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ON A BARGMANN-TYPE TRANSFORM AND A HILBERT SPACE OF HOLOMORPHIC FUNCTIONS

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0. Introduction. In [1], Bargmann studied an integral transform of $L^2(\boldsymbol{R}^n)$ onto a Hilbert space consisting of entire holomorphic functions on $\boldsymbol{C}^n$. His transform may be regarded as a half-form pairing between real and complex polarizations of $\boldsymbol{R}^{2n} \cong \boldsymbol{C}^n$ (see [8, § 2]). In [7], Rawnsley showed that $\mathring{T}^*S^{n-1}$ (the cotangent bundle of the $(n-1)$-sphere minus its zero section) has a Kaehler structure with the Kaehler form equal to the natural symplectic form. Furthermore, he studied in [8] the half-form pairing between real and complex polarizations of $\mathring{T}^*S^{n-1}$, but it is not unitary. Also, we know that there does not exist a distinguished kernel, the definition of which is given in [2, IV. 5], for these polarizations. More precisely, there does exist a "distinguished kernel" defined in a neighborhood of the diagonal of $\mathring{T}^*S^{n-1} \times \mathring{T}^*S^{n-1}$, but it does not extend globally. This "kernel", however, suggests us to consider an integral transform:

$$\mathscr{F}: f \mapsto \hat{f}(z) = \int_{S^{n-1}} e^{x \cdot z} f(x) dS(x) ,$$

where $z \in \boldsymbol{C}^n$, $z^2 = 0$ (for the notations, see Section 1). Incidentally, transformations of the same form as $\mathscr{F}$ have been studied by several authors (see, for example, [3, § 4], [6, § 7], [4, Theorem 2.10], [8, p. 175] and [5, § 4]). In the present note, motivated by these works, we consider the integral transform $\mathscr{F}$ of $L^2(S^{n-1})$ into a space consisting of holomorphic functions on the Kaehler manifold $\mathring{T}^*S^{n-1} = \{z \in \boldsymbol{C}^n \mid z^2 = 0, z \neq 0\}$. $\mathscr{F}$ is injective. In Section 2, we construct, in the case of even-dimensional spheres, a "Plancherel measure" on $\mathring{T}^*S^{n-1}$ to describe the image of $L^2(S^{n-1})$ under this transform. The "inversion formula" is also obtained. As an application, we give in Section 3 an integral representation of a one-parameter group of unitary transformations on $L^2(S^{n-1})$ generated by a pseudo-differential operator $-i\{\Delta + (n-2)^2/4\}^{1/2}$, where Δ is the Laplace-Beltrami operator on S^{n-1} (cf. [8, p. 177]).

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For the sake of simplicity, we shall assume $n \geqq 3$ throughout this paper. In Sections 2 and 3, we furthermore assume that n is odd. The reason why we exclude the case of even n is that, in this case, we cannot identify the function ρ_n which satisfies the equation in Lemma 2.1. The Lie differentiation with respect to a vector field X is denoted by $\mathscr{L}_X$. Volume forms and measures are used interchangeably.

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Added September 6, 1985. R. Wada has informed us that she is able to identify the function ρ_n also for even n, and to remove the assumption in Section 2 that n is odd.

1. Preliminaries. Let $\boldsymbol{R}^n = \{x = (x_1, \cdots, x_n)\}$ be the n-space with the inner product $x \cdot y = \sum x_i y_i$ and the norm $|x| = (x \cdot x)^{1/2}$, and $S^{n-1} = \{x \in \boldsymbol{R}^n \mid |x| = 1\}$ be the unit sphere. The volume element on S^{n-1} is denoted by dS. The volume of S^{n-1} is given by $\mathrm{vol}(S^{n-1}) = 2\pi^{n/2}/\Gamma(n/2)$. Let $L^2(S^{n-1})$ be the Hilbert space of square-integrable functions on S^{n-1} with the following inner product and norm:

$$\langle f, g\rangle_S = \int_{S^{n-1}} \bar{f} g dS\ , \quad \|f\|_S = \langle f, f\rangle_S^{1/2}\ .$$

The subspace of $L^2(S^{n-1})$ consisting of spherical harmonics of degree m is denoted by $H_m(S^{n-1})$, $m = 0, 1, 2, \cdots$. The following is well-known (see, for example, [4, § 3]):

LEMMA 1.1. (i) $\dim H_m(S^{n-1}) = (2m+n-2)\Gamma(m+n-2)/\Gamma(n-1) \times \Gamma(m+1)$.

(ii) *The subspaces* $H_m(S^{n-1})$, $m = 0, 1, 2, \cdots$, *are mutually orthogonal with respect to the inner product* $\langle\ ,\ \rangle_S$.

(iii) *Let* $f_m \in H_m(S^{n-1})$, $m = 0, 1, 2, \cdots$. *Then* $f = \sum f_m$ *belongs to* $L^2(S^{n-1})$ *if and only if* $\sum \|f_m\|_S^2 < \infty$, *and in that case,* $\|f\|_S^2 = \sum \|f_m\|_S^2$.

