3_M (resp. $p^*\Omega^2_S$) to $p^*\Omega^2_S$ (resp. Ω^3_M) which sends $\phi(x, y, \xi)dx \wedge dy \wedge d\xi/\xi$ (resp. $\sum_{\lambda=1}^m \psi_\lambda(x, y)dx \wedge dy \otimes f_\lambda(x, y, \xi)$) to $dx \wedge dy \otimes \phi(x, y, \xi)$ (resp. $\sum_{\lambda=1}^m \psi_\lambda(x, y) f_\lambda(x, y, \xi)dx \wedge dy \wedge d\xi/\xi$) on the stalk at (x, y, ξ). Then it is easy to see $\alpha'' \circ \beta'' = \mathrm{id}_{p^*\Omega^2}$, $\beta'' \circ \alpha'' = \mathrm{id}_{\Omega^3}$. $\square$

LEMMA 2.12. $R^i p_* \Omega^1_M \otimes \Omega^2_S \cong R^i p_*(\Omega^1_M \otimes \Omega^3_M)$, $i = 0, 1$.

PROOF. By Lemma 2.11,

$$R^i p_*(\Omega_M^1 \otimes \Omega_M^3) \cong R^i p_*(\Omega_M^1 \otimes p^*\Omega_S^2) .$$

Since Ω_M^1 has the property (b) of [1, p. 149, Theoreme 4.10],

$$R^i p_*(\Omega_M^1 \otimes p^*\Omega_S^2) \cong R^i p_*\Omega_M^1 \otimes \Omega_S^2 . \qquad \square$$

LEMMA 2.13. $H^1(S, R^0p_*(\Omega_M^1 \otimes \Omega_M^3)) = 0$.

PROOF. Tensoring Ω_S^2 with the exact sequence of Lemma 2.9, we have an exact sequence

$$0 \to \Omega_S^1 \otimes \Omega_S^2 \to R^0p_*\Omega_M^1 \otimes \Omega_S^2 \to \mathcal{O}_S \otimes \Omega_S^2 \to 0$$

because Ω_S^2 is locally free. By Lemma 2.12, the above sequence changes into an exact sequence

$$0 \to \Omega_S^1 \otimes \Omega_S^2 \to R^0p_*(\Omega_M^1 \otimes \Omega_M^3) \to \Omega_S^2 \to 0 .$$

From this exact sequence, we get a cohomology exact sequence

$$\cdots \to H^1(S, \Omega_S^1 \otimes \Omega_S^2) \to H^1(S, R^0p_*(\Omega_M^1 \otimes \Omega_M^3))$$
$$\to H^1(S, \Omega_S^2) \to \cdots .$$

The lemma follows since $H^1(S, \Omega_S^1 \otimes \Omega_S^2) = H^1(S, \Omega_S^2) = 0$. $\square$

LEMMA 2.14. $H^0(S, R^1p_*(\Omega_M^1 \otimes \Omega_M^3)) = 0$.

PROOF. Tensoring Ω_S^2, which is locally free, with the exact sequence of Lemma 2.10 and applying Lemma 2.12, we get an exact sequence

$$0 \to \Omega_S^1 \otimes \Omega_S^2 \to R^1p_*(\Omega_M^1 \otimes \Omega_M^3) \to \Omega_S^2 \to 0 .$$

This gives a cohomology exact sequence

$$0 \to H^0(S, \Omega_S^1 \otimes \Omega_S^2) \to H^0(S, R^1p_*(\Omega_M^1 \otimes \Omega_M^3))$$
$$\to H^0(S, \Omega_S^2) \to \cdots .$$

Since $H^0(S, \Omega_S^1 \otimes \Omega_S^2) = H^0(S, \Omega_S^2) = 0$, we have $H^0(S, R^1p_*(\Omega_M^1 \otimes \Omega_M^3)) = 0$. $\square$

By Lemma 2.13 and Lemma 2.14, Proposition 2.4 is clear.

PROPOSITION 2.15. $\dim H^1(M, \Theta) = 7$.

By the Riemann-Roch theorem, we have

$$\sum_{1=0}^{3} (-1)^i \dim H^i(M, \Theta) = \frac{1}{2}c_3 - \frac{19}{24}c_1c_2 + \frac{1}{2}c_1^3 .$$

From the results we have already got, we get

$$\dim H^1(M, \Theta) = 7 - \frac{1}{2}c_3 + \frac{19}{24}c_1c_2 - \frac{1}{2}c_1^3 .$$

Now we shall calculate the relevant Chern numbers of M.

LEMMA 2.16. $c_3 = 0$.

PROOF. Since M is diffeomorphic to $S^1 \times S^2 \times S^3$, this is clear. □

LEMMA 2.17. $c_1 c_2 = 0$.

PROOF. By the Riemann-Roch theorem,

$$c_1 c_2 = 24 \sum_{i=0}^{3} (-1)^i \dim H^i(M, \mathcal{O}) .$$

Since M is compact, $\dim H^0(M, \mathcal{O}) = 1$. To calculate $\dim H^1(M, \mathcal{O})$, we use the Leray spectral sequence and get

$$0 \to E_2^{1,0} \to H^1(M, \mathcal{O}) \to E_3^{0,1} \to 0 ,$$

where

$$E_2^{i,j} = H^i(S, R^j p_* \mathcal{O}),\ E_3^{0,1} = \mathrm{Ker}(E_2^{0,1} \to E_2^{1,0}) .$$

By Lemma 2.6,

$$E_2^{1,0} = H^1(S, R^0 p_* \mathcal{O}) = H^1(S, \mathcal{O}_S) = 0 .$$

Hence

$$E_3^{0,1} = E_2^{0,1} = H^0(S, \mathcal{O}_S) .$$

On the other hand,

$$E_2^{1,0} = H^1(S, R^0 p_* \mathcal{O}_M) = H^1(S, \mathcal{O}_S) = 0 .$$

Therefore

$$\dim H^1(M, \mathcal{O}) = \dim H^0(S, \mathcal{O}_S) = 1 .$$

As for $H^2(M, \mathcal{O})$, again by the Leray spectral sequence, $H^2(M, \mathcal{O})$ has a filtration with succesive quotients $E_3^{2,0}$, $E_3^{1,1}$ and $E_4^{0,2}$. We have $E_4^{0,2} = 0$ because the fibres of p are of dimension 1. Next we have $E_3^{1,1} = 0$ because

$$E_3^{1,1} = \mathrm{Ker}(E_2^{1,1} \to E_2^{3,0}) \subset E_2^{1,1} = H^1(S, R^1 p_* \mathcal{O}) = H^1(S, \mathcal{O}_S) = 0 .$$

We also have $E_3^{2,0} = 0$ since

$$E_2^{2,0} = H^2(S, R^0 p_* \mathcal{O}) \cong H^2(S, \mathcal{O}_S) = 0 .$$

Therefore $H^2(M, \mathcal{O}) = 0$. Furthermore $H^3(M, \mathcal{O}) \cong H^0(M, \Omega^3) = 0$. □

LEMMA 2.18. $c_1^3 = 0$.

PROOF. It is clear because M is diffeomorphic to $S^1 \times S^2 \times S^3$. □

By the above three lemmas, we have Proposition 2.15.