(iv) *For any* $z \in \boldsymbol{C}^n$ *with* $z^2 = 0$, *the function on* S^{n-1}, $x \mapsto (x \cdot z)^m$, *belongs to* $H_m(S^{n-1})$, *where* $z^2 = \sum z_i^2$ *and* $x \cdot z = \sum x_i z_i$. *Furthermore,* $H_m(S^{n-1})$ *is spanned by these functions.*

For any $1 \leqq i_1, \cdots, i_m \leqq n$, let us define an element of $H_m(S^{n-1})$ by

$$h_{i_1 \cdots i_m} = \left(\prod_{k=0}^{m-1} (2-n-2k)\right)^{-1} [\partial^m |x|^{2-n} / \partial x_{i_1} \cdots \partial x_{i_m}] \,|\, S^{n-1}\ ,$$

where we assume $n \geqq 3$. Note that $H_m(S^{n-1})$ is spanned by these functions.

LEMMA 1.2. (i) $h_{i_1\cdots i_m}(x) = x_{i_1}\cdots x_{i_m} - 1/(2(2m+n-4))\sum_{a\neq b}\delta_{i_a i_b} x_{i_1}\cdots \hat{x}_{i_a}\cdots\hat{x}_{i_b}\cdots x_{i_m} + h'(x)$, *where* $h' \in \bigoplus_{k=0}^{m-4} H_k(S^{n-1})$.

(ii) $x_j h_{i_1\cdots i_m} = h_{i_1\cdots i_m j} + 1/(2m+n-2)\sum_{a=1}^m \delta_{i_a j} h_{i_1\cdots\hat{i}_a\cdots i_m} - 1/((2m+n-2)(2m+n-4))\sum_{a\neq b}\delta_{i_a i_b} h_{i_1\cdots\hat{i}_a\cdots\hat{i}_b\cdots i_m j}$, *where* x_j *denotes the function* $x \mapsto x_j$ *on* S^{n-1}.

For the proof of (ii), recall that the multiplication of x_j is a symmetric operator on $L^2(S^{n-1})$. Then, from (i) and the orthogonality of the subspaces $H_m(S^{n-1})$, $m = 0, 1, 2, \cdots$, we have $x_j h_{i_1\cdots i_m} \in H_{m+1}(S^{n-1}) \oplus H_{m-1}(S^{n-1})$.

LEMMA 1.3. *For any* $m = 1, 2, \cdots$, *we have*

$$\begin{aligned}\langle h_{i_1\cdots i_m}, h_{j_1\cdots j_m}\rangle_S &= \frac{1}{2m+n-2}\sum_{a=1}^m \delta_{i_a j_m}\langle h_{i_1\cdots\hat{i}_a\cdots i_m}, h_{j_1\cdots j_{m-1}}\rangle_S \\ &\quad - \frac{1}{(2m+n-2)(2m+n-4)}\sum_{a\neq b}\delta_{i_a i_b}\langle h_{i_1\cdots\hat{i}_a\cdots\hat{i}_b\cdots i_m j_m}, h_{j_1\cdots j_{m-1}}\rangle_S .\end{aligned}$$

PROOF. Let ξ_j denote the restriction to S^{n-1} of the vector field $\sum_{i=1}^n (\delta_{ji} - x_j x_i)\partial/\partial x_i$ on $\boldsymbol{R}^n$. Then ξ_j is tangent to S^{n-1}. Since $\mathscr{L}_{\xi_j} dS = -(n-1)x_j dS$, we have from

$$\int_{S^{n-1}} \mathscr{L}_{\xi_j}(\bar{f} g dS) = 0$$

that $\xi_j - (n-1)x_j/2$ is a skew-symmetric operator on $(C^\infty(S^{n-1}), \langle\ ,\ \rangle_S)$. Then, by Lemma 1.2, we have

$$\begin{aligned}\Big(\xi_j - \frac{n-1}{2}x_j\Big)h_{i_1\cdots i_m} &= -\Big(m + \frac{n-1}{2}\Big)h_{i_1\cdots i_m j} + \frac{2m+n-3}{2(2m+n-2)}\sum_{a=1}^m \delta_{i_a j} h_{i_1\cdots\hat{i}_a\cdots i_m} \\ &\quad - \frac{2m+n-3}{2(2m+n-2)(2m+n-4)}\sum_{a\neq b}\delta_{i_a i_b} h_{i_1\cdots\hat{i}_a\cdots\hat{i}_b\cdots i_m j} .\end{aligned}$$

Using this formula on both sides of the equation:

$$\Big\langle\Big(\xi_{j_m} - \frac{n-1}{2}x_{j_m}\Big)h_{i_1\cdots i_m}, h_{j_1\cdots j_{m-1}}\Big\rangle_S = -\Big\langle h_{i_1\cdots i_m}, \Big(\xi_{j_m} - \frac{n-1}{2}x_{j_m}\Big)h_{j_1\cdots j_{m-1}}\Big\rangle_S ,$$

we obtain our lemma.

Now, we shall consider an integral transform (cf. [3, § 4] and [6, § 7]): For any $f \in L^2(S^{n-1})$ and $z \in \boldsymbol{C}^n$, let us define

$$\hat{f}(z) = \int_{S^{n-1}} e^{x\cdot z} f(x) dS(x) .$$

Then we have:

LEMMA 1.4. (i) *$\hat{f}$ is an entire function on C^n.*

(ii) $|\hat{f}(z)| \leqq (\mathrm{vol}(S^{n-1}))^{1/2} \|f\|_S e^{|\mathrm{Re}\, z|}$.

(iii) *If $f \in L^2(S^{n-1})$, $f = \sum f_m$ with $f_m \in H_m(S^{n-1})$, then $\sum \hat{f}_m$ converges to $\hat{f}$ uniformly on any bounded set in C^n.*

PROPOSITION 1.5. *If $z^2 = 0$, then*

$$\hat{h}_{i_1\cdots i_m}(z) = \frac{\mathrm{vol}(S^{n-1})\Gamma(n/2)}{2^m \Gamma(m + n/2)} z_{i_1} \cdots z_{i_m} .$$

PROOF. We shall prove this by induction on m. If $m = 0$, then both sides of the equation are equal to $\mathrm{vol}(S^{n-1})$. Now, let $m > 0$ and assume that the proposition holds for $m - 1$. Since, by (i) of Lemma 1.2,

$$\int_{S^{n-1}} x_{j_1} \cdots x_{j_m} h_{i_1\cdots i_m}(x) dS(x) = \langle h_{i_1\cdots i_m}, h_{j_1\cdots j_m} \rangle_S ,$$

we have

$$\hat{h}_{i_1\cdots i_m}(z) = \frac{1}{m!} \sum_{j_1,\cdots,j_m} \langle h_{i_1\cdots i_m}, h_{j_1\cdots j_m} \rangle_S z_{j_1} \cdots z_{j_m} .$$

Then, since $z^2 = 0$, by Lemma 1.3 and the induction assumption, we see that the proposition holds also for m.

Let $\pi: \mathring{T}^*S^{n-1} \to S^{n-1}$ be the bundle consisting of non-zero cotangent vectors to S^{n-1}. The canonical one-form θ on T^*S^{n-1} is defined by $\theta_\alpha(X) = \alpha(\pi_* X)$ for any $\alpha \in \mathring{T}^*S^{n-1}$ and $X \in T_\alpha(\mathring{T}^*S^{n-1})$. The symplectic form and the Liouville volume form on $\mathring{T}^*S^{n-1}$ are given by $\Omega = -d\theta$ and $dM = (-1)^{(n-1)(n-2)/2}((n-1)!)^{-1}\Omega^{n-1}$, respectively. For any real-valued function $h \in C^\infty(\mathring{T}^*S^{n-1})$, the unique vector field X_h on $\mathring{T}^*S^{n-1}$ for which $X_h \rfloor \Omega = dh$ is called the Hamiltonian vector field of h. By means of the metric, we may identify $\mathring{T}^*S^{n-1}$ with the space $\mathring{T}S^{n-1}$ consisting of non-zero tangent vectors to S^{n-1}, which is identified with

$$M = \{(x, y) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \,|\, |x| = 1,\ x\cdot y = 0,\ y \neq 0\} .$$

Furthermore, by an injection $(x, y) \mapsto z = |y|x + \sqrt{-1}y$ of M into C^n, M is identified with a complex cone $\{z \in C^n \,|\, z^2 = 0,\ z \neq 0\}$. This identification gives M a complex structure J. It is known (see [7] and [8, p. 173]) that J is compatible with the symplectic structure, i.e., $(X, Y) \mapsto -\Omega(J(X), Y)$ is a Kaehler metric on M.

Let Holo(M) and $P_m(M)$ denote, respectively, the restrictions to M of entire holomorphic functions and homogeneous polynomials of degree m on $\boldsymbol{C}^n$. For any $\varphi \in \mathrm{Holo}(M)$, there exists a unique $\varphi_m \in P_m(M)$ such that $\varphi = \sum_{m=0}^{\infty} \varphi_m$; uniformly convergent on any bounded set in M. If we define $\psi_{i_1\cdots i_m} \in P_m(M)$ by $\psi_{i_1\cdots i_m}(z) = z_{i_1} \cdots z_{i_m}$, then $P_m(M)$ is spanned by these functions. Since $z_1^2 + \cdots + z_n^2 = 0$, we have $\dim P_m(M) = \dim H_m(S^{n-1})$. (Cf. [5, § 3].)

The unit cotangent bundle $T_1^*S^{n-1}$ to S^{n-1} is identified with $N = \{(x, y) \in M \mid |y| = 1\}$. The canonical volume element on N is denoted by dN. If we define a function $r \in C^\infty(M)$ and a projection $p\colon M \to N$ by $r(x, y) = |y|$ and $p(x, y) = (x, |y|^{-1}y)$, respectively, then we have $dM = p^*dN \wedge r^{n-2}dr$. An inner product in $C^\infty(N)$ is defined by

$$\langle \varphi, \psi \rangle_N = \int_N \bar{\varphi}\psi dN\,.$$

The restriction of $\psi_{i_1\cdots i_m}$ onto N will also be denoted by the same letter.

LEMMA 1.6. (i) *If* $l \neq m$, *then* $\langle \psi_{i_1\cdots i_l}, \psi_{j_1\cdots j_m} \rangle_N = 0$.