2. Small deformations of M. Put

$$W = \boldsymbol{P}^3 - l_0 - l_\infty ,$$
$$B = \{t = (t_1, t_2, \cdots, t_7) \in \boldsymbol{C}^7;\ |t_i| < \delta,\ i = 1, 2, \cdots, 7\}$$

where δ is a sufficiently small positive real number. For $t \in B$, we define a holomorphic automorphism g_t of W by

$$g_t([\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3]) \\ = [\zeta_0 + t_1\zeta_1 : t_2\zeta_0 + (1 + t_3)\zeta_1 : (\alpha + t_4)\zeta_2 + t_5\zeta_3 : t_6\zeta_2 + (\alpha + t_7)\zeta_3] .$$

In particular, $g_0 = g$.

Let $\tilde{g}$ be a holomorphic automorphism of $W \times B$ defined by

$$\tilde{g}(\zeta, t) = (g_t(\zeta), t)$$

and ϖ the projection of $W \times B$ to the second factor. Obviously we have $\varpi \circ \tilde{g} = \varpi$, hence we have the induced map $\mathscr{M} = (W \times B)/\langle \tilde{g} \rangle \rightarrow B$ which we also denote by ϖ.

THEOREM 1. *$(\mathscr{M}, B, \varpi)$ is the complex analytic family which is complete and effectively parametrized at the origin. A complex manifold N is a small deformation of M if and only if N is biholomorphic to $W/\langle g_t \rangle$ for some $t \in B$.*

PROOF. $(\mathscr{M}, B, \varpi)$ is easily seen to be a complex analytic family. Let $\mathfrak{U} = \{U_0, U_W, U_\infty\}$ be the open covering of M defined in §1.2. We define $\theta(\partial/\partial t_i) \in Z^1(\mathfrak{U}, \Theta)$ for $i = 1, 2, \cdots, 7$ as follows:

$$\theta\Big(\frac{\partial}{\partial t_i}\Big)(U_0 \cap U_W) = \theta\Big(\frac{\partial}{\partial t_i}\Big)(U_W \cap U_\infty) = 0 ,$$
$$\theta\Big(\frac{\partial}{\partial t_1}\Big)(U_0 \cap U_\infty) = -x_0\Big(x_0\frac{\partial}{\partial x_0} + y_0\frac{\partial}{\partial y_0} + z_0\frac{\partial}{\partial z_0}\Big) ,$$
$$\theta\Big(\frac{\partial}{\partial t_2}\Big)(U_0 \cap U_\infty) = \frac{\partial}{\partial x_0} , \quad \theta\Big(\frac{\partial}{\partial t_3}\Big)(U_0 \cap U_\infty) = x_0\frac{\partial}{\partial x_0} ,$$
$$\theta\Big(\frac{\partial}{\partial t_4}\Big)(U_0 \cap U_\infty) = \frac{y_0}{\alpha}\frac{\partial}{\partial y_0} , \quad \theta\Big(\frac{\partial}{\partial t_5}\Big)(U_0 \cap U_\infty) = \frac{z_0}{\alpha}\frac{\partial}{\partial y_0} ,$$
$$\theta\Big(\frac{\partial}{\partial t_6}\Big)(U_0 \cap U_\infty) = \frac{y_0}{\alpha}\frac{\partial}{\partial z_0} , \quad \theta\Big(\frac{\partial}{a t_7}\Big)(U_0 \cap U_\infty) = \frac{z_0}{\alpha}\frac{\partial}{\partial z_0} ,$$

with respect to the system of local coordinates $(x_0, y_0, z_0) = (\zeta_1/\zeta_0, \zeta_2/\zeta_0, \zeta_3/\zeta_0)$. Here $\theta(\partial/\partial t_i)(U_0 \cap U_W)$ means the value of $\theta(\partial/\partial t_i)$ on $U_0 \cap U_W$. Then it is easy to see that $[\theta(\partial/\partial t_i)]$ $(i = 1, 2, \cdots, 7)$ are linearly independent in $H^1(\mathfrak{U}, \Theta)$, where $[\theta(\partial/\partial t_i)]$ denotes the cohomology class represented by $\theta(\partial/\partial t_i)$. Indeed, suppose $\sum_{i=1}^7 \alpha_i[\theta(\partial/\partial t_i)] = 0$ for complex numbers α_i.

This is equivalent to the existence of an element $v \in H^0(U_0 \cup U_W \cup U_\infty, \Theta)$ such that

$$\sum_{i=1}^{7} \alpha_1 \theta\Big(\frac{\partial}{\partial t_i}\Big)(U_0 \cap U_\infty) = v\,|_{U_0 \cap U_\infty} - g_* v\,|_{U_0 \cap U_\infty} .$$

Writing this equation explicitly, we have

$$\begin{aligned}
&- \alpha_1 x_0\Big(x_0 \frac{\partial}{\partial x_0} + y_0 \frac{\partial}{\partial y_0} + z_0 \frac{\partial}{\partial z_0}\Big) + \alpha_2 \frac{\partial}{\partial x_0} + \alpha_3 x_0 \frac{\partial}{\partial x_0} \\
&\quad + \frac{1}{\alpha}\Big\{\alpha_4 y_0 \frac{\partial}{\partial y_0} + \alpha_5 z_0 \frac{\partial}{\partial y_0} + \alpha_6 y_0 \frac{\partial}{\partial z_0} + \alpha_7 z_0 \frac{\partial}{\partial z_0}\Big\} \\
&= (1 - 1/\alpha)(c_1 y_0 + d_1 z_0 + f x_0 y_0 + g x_0 z_0) \frac{\partial}{\partial x_0} \\
&\quad + \{(1-\alpha)(a_2 + b_2 x_0) + (1 - 1/\alpha)(f y_0 + g y_0 z_0)\} \frac{\partial}{\partial y_0} \\
&\quad + \{(1-\alpha)(a_3 + b_3 x_0) + (1 - 1/\alpha)(f y_0 z_0 + g z_0)\} \frac{\partial}{\partial z_0} .
\end{aligned}$$

Then we have $\alpha_i = 0$ for $i = 1, 2, \cdots, 7$. This shows the linear independence of $[\theta(\partial/\partial t_i)]$ $(i = 1, 2, \cdots, 7)$. Lastly it is easy to see that $\rho_0(\cdot) = i([\theta(\cdot)])$, where i is the inclusion map of $H^1(\mathfrak{U}, \Theta)$ to $H^1(M, \Theta)$ and ρ_0 is the Kodaira-Spencer map. The above result shows that ρ_0 is bijective because $H^1(M, \Theta)$ is 7-dimensional. □

§ 3. Small deformations of $M(n)(n \geqq 2)$.

1. Cohomology groups of $M(n)$.

PROPOSITION 3.1. $H^3(M, \Theta) = 0$.

PROOF. By the Kodaira-Serre duality,

$$H^3(M(n), \Theta) \cong H^0(M(n), \Omega^1 \otimes \Omega^3) .$$

But by [7, p. 7, Proposition 2.3]

$$H^0(X, (\Omega^1)^{\otimes m_1} \otimes (\Omega^2)^{\otimes m_2} \otimes (\Omega^3)^{\otimes m_3}) = 0$$

for any Class L manifold X and for non-negative integers m_1, m_2, m_3 such that $m_1 + m_2 + m_3 > 0$. □

PROPOSITION 3.2. $\dim H^0(M(n), \Theta) = 3 (n \geqq 2)$.

PROOF. We first prove the assertion for $n = 2$. We have defined $M(2)$ by $Z(M^1, M^2, \iota^1, \iota^2)$. We denote by ι^{-1} the inverse mapping of ι considered as a mapping of $N(\varepsilon)$ to $\iota(N(\varepsilon)) \subset \pi(U_W) \subset M$. Then $s = \iota^2 \circ \sigma \circ (\iota^1)^{-1}$

of $\iota^1(N(\varepsilon))\subset M^1$ to $\iota^2(N(\varepsilon))\subset M^2$ is expressed in terms of the local coordinates induced by the homogeneous coordinates in $\boldsymbol{P}^3$ as

$$s([\zeta_0:\zeta_1:\zeta_2:\zeta_3])=[\mu\zeta_0+\nu\zeta_2:\mu\zeta_1+\nu\zeta_3:-(\nu\zeta_0+\mu\zeta_2):-(\nu\zeta_1+\mu\zeta_3)]\,,$$

where $\mu=1+\lambda^2$, $\nu=1-\lambda^2$. In the above, π and σ are mappings defined at the end of §1. By the Mayer-Vietoris exact sequence of cohomology groups with coefficient in Θ, we see that an element of $H^0(M(2),\Theta)$ is identified with an element $v\in H^0((M^1)^{\sharp},\Theta)$ such that $s_*(v|_{\iota^1(N(\varepsilon))})$ is the restriction of an element v' of $H^0((M^2)^{\sharp},\Theta)$ to $\iota^2(N(\varepsilon))$, i.e., $v'|_{\iota^2(N(\varepsilon))}=s_*(v|_{\iota^1(N(\varepsilon))})$. On $\iota^1(N(\varepsilon))\cap\pi(\{\zeta_0\neq 0\})$, v has the form

$$(a_1+a_2x_0+dx_0)\frac{\partial}{\partial x_0}+(b_1y_0+b_2z_0+dx_0y_0)\frac{\partial}{\partial y_0}+(c_1y_0+c_2z_0+dx_0z_0)\frac{\partial}{\partial z_0}$$

because $H^0((M^1)^{\sharp},\Theta)=H^0(M^1,\Theta)$. So does v' on the similar domain. In the following, we denote the coordinates and coefficients concerned with M^2 by letters with primes, for instance, ζ', a'. Calculating $4\lambda^2\{s_*(v|_{\iota^1(N(\varepsilon))})-v'|_{\iota^2(N(\varepsilon))}\}$ in terms of the local coordinates x_0, y_0, z_0 and x_0', y_0', z_0', we have

$$\begin{aligned}&\{(\mu^2a_1-\nu^2c_1-4\lambda^2a_1')+(\mu^2a_2-\nu^2c_2+\nu^2b_1-4\lambda^2a_2')x_0'+\mu\nu(a_1-c_1)y_0'\\&\quad+\mu\nu(a_2-c_2)z_0'+(\nu^2b_2+\mu^2d-4\lambda^2d')x_0'^2+\mu\nu b_1x_0'y_0'+\mu\nu(b_2\\&\quad+d)x_0'z_0'\}\partial/\partial x_0'+\{\mu\nu b_1+\mu\nu(b_2+d)x_0'+((\mu^2+\nu^2)b_1-4\lambda^2b_1')y_0'\\&\quad+(\mu^2b_2+\nu^2d-4\lambda^2b_2')z_0'+(\nu^2b_2+\mu^2d-4\lambda^2d')x_0'y_0'+\mu\nu b_1y_0'^2\\&\quad+\mu\nu(b_2+d)y_0'z_0'\}\partial/\partial y_0'+\{\mu\nu(c_1-a_1)+\mu\nu(c_2-a_2)x_0'+(\mu^2c_1\\&\quad-\nu^2a_1-4\lambda^2)y_0'+(\mu^2c_2-\nu^2a_2+\nu^2b_1-4\lambda^2c_2')z_0'+(\nu^2b_2+\mu^2d\\&\quad-4\lambda^2d')x_0'z_0'+\mu\nu b_1y_0'z_0'+\mu\nu(b_2+d)z_0'^2\}\partial/\partial z_0'\,.\end{aligned}$$

Thus the equation $s_*(v|_{\iota^1(N(\varepsilon))})=v'|_{\iota^2(N(\varepsilon))}$ is equivalent to the relations among coefficients

$$\begin{aligned}&a_1=a_1'=c_1=c_1'\,,\quad a_2=a_2'=c_2=c_2'\,,\\&b_1=b_1'=0\,,\quad b_2=b_2'=-d=-d'\,.\end{aligned}$$

This concludes $\dim H^0(M(2),\Theta)=3$.

We now prove the assertion for $n\geqq 3$. It is easy to check that an element, $v=(a+bx_0+cx_0^2)\partial/\partial x_0+(-cz_0+cx_0y_0)\partial/\partial y_0+(ay_0+bz_0+cx_0z_0)\partial/\partial z_0$, of $H^0(N(\varepsilon),\Theta)$ is σ_*-invariant and τ_*-invariant, i.e., $\sigma_*v=v$ and $\tau_*v=v$. Since $M(3)=Z(M(2),M^3,\iota^2|_{N(\varepsilon)}\circ\tau,\iota^3)$ and $\iota=\pi\circ\tau$, the above facts imply that every element of $H^0(M(2)^{\sharp},\Theta)$ has the extension to $M(3)$ and to $M(n)$ for any $n\geqq 4$. □

Proposition 3.3. $\dim H^1(M(n),\Theta)=15n-12$.

We first note that the embedding ι of $U_\varepsilon \subset \boldsymbol{P}^3$ into M is naturally extended to an automorphism τ of $\boldsymbol{P}^3$ when we consider M as a manifold obtained by identification of $\zeta \in U_0 \cap g^{-1}(U_\infty)$ with $g(\zeta) \in g(U_0) \cap U_\infty$ in $U_0 \cup U_W \cup U_\infty \subset \boldsymbol{P}^3$. Here U_0, U_W, U_∞ have already been defined in §1. 2. We denote $U_0 \cup U_W \cup U_\infty$ by $\widetilde{M(1)}$, $U_0 \cap g^{-1}(U_\infty)$ by $N(1)_1$, and $g(U_0) \cap U_\infty$ by $N(1)_2$. $\boldsymbol{P}^3 - \widetilde{M(1)}$ has two connected components: $K(1)_1$ containing l_0 and $K(1)_2$ containing l_∞. From now on, we denote g by $g_{(1)1}$.

Assume that $M(n)$, $N(n)_i$, $K(n)_i$ $(1 \leqq i \leqq 2n)$, and $g_{(n)j}(1 \leqq j \leqq n)$ are defined for n so that $\widetilde{M(n)}$, $N(n)_i$, $K(n)_i$ are subsets in $\boldsymbol{P}^3$ and that each $g_{(n)j}$ is a holomorphic automorphism of $\boldsymbol{P}^3$, which induces an isomorphism of $N(n)_{2j-1}$ to $N(n)_{2j}$ for any $1 \leqq j \leqq n$. Assume also that $M(n)$ is constructed by identification of $\zeta^j \in N(n)_{2j-1}$ with $g_{(n)j}(\zeta^j) \in N(n)_{2j}$ $(1 \leqq j \leqq n)$ in $\widetilde{M(n)}$ and that the embedding $\iota_n : U_\varepsilon \to M(n)$ lifts to an open embedding into $\widetilde{M(n)}$ and extends to an automorphism $\tilde{\iota}_n$ of $\boldsymbol{P}^3$. We define $\widetilde{M(n+1)}$ by $\tilde{\iota}_n^{-1}(\widetilde{M(n)}) - \cup_{i=1}^2 \sigma \circ \tau^{-1}((K(1)_i)$, and $N(n+1)_i$ (resp. $K(n+1)_i$) by $\tilde{\iota}_n^{-1}(N(n)_i)$ (resp. $\tilde{\iota}_n^{-1}(K(n)_i)$ for $1 \leqq i \leqq 2n$ and by $\sigma \circ \tau^{-1}(N(1)_{i-2n})$ (resp. $\sigma \circ \tau^{-1}(K(1)_{i-2n})$ for $i = 2n+1, 2n+2$. We also define $g_{(n+1)j}$ by $\iota_n^{-1} \circ g_{(n)j} \circ \iota_n$ for $1 \leqq j \leqq n$ and by $\sigma \circ \tau^{-1} \circ g_{(1)1} \circ \tau \circ \sigma$ for $j = n+1$. Then we can easily see that every $g_{(n+1)j}$ is an automorphism of $\boldsymbol{P}^3$, which induces an isomorphism of $N(n+1)_{2j-1}$ to $N(n+1)_{2j}$ and that we obtain $M(n+1)$ by identifying $\zeta^j \in N(n+1)_{2j-1}$ with $g_{(n+1)j}(\zeta^j) \in N(n+1)_{2j}$ in $\widetilde{M(n+1)}$ for $1 \leqq j \leqq n+1$.