(ii) *For any* $m = 1, 2, \cdots$, *we have*

$$\begin{aligned}\frac{(m+n-3)(2m+n-2)}{2}&\langle \psi_{i_1\cdots i_m}, \psi_{j_1\cdots j_m} \rangle_N \\ &= (2m+n-4) \sum_{a=1}^{m} \delta_{i_a j_m} \langle \psi_{i_1\cdots \hat{i}_a \cdots i_m}, \psi_{j_1\cdots j_{m-1}} \rangle_N \\ &\quad - \sum_{a\neq b} \delta_{i_a i_b} \langle \psi_{i_1\cdots \hat{i}_a \cdots \hat{i}_b \cdots i_m j_m}, \psi_{j_1\cdots j_{m-1}} \rangle_N\,.\end{aligned}$$

PROOF. (i) The Hamiltonian vector field of the function r is given by

$$X_r|_{(x,y)} = \sum_{i=1}^{n} \left(\frac{1}{|y|} y_i \frac{\partial}{\partial x_i} - |y| x_i \frac{\partial}{\partial y_i} \right).$$

X_r is tangent to N. Denoting the restriction of X_r to N also by X_r, we have $\mathscr{L}_{X_r} dN = 0$. It follows that $\langle X_r\varphi, \psi \rangle_N = -\langle \varphi, X_r\psi \rangle_N$ for any $\varphi, \psi \in C^\infty(N)$. On the other hand, from $X_r z_i = -\sqrt{-1} z_i$ we have $X_r \psi_{i_1\cdots i_m} = -\sqrt{-1}\, m \psi_{i_1\cdots i_m}$. Then (i) follows immediately.

(ii) The Hamiltonian vector field X_j of the function $(x, y) \mapsto y_j$ on M is given by

$$X_j|_{(x,y)} = \sum_{i=1}^{n} \left\{ (\delta_{ji} - x_j x_i) \frac{\partial}{\partial x_i} + (x_j y_i - y_j x_i) \frac{\partial}{\partial y_i} \right\}.$$

Since $[X_j, \sum_{k=1}^{n} y_k (\partial/\partial y_k)] = 0$, X_j induces a tangent vector field η_j to N. η_j is given by

$$\eta_j|_{(x,y)} = \sum_{i=1}^{n} \left\{(\delta_{ji} - x_j x_i)\frac{\partial}{\partial x_i} - y_j x_i \frac{\partial}{\partial y_i}\right\} .$$

Since $\mathscr{L}_{\eta_j} dN = -2^{-1}(n-1)(z_j + \bar{z}_j)dN$, we have from

$$\int_N \mathscr{L}_{\eta_j}(\bar{\varphi}\psi dN) = 0 ,$$

$$\langle \eta_j\varphi, \psi\rangle_N + \langle \varphi, \eta_j\psi\rangle_N = \frac{n-1}{2}(\langle z_j\varphi, \psi\rangle_N + \langle \varphi, z_j\psi\rangle_N)$$

for any $\varphi, \psi \in C^\infty(N)$, where z_j denotes the function $(x, y) \mapsto z_j = x_j + \sqrt{-1}y_j$ on N. If we put $j = j_m$, $\varphi = \psi_{i_1\cdots i_m}$ and $\psi = \psi_{j_1\cdots j_{m-1}}$, then using (i) we have

$$\langle \eta_{j_m}\psi_{i_1\cdots i_m}, \psi_{j_1\cdots j_{m-1}}\rangle_N + \langle \psi_{i_1\cdots i_m}, \eta_{j_m}\psi_{j_1\cdots j_{m-1}}\rangle_N = \frac{n-1}{2}\langle \psi_{i_1\cdots i_m}, \psi_{j_1\cdots j_m}\rangle_N .$$

Now, since $\eta_j(z_k) = \delta_{jk} - (1/2)z_j z_k - (1/2)z_j\bar{z}_k$, we have

$$\begin{aligned}&\langle \eta_{j_m}\psi_{i_1\cdots i_m}, \psi_{j_1\cdots j_{m-1}}\rangle_N \\ &\quad = \sum_{a=1}^{m} \delta_{i_a j_m}\langle \psi_{i_1\cdots \hat{i}_a\cdots i_m}, \psi_{j_1\cdots j_{m-1}}\rangle_N - \frac{1}{2}\sum_{a=1}^{m}\langle \psi_{i_1\cdots\hat{i}_a\cdots i_m j_m}, \psi_{j_1\cdots j_{m-1}i_a}\rangle_N\end{aligned}$$

and

$$\langle \psi_{i_1\cdots i_m}, \eta_{j_m}\psi_{j_1\cdots j_{m-1}}\rangle_N = -\frac{m-1}{2}\langle \psi_{i_1\cdots i_m}, \psi_{j_1\cdots j_m}\rangle_N .$$

It follows that

$$\begin{aligned}&\frac{m+n-2}{2}\langle \psi_{i_1\cdots i_m}, \psi_{j_1\cdots j_m}\rangle_N \\ &\quad = \sum_{a=1}^{m} \delta_{i_a j_m}\langle \psi_{i_1\cdots \hat{i}_a\cdots i_m}, \psi_{j_1\cdots j_{m-1}}\rangle_N - \frac{1}{2}\sum_{a=1}^{m}\langle \psi_{i_1\cdots\hat{i}_a\cdots i_m j_m}, \psi_{j_1\cdots j_{m-1}i_a}\rangle_N ,\end{aligned}$$

from which we obtain (ii).