LEMMA 3.4. *If $\tilde{v} \in H^1(\widetilde{M(n)}, \Theta)$ is the lifting of $v \in H^1(M(n), \Theta)$, then* $\tilde{v} = 0$.

PROOF. Since $\tilde{v}$ is the lifting, it satisfies the conditions

$$\tilde{v}|_{N(n)_{2j}} = (g_{(n)j})_* \tilde{v}|_{N(n)_{2j-1}}, \quad 1 \leqq j \leqq n .$$

Let $v|_{N(n)_i}$ decompose into $a_{(n)i} + b_{(n)i}$ $(1 \leqq i \leqq n)$ where $a_{(n)i}$ is the restriction of an element $\tilde{a}_{(n)i}$ of $H^1(K(n)_i \cup N(n)_i, \Theta)$ and $b_{(n)i}$ is the restriction of $\tilde{b}_{(n)i} \in H^1(\boldsymbol{P}^3 - K(n)_i, \Theta)$. This decomposition is possible and unique by the Mayer-Vietoris exact sequence for the pair $(\boldsymbol{P}^3 - K(n)_i, K(n)_i \cup N(n)_i)$. Then the relations among $a_{(n)i}$ and $b_{(n)i}$ are as follows:

$$a_{(n)2j} = (g_{(n)j})_* b_{(n)2j-1}, \quad b_{(n)2j} = (g_{(n)j})_* a_{(n)2j-1},$$

for $1 \leqq j \leqq n$.

Let $L(n) := M(n) - \cup_{i=1}^{2n} N(n)_i$. Consider the commutative diagram of cohomology groups with coefficients in Θ:

$$\begin{array}{ccccccccc} \longrightarrow & H^1(\widetilde{M(n)}) & \xrightarrow{\alpha} & \bigoplus_{i=1}^{2n} H^1(N(n)_i) & \xrightarrow{\delta_L} & H^1_{L(n)}(\widetilde{M(n)}) & \longrightarrow & & \\ & \uparrow & & \uparrow & & \rho\uparrow\cong & & & \\ \longrightarrow & 0 & \longrightarrow & \bigoplus_{i=1}^{2n} H^1(N(n)_i \cup K(n)_i) & \underset{\cong}{\xrightarrow{\tilde{\delta}_L}} & H^1_{L(n)}(\boldsymbol{P}^3) & \longrightarrow & 0 & \longrightarrow \end{array}$$

The first row is the local cohomology exact sequence for the pair $(\widetilde{M(n)}, L(n))$ and the second is that for the pair $(\boldsymbol{P}^3, L(n))$.

By the relations among $a_{(n)i}$ and $b_{(n)i}$,

$$\begin{aligned} 0 = \delta_L(\alpha(\theta)) &= \delta_L\Big(\bigoplus_{i=1}^{2n} (a_{(n)i} + b_{(n)i})\Big) = \delta_L\Big(\bigoplus_{j=1}^{n} a_{(n)2j-1} + \bigoplus_{j=1}^{n} (g_{(n)j})_* b_{(n)2j-1}\Big) \\ &= \bigoplus_{j=1}^{n} \delta_L(a_{(n)2j-1}) + \bigoplus_{j=1}^{n} \delta_L((g_{(n)j})_* b_{(n)2j-1}) \,. \end{aligned}$$

By the commutativity of the diagram above, the last line of the above equation is equal to

$$\begin{aligned} &\rho\Big(\tilde{\delta}_L\Big(\bigoplus_{j=1}^{n} \tilde{a}_{(n)2j-1}\Big) + \tilde{\delta}_L\Big(\bigoplus_{j=1}^{n} ((g_{(n)j})_* \tilde{b}_{(n)2j-1})\Big) \\ &\qquad = \rho\circ\tilde{\delta}_L\Big(\Big(\bigoplus_{j=1}^{n} a_{(n)2j-1} + \bigoplus_{j=1}^{n} (g_{(n)j})_* b_{(n)2j-1}\Big)\Big) \,. \end{aligned}$$

Since $\tilde{\delta}_L$ and ρ are isomorphisms, we have

$$\bigoplus_{j=1}^{n} \tilde{a}_{(n)2j-1} + \bigoplus_{j=1}^{n} (g_{(n)j})_* \tilde{b}_{(n)2j-1} = 0 \,.$$

Since all the terms on the left are from the distinct components of $\bigoplus_{i=1}^{n} H^1(N(n)_i \cup K(n)_i)$, we have

$$\tilde{a}_{(n)2j-1} = (g_{(n)j})_* \tilde{b}_{(n)2j-1} = 0$$

for $j = 1, \cdots, n$. This is equivalent to

$$\tilde{a}_{(n)i} = \tilde{b}_{(n)i} = 0,\ i = 1, \cdots, 2n \,.$$

Hence we have $\alpha(v) = 0$. The proof of the lemma is complete once we show that α is injective, i.e., the following sequence is exact:

$$0 \to H^0(\widetilde{M(n)}) \to \bigoplus_{i=1}^{2n} H^0(N(n)_i) \to H^1_{L(n)}(\widetilde{M(n)}) \to 0 \,.$$

First consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\widetilde{M(n)}) & \longrightarrow & \bigoplus_{i=1}^{2n} H^0(N(n)_i) & \longrightarrow & H^1_{L(n)}(\widetilde{M(n)}) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H^0(\boldsymbol{P}^3) & \longrightarrow & \bigoplus_{i=1}^{2n} H^0(N(n)_i \cup K(n)_i) & \longrightarrow & H^1_{L(n)}(\boldsymbol{P}^3) & \longrightarrow & 0 \,. \end{array}$$

Both rows are the local cohomology exact sequences in view of facts $H^0_{L(n)}(\widetilde{M(n)}) = H^0_{L(n)}(\boldsymbol{P}^3) = 0$ and $H^1(\boldsymbol{P}^3) = 0$. We are done since any holomorphic vector field on $\widetilde{M(n)}$ (resp. $N(n)_i$) can be extended uniquely to one on $\boldsymbol{P}^3$ (resp. $N(n)_i \cup K(n)_i$). □

LEMMA 3.5. *The restrictions of any element of* $H^1(M(n), \Theta)$ *to* $\iota_{n-1}(N(\varepsilon))$, $\iota_n(U_\varepsilon)$, *and* $\iota_n(N(\varepsilon))$ *are zero.*

PROOF. Consider the following commutative diagram

$$\begin{array}{ccc} H^1(M(n), \Theta) & \xrightarrow{\beta} & H^1(\widetilde{M(n)}, \Theta) \\ \downarrow & & \downarrow \\ H^1(\iota_{n-1}(N(\varepsilon)), \Theta) & \xrightarrow{\mathrm{id}} & H^1(\iota_{n-1}(N(\varepsilon)), \Theta) \end{array} .$$

Since β is the zero map, we have the conclusion for $\iota_{n-1}(N(\varepsilon))$. The other two are obvious because $\iota_n(N(\varepsilon)) \subset \iota_n(U_\varepsilon) \subset \iota_{n-1}(N(\varepsilon))$. □

LEMMA 3.6. *The following sequence is exact*:

$$0 \to H^0(M(n), \Theta) \to H^0(M(n)^\#, \Theta) \oplus H^0(\iota_n(U_\varepsilon), \Theta) \to H^0(N(\varepsilon), \Theta) \to 0.$$

PROOF. As is already proved, $H^0(M(n), \Theta)$ is isomorphic to $H^0(M(n)^\#, \Theta)$ by the restriction map, and so is $H^0(\iota_n(U_\varepsilon), \Theta)$ to $H^0(N(\varepsilon), \Theta)$. □

LEMMA 3.7. $H^1(M(n), \Theta)$ *is isomorphic to the subgroup* $H^1(M(n), \Theta)^\#$ *in* $H^1(M(n)^\#, \Theta)$ *consisting of elements whose restrictions to* $\iota_n(N(\varepsilon))$ *are zero.*

PROOF. By the above lemma, we have an exact sequence:

$$0 \to H^1(M(n), \Theta) \to H^1(M(n)^\#, \Theta) \oplus H^1(\iota_n(U_\varepsilon), \Theta) \to H^1(N(\varepsilon), \Theta) .$$

By Lemma 3.5, the first factor of the image of an element of $H^1(M(n), \Theta)$ is contained in $H^1(M(n), \Theta)^\#$ and the second component of the image is zero. So the restriction map of $H^1(M(n), \Theta)$ to $H^1(M(n)^\#, \Theta)$ induces a map of $H^1(M(n), \Theta)$ to $H^1(M(n), \Theta)^\#$. The above exact sequence proves the injectivity of the map.

The surjectivity is proved by chasing the sequence. A pair of an element of $H^1(M(n), \Theta)^\#$ and zero of $H^1(\iota_n(U_\varepsilon))$ is mapped to zero in $H^1(N(\varepsilon), \Theta)$. By the exactness of the sequence, there exists an element of $H^1(M(n), \Theta)$ mapped to the pair. □

PROOF OF PROPOSITION 3.3. We first claim that $\mathrm{Im}\,(H^1(M(n), \Theta) \to H^1(M(n-1)^\#, \Theta) \oplus H^1(M^{n\#}, \Theta))$ is isomorphic to $H^1(M(n-1), \Theta) \oplus H^1(M^n, \Theta)$. The image is contained in $H^1(M(n-1), \Theta)^\# \oplus H^1(M^n, \Theta)^\#$ by Lemma 3.5.

Conversely any pair in $H^1(M(n-1), \Theta)^\# \oplus H^1(M^n, \Theta)^\#$ is the image of an element of $H^1(M(n), \Theta)$ because the image of the pair in $H^1(N(\varepsilon), \Theta)$ is zero. Hence the image of $H^1(M(n), \Theta)$ is equal to $H^1(M(n-1), \Theta)^\# \oplus H^1(M^n, \Theta)^\#$ and isomorphic to $H^1(M(n-1), \Theta) \oplus H^1(M^n, \Theta)$ by Lemma 3.7.

By the above claim, we have an exact sequence

$$0 \to \operatorname{Coker}(H^0(M(n-1)^\#, \Theta) \oplus H^0(M^{n\#}, \Theta) \to H^0(N(\varepsilon), \Theta)$$
$$\to H^1(M(n), \Theta) \to H^1(M(n-1), \Theta) \oplus H^1(M^n, \Theta) \to 0 .$$

If $n = 2$, the dimension of the cokernel is 4 because $\dim H^0(M(2), \Theta) = 3$, $\dim H^0(M(1)^\#, \Theta) = 7$, $\dim H^0(N(\varepsilon), \Theta) = 15$. Therefore

$$\dim H^1(M(2)) = \dim (H^1(M^1) \oplus H^1(M^2)) + 4 = 14 + 4 = 18 ,$$

which proves the assertion for $n = 2$. If $n \geqq 3$, we see that the cokernel is 8 dimensional, because $\dim H^0(M(n), \Theta) = \dim H^0(M(n-1)^\#, \Theta) = 3$, $\dim H^0(M^{n\#}, \Theta) = 7$, and $\dim H^0(N(\varepsilon)) = 15$. Then by induction on n, we have

$$\begin{aligned}\dim H^1(M(n), \Theta) &= \dim (H^1(M(n-1), \Theta) \oplus H^1(M, \Theta)) + 8 \\ &= 15(n-1) - 12 + 7 + 8 = 15n - 12 .\end{aligned}$$

□

PROPOSITION 3.8 $H^2(M(n), \Theta) = 0$ $(n \geqq 2)$

PROOF. By the Riemann-Roch theorem, we have

$$\begin{aligned}&3 - 15n + 12 + \dim H^2(M(n), \Theta) \\ &= \frac{1}{2}c_1^3[M(n)] - \frac{19}{24}c_1c_2[M(n)] + \frac{1}{2}c_3[M(n)] .\end{aligned}$$

Due to [6, p. 6, Proposition 2.2], we have

$$c_I[X_1 \# X_2] = c_I[X_1] + c_I[X_2] - c_I[\boldsymbol{P}^3] .$$

for any Class L manifolds X_1 and X_2, where c_I is the Chern number. Hence

$$c_I[M(n)] = nc_I[M] - (n-1)c_I[\boldsymbol{P}^3] = -(n-1)c_I[\boldsymbol{P}^3]$$

because $c_I[M] = 0$. Therefore, with the well-known fact on the cohomology groups of $\boldsymbol{P}^3$ with coefficients in Θ, we have

$$\begin{aligned}\dim H^2(M(n), \Theta) &= 15n - 15 - (n-1)\textstyle\sum(-1)^i \dim H^i(\boldsymbol{P}^3, \Theta) \\ &= 15n - 15 - (n-1)(15 - 0 + 0 - 0) = 0 .\end{aligned}$$

□

2. Small deformations of $M(2)$. Let δ be a sufficiently small positive real number and $B(t')$ a domain in $\boldsymbol{C}^4$ defined by

$$B(t') = \{t' = (t'_1, t'_2, t'_3, t'_4) \in \boldsymbol{C}^4;\ |t'_i| < \delta \quad (i = 1, 2, 3, 4)\} ,$$

Let $\mathfrak{U}^j = \{U_0^j, U_W^j, U_\infty^j\}$, the j-the copy of $\mathfrak{U}$, be the open covering of

M^j for $j \in N$. We define a holomorphic open embedding $s_{t'}$ of $\iota(N(\varepsilon)) \subset \pi(U^1_W) \subset M^1$ into $\pi(U^2_W) \subset M^2 \subset M^2$ by

$$\begin{aligned} s_{t'}([\zeta_0\colon \zeta_1\colon \zeta_2\colon \zeta_3]) \\ = [\mu\zeta_0 + \nu\zeta_2\colon \mu\nu_1 + \nu\zeta_3\colon (\mu t'_1 - \nu)\zeta_0 + \mu t'_2\zeta_1 + (\nu t'_1 - \mu)\zeta_2 \\ + \nu t'_2\zeta_3\colon \mu t'_3\zeta_0 + (\mu t'_4 - \nu)\zeta_1 + \nu t'_3\zeta_2 + (\nu t'_4 - \mu)\zeta_3] \end{aligned}$$

where $\mu = 1 + \lambda^2$ and $\nu = 1 - \lambda^2$. In the above, the local coordinates on $\pi(U^j_W)$ are taken as those of P^3 since $\pi(U^j_W)$ is isomorphic to U^j_W. We restrict s'_t to $\iota^1(N(\varepsilon)) \cap s_{t'}^{-1}(\iota^2(N(\varepsilon)))$ which we shall simply denote by s'_t. Then $s_{t'}$ becomes a holomorphic open embedding of $\iota^1(N(\varepsilon)) \cap s_{t'}^{-1}(\iota^2(N(\varepsilon)))$ into U^2_W. Note that $s_0 = s = \iota^2 \circ \sigma \circ (\iota^1)^{-1}$.