LEMMA 1.7. *For any* $m = 0, 1, 2, \cdots$, *we have*

$$\langle h_{i_1\cdots i_m}, h_{j_1\cdots j_m}\rangle_S = c_m\langle \hat{h}_{i_1\cdots i_m}, \hat{h}_{j_1\cdots j_m}\rangle_N ,$$

where

$$c_m = \frac{\Gamma(m+n/2)\Gamma(m+1)\dim H_m(S^{n-1})}{(\mathrm{vol}(S^{n-1}))^2\mathrm{vol}(S^{n-2})\Gamma(n/2)} .$$

PROOF. By Proposition 1.5,

$$\hat{h}_{i_1\cdots i_m} = \frac{\mathrm{vol}(S^{n-1})\Gamma(n/2)}{2^m\Gamma(m+n/2)}\psi_{i_1\cdots i_m} .$$

Hence it suffices to show that

$$\langle h_{i_1\cdots i_m}, h_{j_1\cdots j_m}\rangle_S = c'_m\langle \psi_{i_1\cdots i_m}, \psi_{j_1\cdots j_m}\rangle_N ,$$

where

$$c'_m = \frac{\Gamma(n/2)\Gamma(m+1)\dim H_m(S^{n-1})}{2^{2m}\operatorname{vol}(S^{n-2})\Gamma(m+n/2)} .$$

We shall show this by induction on m. If $m=0$, then both sides of the equation are equal to $\operatorname{vol}(S^{n-1})$. Now, let $m>0$ and assume that the equality holds for $m-1$. Then by Lemma 1.3 and (ii) of Lemma 1.6, we have

$$\begin{aligned}&\langle h_{i_1\cdots i_m}, h_{j_1\cdots j_m}\rangle_S\\ &= \frac{c'_{m-1}}{(2m+n-2)(2m+n-4)}\Big\{(2m+n-4)\sum_{a=1}^{m}\delta_{i_a j_m}\langle \psi_{i_1\cdots \hat{i}_a\cdots i_m}, \psi_{j_1\cdots j_{m-1}}\rangle_N\\ &\quad - \sum_{a\neq b}\delta_{i_a i_b}\langle \psi_{i_1\cdots \hat{i}_a\cdots \hat{i}_b\cdots i_m j_m}, \psi_{j_1\cdots j_{m-1}}\rangle_N\Big\}\\ &= \frac{(m+n-3)c'_{m-1}}{2(2m+n-4)}\langle \psi_{i_1\cdots i_m}, \psi_{j_1\cdots j_m}\rangle_N = c'_m\langle \psi_{i_1\cdots i_m}, \psi_{j_1\cdots j_m}\rangle_N .\end{aligned}$$

PROPOSITION 1.8 (cf. [5, § 4]). *For any $x\in S^{n-1}$, we have*

$$\int_N (x\cdot\bar{z})^m \psi_{i_1\cdots i_m}(z)\,dN(z) = \frac{\operatorname{vol}(S^{n-1})\operatorname{vol}(S^{n-2})2^m}{\dim H_m(S^{n-1})} h_{i_1\cdots i_m}(x) .$$

This proposition is proved by induction on m, where we use (ii) of Lemma 1.2 and (ii) of Lemma 1.6.

2. Hilbert space $P(M)$ and integral transform $\mathscr{F}$.

From now on, we shall assume that $n = 3, 5, 7, \cdots$.

LEMMA 2.1. *There exists a unique polynomial ρ_n which satisfies*

$$\int_0^\infty r^{2m+n-2}e^{-2r}\rho_n(r)dr = c_m$$

for all $m = 0, 1, 2, \cdots$.

PROOF. If there exists a polynomial $\rho_n(r)=\sum_k a_{n,k}r^k$ which satisfies the condition in our lemma, then the coefficients must satisfy $\sum_k a_{n,k}2^{-(k+2m+n-1)}\Gamma(k+2m+n-1)=c_m$ for all m. This condition is rewritten as

$$\begin{aligned}&\sum_k a_{n,k}\frac{\Gamma(k+2m+n-1)}{2^{k+1}\Gamma(2m+n-1)}\\ &\qquad = \frac{\pi^{1/2}(2m+n-2)\Gamma(m+n-2)}{(\operatorname{vol}(S^{n-1}))^2\operatorname{vol}(S^{n-2})\Gamma(n-1)\Gamma(n/2)\Gamma(m+(n-1)/2)}\end{aligned}$$

for all $m = 0, 1, 2, \cdots$. Since n is odd, both sides of the equation above are polynomials of m. Hence, $a_{n,k}$ are determined uniquely. The existence of ρ_n also follows from the above equation.

Note that the degree of the polynomial ρ_n is $(n-1)/2$, and the coefficient of the highest degree is positive. For example, we have $\rho_3(r) = a_{3,1}(r - 1/2)$, $\rho_5(r) = a_{5,2}(r^2 - r)$ and $\rho_7(r) = a_{7,3}(r^3 - r^2 - r/2)$. Unfortunately, since $a_{n,0} = 0$ and $a_{n,1} < 0$ for $n \geqq 5$, $\rho_n|(0, \infty)$ is not a positive function. It is to be desired that there exists a positive function on $(0, \infty)$ which satisfies the equation in Lemma 2.1. We also remark that for even n, there does not exist any polynomial which satisfies the condition in Lemma 2.1. This is the reason why we restrict our attention to the case of odd n.