Now we construct a complex manifold $\mathscr{M}(2)$ as follows. First take two copies of $(\mathscr{M}, B, \varpi)$, $(\mathscr{M}^j, B^j, \varpi^j)$ for $j = 1, 2$. We write (x^j, t_j) a point of $\mathscr{M}^j$. Let π^j be the natural projection of $W \times B^j$ to $\mathscr{M}^j$ and π^j_{tj} be the restriction of π^j to $W \times \{t^j\}$. From Theorem 1, $M^j_{tj} = (\varpi^j)^{-1}(t^j)$ contains a domain $\pi^j_{tj}(U^j_W)$ biholomorphic to U^j_W, which contains $\pi^j_{tj}\tau(U_{1/\varepsilon})$. Put $\mathscr{M}^{i\#} = \mathscr{M}^j - \overline{\pi^j(\tau(U_{1/\varepsilon}) \times B^j)}$. We define $\mathscr{M}(2) = \mathscr{M}^{1\#} \times B^2 \times B(t') \cup \mathscr{M}^{2\#} \times B^1 \times B(t')$ by identifying

$$((x^1, t^1), t^2, t') \in \pi^1(\tau(N(\varepsilon)) \times B^1) \times B^2 \times B(t') \subset \mathscr{M}^{1\#} \times B^2 \times B(t')$$

with

$$((x^2, \tilde{t}^2), \tilde{t}^1, \tilde{t}') \in \pi^2(\tau(N(\varepsilon)) \times B^2) \times B^1 \times B(t') \subset \mathscr{M}^{2\#} \times B^1 \times B(t')$$

if and only if

$$x^2 = s_{t'}(x^1),\ t^1 = \tilde{t}^1,\ t^2 = \tilde{t}^2,\ t' = \tilde{t}'\ .$$

We define the projection ϖ of $\mathscr{M}(2)$ to $B^1 \times B^2 \times B(t')$ by

$$\varpi\colon ((x^1, t^1), t^2, t') \mapsto (t^1, t^2, t')$$

and

$$\varpi\colon ((x^2, t^2), t^1, t') \mapsto (t^1, t^2, t')\ .$$

Then it is clear that $(\mathscr{M}(2), B^1 \times B^2 \times B(t'), \varpi)$ becomes a complex analytic family.

THEOREM 2. $(\mathscr{M}(2), B^1 \times B^2 \times B(t'), \varpi)$ *is the complete, effectively parametrized complex analytic family of small deformations of* $M(2)$.

PROOF. $(\mathfrak{U}^j)^\#\colon = \{U^j_0, U^j_W - \overline{\iota^j(U_{1/\varepsilon})}, U^j_\infty\}$ is a covering of $(M^j)^\#$. We denote $U^j_W - \overline{\iota^j(U_{1/\varepsilon})}$ by $(U^j_W)^\#$ for simplicity. We take $\mathfrak{U}(2) = (\mathfrak{U}^1)^\# \cup (\mathfrak{U}^2)^\#$ as a covering of $M(2)$. We define a linear map $\theta\colon T_0(B^1 \times B^1 \times B(t')) \to Z^1(\mathfrak{U}(2), \Theta)$ as follows: $\theta(\partial/\partial t^j_i)$ is equal to the vector field listed in the proof of Theorem 1 on $U_0^j \cap U_\infty^j$ and takes the value zero on other intersections of any distinct two members of $\mathfrak{U}(2)$ for $i = 1, \cdots, 7$ and $j = 1, 2$. As for $\theta(\partial/\partial t')$, we define

$$\theta(\partial/\partial t_1')((U_W^1)^\# \cap (U_W^2)^\#) = \partial/\partial y_0' ,$$
$$\theta(\partial/\partial t_2')((U_W^1)^\# \cap (U_W^2)^\#) = x_0'\partial/\partial y_0' ,$$
$$\theta(\partial/\partial t_3')((U_W^1)^\# \cap (U_W^2)^\#) = \partial/\partial z_0' ,$$
$$\theta(\partial/\partial t_4')((U_W^1)^\# \cap (U_W^2)^\#) = x_0'\partial/\partial z_0' ,$$

and $\theta(\partial/\partial t_k')$ takes the value zero on other intersections for $k = 1, 2, 3, 4$. Then it is easy to see that $i([\theta(\cdot)]) = \rho_0(\cdot)$ where i is the inclusion of $H^1(\mathfrak{U}(2), \Theta)$ to $H^1(M(2), \Theta)$ and ρ_0 is the Kodaira-Spencer map. $[\theta(\partial/\partial t_i^j)]$ and $[\theta(\partial/\partial t_k')]$ are linearly independent. Indeed, suppose that we have an equation

$$\sum_{i=1}^{7} \alpha_i^1\theta(\partial/\partial t_i^1) + \sum_{i=1}^{7} \alpha_i^2\theta(\partial/\partial t_i^2) + \sum_{k=1}^{4} \beta_k\theta(\partial/\partial t_k') = \delta v$$

where v is an element of $C^0(\mathfrak{U}(2), \Theta)$ and α_i^j, β_k are complex numbers. By Theorem 1, we have $\alpha_i^j = 0$ for all i, j. So the above equation reduces to the equation

$$\sum_{k=1}^{4} \beta_k\theta(\partial/\partial t_k') = s_* v^1 - v^2 ,$$

where v^j belongs to $H^0((M^j)^\#, \Theta)$ for $j = 1, 2$. Using the calculation in the proof of Proposition 3.2, the above equation becomes

$$\begin{aligned}
&\beta_1\partial/\partial y_0' + \beta_2 x_0'\partial/\partial y_0' + \beta_3\partial/\partial z_0' + \beta_4 x_0'\partial/\partial z_0' \\
&\quad = \{(\mu^2 a_1 - \nu^2 c_1 - 4\lambda^2 a_1') + (\mu^2 a_2 - \nu^2 c_2 + \nu^2 b_1 - 4\lambda^2 a_2')x_0' \\
&\qquad + \mu\nu(a_1 - c_1)y_0' + \mu\nu(a_2 - c_2)z_0' + (\nu^2 b_2 + \mu^2 d - 4\lambda^2 d')x_0'^2 \\
&\qquad + \mu\nu b_1 x_0' y_0' + \mu\nu(b_2 + d)x_0' z_0'\}\partial/\partial x_0' + \{\mu\nu b_1 + \mu\nu(b_2 + d)x_0' \\
&\qquad + ((\mu^2 + \nu^2)b_1 - 4\lambda^2 b_1')y_0' + (\mu^2 b_2 + \nu^2 d - 4\lambda^2 b_2')z_0' + (\nu^2 b_2 + \mu^2 d \\
&\qquad - 4\lambda^2 d')x_0' y_0' + \mu\nu b_1 y_0'^2 + \mu\nu(b_2 + d)y_0' z_0'\}\partial/\partial y_0' + \{\mu\nu(c_1 - a_1) \\
&\qquad + \mu\nu(c_2 - a_2)x_0' + (\mu^2 c_1 - \nu^2 a_1 - 4\lambda^2 c_1')y_0' \\
&\qquad + (\mu^2 c_2 - \nu^2 a_2 + \nu^2 b_1 - 4\lambda^2 c_2')z_0' + (\nu^2 b_2 + \mu^2 d - 4\lambda^2 d')x_0' z_0' \\
&\qquad + \mu\nu b_1 y_0' z_0' + \mu\nu(b_2 + d)z_0'^2\}\partial/\partial z_0'
\end{aligned}$$

where a_i, b_i, c_i, d, a_i', b_i', c_i', and d' are complex numbers. This shows us that all β_i vanish and the image of $[\theta(\cdot)]$ spans an 18-dimensional vector subspace in $H^1(\mathfrak{U}(2), \Theta)$, which in turn is a subspace of 18-dimentional $H^1(M(2), \Theta)$. Hence $\rho_0(\cdot) = i([\theta(\cdot)])$ is bijective. □

3. Small deformations of $M(n)(n \geqq 3)$. We construct the complete, effectivzly parametrized complex analytic family $\mathscr{M}(n)$ of small deformations of $M(n)$ inductively. Let

$$B(t'') = \{t'' = (t''_1, \cdots, t''_8) \in \boldsymbol{C}^8; |t''_i| < \delta \quad (i = 1, \cdots, 8)\} .$$

We define a holomorphic open embedding $r_{t''}$ of $\iota^{n-1}|_{N(\varepsilon)} \circ \tau(N(\varepsilon)) \subset M(n-2)^{\#} \cap M^{n-1\#} \subset M(n-1)$ into $\iota^n(N(\varepsilon)) \subset M^{n\#}$ by

$$\begin{aligned} r_{t''}([\zeta_0: \zeta_1: \zeta_2: \zeta_3]) &= [\lambda(1 - t''_1)\zeta_0 - \lambda t''_2\zeta_1 + (1 + t''_1)\zeta_2 + t''_2\zeta_3: \lambda t''_3\zeta_0 + \lambda(1 + t''_4)\zeta_1 \\ &\quad - t''_3\zeta_2 + (1 - t''_4)\zeta_3: \lambda(1 + t''_5)\zeta_0 + \lambda t''_6\zeta_1 + (-1 + t''_5)\zeta_2 + t''_6\zeta_3: \\ &\quad \lambda t''_7\zeta_0 + \lambda(1 + t''_8)\zeta_1 + t''_7\zeta_2 - (1 + t''_8)\zeta_3] \end{aligned}$$

with respect to the system of local coordinates induced by the homogeneous coordinates of $\boldsymbol{P}^3$.

We have already constructed $\mathscr{M}(2)$ in §3. 2. $\mathscr{M}(2)$ contains $\pi(\tau(N(\varepsilon)) \times B^1) \times B^2 \times B(t')$ such that

$$\pi(\tau \circ \tau(U_\varepsilon) \times B^1) \times B^2 \times B(t') \cap M(2) = \iota^2 \circ \tau(U_\varepsilon) \subset M^{1\#} \cap M^{2\#} .$$

Here $M(2)$ is identified with the fibre $\varpi^{-1}(0)$. Assume that $\mathscr{M}(n)$ is constructed with the parameter space $B(n)$ and that $\mathscr{M}(n)$ contains $\iota^n|_{N(\varepsilon)} \circ \tau(U_\varepsilon) \times B(n)$ with the property

$$(*)_n \qquad (\iota^n|_{N(\varepsilon)} \circ \tau(U_\varepsilon) \times B(n)) \cap M(n) = \iota^n|_{N(\varepsilon)} \circ \tau(U_\varepsilon) \subset M(n-1)^{\#} \cap M^{\#} .$$

We denote $\mathscr{M}(n) - (\overline{\iota^n|_{N(\varepsilon)} \circ \tau(U_{1/\varepsilon}) \times B(n)})$ by $\mathscr{M}(n)^{\#}$. We construct $\mathscr{M}(n+1)$ from $\mathscr{M}(n)^{\#}$ and $\mathscr{M}^{n+1\#}$ by identifying

$$((x, t), t^{n+1}, t'') \in \mathscr{M}(n)^{\#} \times B^{n+1} \times B(t'')$$

with

$$((x^{n+1}, \tilde{t}^{n+1}), \tilde{t}, \tilde{t}'') \in \mathscr{M}^{n+1\#} \times B(n) \times B(t'')$$

if and only if

$$x^{n+1} = r_{t''}(x),\ t = \tilde{t},\ t^{n+1} = \tilde{t}^{n+1},\ t'' = \tilde{t}'' .$$

It is clear that $\mathscr{M}(n+1)$ contains $\iota^{n+1}|_{N(\varepsilon)} \circ \tau(U_\varepsilon) \times B(n+1)$ with the property $(*)_{n+1}$. Hence we get $\mathscr{M}(n)$ for any $n \in \boldsymbol{N}$. We project $\mathscr{M}(n+1)$ onto $B(n+1) = B(n) \times B^{n+1} \times B(t'')$ by

$$\begin{aligned} \varpi&: ((x, t), t^{n+1}, t'') \mapsto (t, t^{n+1}, t'') \\ \varpi&: ((x^{n+1}, t^{n+1}), t, t'') \mapsto (t, t^{n+1}, t'') . \end{aligned}$$

Then $(\mathscr{M}(n+1), B(n+1), \varpi)$ is a complex analytic family with $\varpi^{-1}(0) = M(n+1)$.

THEOREM 3. *$(\mathscr{M}(n), B^1 \times \cdots \times B^n \times B(t') \times \underbrace{B(t'') \times \cdots \times B(t'')}_{n-2}, \varpi)$ is the* *complete, effectively parametrized complex analytic family of small deformations of $M(n)$.*

PROOF. We define the covering $\mathfrak{U}(n)$ *of* $M(n)$ inductively. We have already defined $\mathfrak{U}(1) = \mathfrak{U}$ and $\mathfrak{U}(2)$ in the proof of Theorem 2. Put

$$(U_W^{1\#})^\# = U_W^{1\#} - \overline{\iota^1|_{N(\varepsilon)}\circ\tau(U_\varepsilon)} ,$$
$$(U_W^{2\#})^\# = U_W^{2\#} - \overline{\iota^2|_{N(\varepsilon)}\circ\tau(U_{1/\varepsilon})} .$$

We define $\mathfrak{U}(3)$ to be

$$\{U_0^1, (U_W^{1\#})^\#, U_\infty^1, U_0^2, (U_W^{2\#})^\#, U_\infty^2\} \cup \mathfrak{U}^{3\#} .$$