Now, for any $\varphi, \psi \in \mathrm{Holo}(M)$, let us define

$$\langle \varphi, \psi \rangle_M = \int_M \overline{\varphi(z)}\psi(z)d\mu_n(z) ,$$

where $d\mu_n(z) = e^{-2|y|}\rho_n(|y|)dM(z)$, $z = |y|x + \sqrt{-1}y \in M$ (cf. [8, p. 174]). Although the measure $d\mu_n$ is not positive, we have:

THEOREM 2.2. *For any $\varphi \in P_l(M)$ and $\psi \in P_m(M)$,*

$$\langle \varphi, \psi \rangle_M = c_m \langle \varphi, \psi \rangle_N ,$$

where φ and ψ on the right hand side stand for the restrictions of φ and ψ onto N, respectively. In particular, $\langle\ ,\ \rangle_M$ is positive definite on $P_m(M)$, and $f \mapsto \hat{f}$ is a unitary isomorphism of $(H_m(S^{n-1}), \langle\ ,\ \rangle_S)$ onto $(P_m(M), \langle\ ,\ \rangle_M)$.

PROOF. Since $dM = p^*dN \wedge r^{n-2}dr$, we have, by (i) of Lemma 1.6 and Lemma 2.1,

$$\langle \varphi, \psi \rangle_M = \int_0^\infty r^{l+m+n-2}e^{-2r}\rho_n(r)dr \int_N \bar{\varphi}\psi dN = c_m \langle \varphi, \psi \rangle_N .$$

Then, the unitarity of $f \mapsto \hat{f}$ follows from Lemma 1.7.

The following lemma is due to Bargmann [1, p. 190].

LEMMA 2.3. *Let $S = \sum_{k=1}^\infty b_k$ be a series with non-negative real terms, let $\gamma_k(t)$, $t > 0$, be so chosen that (1) $0 \leqq \gamma_k(t) \leqq 1$, (2) $\lim_{t\to\infty} \gamma_k(t) = 1$, and set $S(t) = \sum \gamma_k(t)b_k$. S converges if and only if $S(t)$ are uniformly bounded, and in that case $S = \lim S(t)$.*

PROPOSITION 2.4. *Let $\varphi \in \mathrm{Holo}(M)$, $\varphi = \sum \varphi_m$ with $\varphi_m \in P_m(M)$. Then*

$$\langle \varphi, \varphi \rangle_M = \sum \langle \varphi_m, \varphi_m \rangle_M ,$$

i.e., either both sides are infinite, or both sides are finite and equal.

PROOF. For any $\sigma > 0$, let

$$I(\sigma) = \int_{M(\sigma)} |\varphi|^2 d\mu_n ,$$

where $M(\sigma) = \{z = |y|x + \sqrt{-1}y \in M \mid |y| \leqq \sigma\}$. Then $\sigma \mapsto I(\sigma)$ is, for large σ, monotone increasing and $\langle \varphi, \varphi \rangle_M = \lim_{\sigma \to \infty} I(\sigma)$. Since $\sum \varphi_m$ converges uniformly to φ on $M(\sigma)$, we have by (i) of Lemma 1.6 and Theorem 2.2,

$$\begin{aligned} I(\sigma) &= \sum_{l,m=0}^{\infty} \int_{M(\sigma)} \overline{\varphi_l(z)} \varphi_m(z) d\mu_n(z) = \sum_{l,m=0}^{\infty} \int_0^{\sigma} r^{l+m+n-2} e^{-2r} \rho_n(r) dr \int_N \overline{\varphi_l} \varphi_m dN \\ &= \sum_{m=0}^{\infty} \int_0^{\sigma} r^{2m+n-2} e^{-2r} \rho_n(r) dr \langle \varphi_m, \varphi_m \rangle_N = \sum_{m=0}^{\infty} \frac{c_m(\sigma)}{c_m} \langle \varphi_m, \varphi_m \rangle_M , \end{aligned}$$

where

$$c_m(\sigma) = \int_0^{\sigma} r^{2m+n-2} e^{-2r} \rho_n(r) dr .$$

Since there exists $\sigma_n > 0$ such that $c_m(\sigma) > 0$ for all $\sigma > \sigma_n$ and $m = 0, 1, 2, \cdots$, applying Lemma 2.3, we have the desired result.

Now, let us define

$$P(M) = \{\varphi \in \mathrm{Holo}(M) \mid \langle \varphi, \varphi \rangle_M < \infty\} .$$

Then it follows from Theorem 2.2 and Proposition 2.4 that $\langle\ ,\ \rangle_M$ is a Hermitian inner product in $P(M)$. The corresponding norm is denoted by $\|\ \|_M$.

THEOREM 2.5. *$\mathscr{F}: f \mapsto \hat{f}$ is a unitary isomorphism of $(L^2(S^{n-1}), \langle\ ,\ \rangle_S)$ onto $(P(M), \langle\ ,\ \rangle_M)$.*

PROOF. Let $f \in L^2(S^{n-1})$, $f = \sum f_m$ with $f_m \in H_m(S^{n-1})$. Then, by (iii) of Lemma 1.4, Proposition 2.4, Theorem 2.2 and (iii) of Lemma 1.1, we have

$$\|\hat{f}\|_M^2 = \sum \|\hat{f}_m\|_M^2 = \sum \|f_m\|_S^2 = \|f\|_S^2 < \infty .$$

It follows that $\hat{f} \in P(M)$ and that $\mathscr{F}$ is unitary. The surjectivity of $\mathscr{F}$ is also shown easily.