The former set is for $M(2)^\#$ and the latter is for $M^{3\#}$. Then $\iota^3|_{N(\varepsilon)}\circ\tau(U_\varepsilon)\subset M(2)^\#\cap M^{3\#}$ intersects only $(U_W^{2\#})^\#$ and $U_W^{3\#}$. Assume that $\mathfrak{U}(n)$ is defined so that

$$(**)_n \quad \begin{cases} \text{any distinct three of } \mathfrak{U}(n) \text{ do not intersect and } \iota^n|_{N(\varepsilon)}\circ\tau(U_\varepsilon)\subset \\ M(n-1)^\#\cap M^{n\#} \text{ intersects only } (U_W^{(n-1)\#})^\# \text{ and } U_W^{n\#} \text{of } \mathfrak{U}(n). \end{cases}$$

Put

$$((U_W^{(n-1)\#})^\#)^\# = ((U^{(n-1)})^\#)^\# - \overline{\iota^{n-1}|_{N(\varepsilon)}\circ\tau\circ\sigma\circ\tau(U_\varepsilon)} ,$$
$$(U_{W}^{n\#})^\# = U_W^{n\#} - \overline{\iota^n|_{N(\varepsilon)}\circ\tau(U_{1/\varepsilon})} .$$

We define $\mathfrak{U}(n)^\#$ to be

$$(\mathfrak{U}(n) - \{(U_W^{(n-1)\#})^\#, U_W^{n\#}\}) \cup \{((U_W^{(n-1)\#})^\#)^\#, (U_W^{n\#})^\#\} ,$$

and $\mathfrak{U}(n+1)$ to be $\mathfrak{U}(n)^\#\cup(\mathfrak{U}^{n+1})^\#$. Then $\mathfrak{U}(n+1)$ has the propety $(**)_{n+1}$. Therefore $\mathfrak{U}(n)$ is defined for any $n\in\boldsymbol{N}$ with the property $(**)_n$.

Now we proved that $(\mathcal{M}(n), B^{(n)}, \varpi)$ is the complete, effectively parametrized family of small deformation of $M(n)$ by induction. We have already shown that

$$(***)_2 \quad \begin{cases} H^1(\mathfrak{U}(2), \Theta) \cong H^1(M(2), \Theta) , \\ \rho_0 \colon T_0(B(2)) \overset{\cong}{\to} H^1(M(2), \Theta) \end{cases}$$

Assume $(***)_n$ and that $\theta^{(n)}\colon T_0(B(n))\to Z^1(\mathfrak{U}(n), \Theta)$ is defined so that $i([\theta^{(n)}(\cdot)])=\rho_0(\cdot)$, where i is the inclusion map of $H^1(\mathfrak{U}(n), \Theta)$ in $H^1(M(n), \Theta)$. We define $\theta^{(n+1)}$ of $T_0(B(n+1))$ to $Z^1(\mathfrak{U}(n+1), \theta)$ as follows. Let $\theta^{(n+1)}(\partial/\partial t_i^{(n)})$ take the same value as $\theta^{(n)}(\partial/\partial t_i^{(n)})$ on the intersections of any distinct two members of $\mathfrak{U}(n)^\#$ and take zero on other intersections, where $t_i^{(n)}$ is the parameter of $B(n)$ *f*or $i = 1, \cdots, 15n-12$. Let $\theta^{(n+1)}(\partial/\partial t_i^{n+1})$ take the value $\theta^{(1)}(\partial/\partial t_i)$ on the intersection of any distinct two of $(\mathfrak{U}^{n+1})^\#$ and take zero on other intersections, for $i = 1, \cdots, 7$. As for $\theta^{(n+1)}(\partial/\partial t_i')$, let

$$\theta^{(n+1)}(\partial/\partial t_1'')((U_W^{n\#})^\# \cap U_W^{n+1\#}) = y_0(x_0\partial/\partial x_0 + y_0\partial/\partial y_0 + z_0\partial/\partial z_0) ,$$
$$\theta^{(n+1)}(\partial/\partial t_2'')((U_W^{n\#})^\# \cap U_W^{n+1\#}) = z_0(x_0\partial/\partial x_0 + y_0\partial/\partial y_0 + z_0\partial/\partial z_0) ,$$
$$\theta^{(n+1)}(\partial/\partial t_3'')((U_W^{n\#})^\# \cap U_W^{n+1\#}) = y_0\partial/\partial x_0 ,$$
$$\theta^{(n+1)}(\partial/\partial t_4'')((U_W^{n\#})^\# \cap U_W^{n+1\#}) = z_0\partial/\partial x_0 ,$$
$$\theta^{(n+1)}(\delta/\delta t_5''(((U_W^{n\#})^\# \cap U_W^{n+1\#}) = \partial/\partial y_0 ,$$
$$\theta^{(n+1)}(\partial/\partial t_6'')((U_W^{n\#})^\# \cap U_W^{n+1\#}) = x_0\partial/\partial y_0 ,$$
$$\theta^{(n+1)}(\partial/\partial t_7'')((U_W^{n\#})^\# \cap U_W^{n+1\#}) = \partial/\partial z_0 ,$$
$$\theta^{(n+1)}(\partial/\partial t_8'')((U_W^{n\#}) \cap U_W^{n+1\#}) = x_0\partial/\partial z_0 ,$$

and $\theta^{(n+1)}(\partial/\partial t_1'')$ take zero on other intersection of any distinct two of $\mathfrak{U}(n+1)$. In the following, we write θ instead of $\theta^{(n+1)}$ for simplicity. Suppose that we have $v \in C^0(\mathfrak{U}(n+1), \Theta)$ such that

$$\sum_{i=1}^{15n-12} \alpha_i \theta(\partial/\partial t_i^{(n)}) + \sum_{i=1}^{7} \beta_i \theta(\partial/\partial t_i^{n+1}) + \sum_{i=1}^{8} \gamma_i \theta(\partial/\partial t_i'') = \delta v .$$

By induction hypothesis, the above equality reduces to

$$\sum_{i=1}^{8} \gamma_i \theta(\partial/\partial t_i'') = r_* v'' - v'$$

where $v' \in H^0(M^{n\#}, \Theta)$, $v'' \in H^0(M(n)^\#, \Theta)$ and $r = r_0$. Using the calculations in the proof of Propositions 3.2, we have

$$\begin{aligned}(\gamma_1 y_0 + \gamma_2 z_0)&(x_0\partial/\partial x_0 + y_0\partial/\partial y_0 + z_0\partial/\partial z_0) + \gamma_3 y_0 y_0 \partial/\partial x_0 + \gamma_4 z_0 \partial/\partial x_0 + \gamma_5 \partial/\partial y_0 \\ &+ \gamma_6 x_0 \partial/\partial y_0 + \gamma_7 \partial/\partial z_0 + \gamma_8 x_0 \partial/\partial z_0 \\ &= \{a - a_1 + (b - a_2)x_0 + (c - d)x_0^2\}\partial/\partial x_0 + \{b_1 y_0 + (-c - b_2)z_0 \\ &+ (c - d)x_0 y_0\}\partial/\partial y_0 + \{(a - c_1)y_0 + (b - c_2)z_0 + (c - d)x_0 z_0)\partial/\partial z_0 .\end{aligned}$$

This asserts that $[\theta(\partial/\partial t_1^{(n)})], \cdots, [\theta(\partial/\partial t_{15n-12}^{(n)})]$, $[\theta(\partial/\partial t_1^{n+1})], \cdots, [\theta(\partial/\partial t_7^{n+1})]$, $[\theta(\partial/\partial t_1'')], \cdots, [\theta(\partial/\partial t_8'')]$ are linearly independent. It is easily seen that that $\rho_0(\cdot) = i([\theta(\cdot)])$. Since $\dim H^1(M(n+1), \Theta) = 15n + 3$, ρ_0 is bijective. □

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