We have from Theorem 2.5 and (ii) of Lemma 1.4 the following:

COROLLARY 2.6. (i) *$(P(M), \langle\ ,\ \rangle_M)$ is a Hilbert space.* (ii) *For any $\varphi \in P(M)$ and $z = |y|x + \sqrt{-1}y \in M$,*

$$|\varphi(z)| \leqq (\mathrm{vol}(S^{n-1}))^{1/2} \|\varphi\|_M e^{|y|} .$$

From (ii) of Corollary 2.6, it follows that, for a fixed $w \in M$, the map $\varphi \mapsto \varphi(w)$ defines a bounded linear functional on $P(M)$. It is necessarily of the form

$$\varphi(w) = \langle e_w, \varphi \rangle_M$$

with a uniquely defined $e_w \in P(M)$. If we define function on $M \times M$ by

$$K(w, z) = \int_{S^{n-1}} e^{x \cdot w} e^{x \cdot \bar{z}} dS(x) ,$$

then $\overline{K(w, z)} = K(z, w)$ and $\overline{K(w, \cdot)} \in P(M)$ immediately from the definition.

LEMMA 2.7 (cf. [1, §1c]).

$$e_w(z) = \overline{K(w, z)} .$$

PROOF. It is sufficient to show that

$$\langle \overline{K(w, \cdot)}, \psi_{i_1 \cdots i_m} \rangle_M = \psi_{i_1 \cdots i_m}(w) .$$

Making use of Theorem 2.2, Lemma 1.6 and Propositions 1.8 and 1.5, we have

$$\begin{aligned}
\langle \overline{K(w, \cdot)}, \psi_{i_1 \cdots i_m} \rangle_M &= \int_M \left(\int_{S^{n-1}} e^{x \cdot w} e^{x \cdot \bar{z}} dS(x) \right) \psi_{i_1 \cdots i_m}(z) d\mu_n(z) \\
&= \int_{S^{n-1}} e^{x \cdot w} \left(\int_M e^{x \cdot \bar{z}} \psi_{i_1 \cdots i_m}(z) d\mu_n(z) \right) dS(x) \\
&= \frac{1}{m!} \int_{S^{n-1}} e^{x \cdot w} \left(\int_M (x \cdot \bar{z})^m \psi_{i_1 \cdots i_m}(z) d\mu_n(z) \right) dS(x) \\
&= \frac{c_m}{m!} \int_{S^{n-1}} e^{x \cdot w} \left(\int_N (x \cdot \bar{z})^m \psi_{i_1 \cdots i_m}(z) \, dN(z) \right) dS(x) \\
&= \frac{c_m \operatorname{vol}(S^{n-1}) \operatorname{vol}(S^{n-2}) 2^m}{m! \dim H_m(S^{n-1})} \int_{S^{n-1}} e^{x \cdot w} h_{i_1 \cdots i_m}(x) dS(x) \\
&= \psi_{i_1 \cdots i_m}(w) .
\end{aligned}$$

K is the reproducing kernel for $P(M)$, i.e.,

$$\varphi(w) = \int_M K(w, z) \varphi(z) d\mu_n(z) .$$

Now, we shall consider the inverse operator $\mathscr{F}^{-1}$. Let $P^{(\lambda)}(M) = \{\varphi \in \mathrm{Holo}(M) \mid \text{for a suitable } c > 0,\ |\varphi(z)| \leqq c e^{\lambda |y|} \text{ for all } z = |y| x + \sqrt{-1} y \in M\}$ $(0 < \lambda < 1)$. Then $P^{(\lambda)}(M)$ is a subspace of $P(M)$. If, for each $\varphi \in P(M)$, we define $\varphi^{(\lambda)}$ by $\varphi^{(\lambda)}(z) = \varphi(\lambda z)$, then $\varphi^{(\lambda)} \in P^{(\lambda)}(M)$.

LEMMA 2.8 (cf. [1, p. 197]). (i) *$\varphi \in P(M)$ if and only if all $\varphi^{(\lambda)} \in P(M)$, $0 < \lambda < 1$, and their norms $\|\varphi^{(\lambda)}\|_M$ are uniformly bounded.*

(ii) *If $\varphi \in P(M)$, then $\|\varphi - \varphi^{(\lambda)}\|_M \to 0$ as $\lambda \to 1$.*

PROOF. Let $\varphi \in \mathrm{Holo}(M)$, $\varphi = \sum \varphi_m$ with $\varphi_m \in P_m(M)$. Then we have $\varphi^{(\lambda)}(z) = \varphi(\lambda z) = \sum \lambda^m \varphi_m(z)$. It follows from Proposition 2.4 that $\|\varphi^{(\lambda)}\|_M^2 = \sum \lambda^{2m} \|\varphi_m\|_M^2$. Then by Lemma 2.3 we have (i). (ii) follows immediately from $\|\varphi - \varphi^{(\lambda)}\|_M^2 = \sum (1-\lambda^m)^2 \|\varphi_m\|_M^2$.

THEOREM 2.9 (cf. [1, p. 202]). *If $\varphi \in P^{(\lambda)}(M)$ for some λ, $0 < \lambda < 1$, then*

$$(\mathscr{F}^{-1}\varphi)(x) = \int_M e^{x\cdot\bar{z}}\varphi(z)d\mu_n(z) ,$$

for any $x \in S^{n-1}$.

PROOF. Since $\varphi \in P^{(\lambda)}(M)$, the integration converges absolutely. It suffices to prove that

$$\int_{S^{n-1}} e^{x\cdot w}\left(\int_M e^{x\cdot\bar{z}}\varphi(z)d\mu_n(z)\right)dS(x) = \varphi(w) ,$$

which we show easily by interchanging integrations and using the reproducing property of K.

COROLLARY 2.10 (cf. [1, (2.14)]). *For any $\varphi \in P(M)$,*

$$(\mathscr{F}^{-1}\varphi)(x) = \operatorname*{Lim}_{\lambda\to 1} \int_M e^{x\cdot\bar{z}}\varphi(\lambda z)d\mu_n(z) ,$$

where Lim *means the strong convergence in $L^2(S^{n-1})$.*

We also have another explicit expression for $\mathscr{F}^{-1}$.

THEOREM 2.11 (cf. [1, (2.15)]). *For any $\varphi \in P(M)$,*

$$(\mathscr{F}^{-1}\varphi)(x) = \operatorname*{Lim}_{\sigma\to\infty} \int_{M(\sigma)} e^{x\cdot\bar{z}}\varphi(z)d\mu_n(z) .$$

PROOF. Let $\varphi = \sum \varphi_m$ with $\varphi_m \in P_m(M)$. Define, for $x \in S^{n-1}$,

$$f^{(\sigma)}(x) = \int_{M(\sigma)} e^{x\cdot\bar{z}}\varphi(z)d\mu_n(z)$$

and

$$f_m^{(\sigma)}(x) = \int_{M(\sigma)} e^{x\cdot\bar{z}}\varphi_m(z)d\mu_n(z) .$$

Then, by Propositions 1.5 and 1.8, we have for any $w \in M$,

$$\begin{aligned}(\mathscr{F}f_m^{(\sigma)})(w) &= \int_{S^{n-1}} e^{x\cdot w}\left(\int_{M(\sigma)} e^{x\cdot\bar{z}}\varphi_m(z)d\mu_n(z)\right)dS(x) \\ &= \frac{c_m(\sigma)}{m!}\int_{S^{n-1}} e^{x\cdot w}\left(\int_N (x\cdot\bar{z})^m\varphi_m(z)\,dN(z)\right)dS(x) = \frac{c_m(\sigma)}{c_m}\varphi_m(w) .\end{aligned}$$

By the uniform convergence of $\varphi = \sum \varphi_m$ on $M(\sigma)$, we have

$$
\begin{aligned}
(\mathscr{F}f^{(\sigma)})(w) &= \int_{S^{n-1}\times M(\sigma)} e^{x\cdot w}e^{x\cdot\bar z}\varphi(z)d\mu_n(z)dS(x) \\
&= \sum \int_{S^{n-1}\times M(\sigma)} e^{x\cdot w}e^{x\cdot\bar z}\varphi_m(z)d\mu_n(z)dS(x) \\
&= \sum (\mathscr{F}f_m^{(\sigma)})(w) = \sum \frac{c_m(\sigma)}{c_m}\varphi_m(w) .
\end{aligned}
$$

It follows from Proposition 2.4 that

$$
\|\varphi - \mathscr{F}f^{(\sigma)}\|_M^2 = \sum\left(1 - \frac{c_m(\sigma)}{c_m}\right)^2 \|\varphi_m\|_M^2 \to 0
$$

as $\sigma \to \infty$. Here recall that there exists a constant $\sigma_n > 0$ such that $c_m(\sigma) > 0$ for any $\sigma > \sigma_n$ and $m = 0, 1, 2, \cdots$. Since $\mathscr{F}$ is a unitary isomorphism, we have $\mathscr{F}^{-1}\varphi = \mathrm{Lim}_{\sigma\to\infty} f^{(\sigma)}$.

3. An application. The mapping $\mathscr{F}$ establishes a unitary isomorphism between the linear operators on $P(M)$ and those on $L^2(S^{n-1})$. In this section, we shall consider a one-parameter group of unitary transformations, which is easily analyzed on $P(M)$, and translate the results into the language of $L^2(S^{n-1})$ (see [1, § 3] and [8, *p.* 177]).

The one-parameter group of canonical transformations on M generated by the Hamiltonian vector field X_r is given by $\phi_t\colon z \mapsto e^{it}z$. Since $X_r r = 0$ and $\mathscr{L}_{X_r} dM = 0$, ϕ_t preserves the measure $d\mu_n$ as well as the complex structure J on M. Hence ϕ_t induces a unitary transformation $\varphi \mapsto \varphi\circ\phi_{-t}$ on $P(M)$. Let us define a one-parameter group $\{V_t \mid t \in \boldsymbol{R}\}$ of unitary transformations on $P(M)$ by

$$
(V_t\varphi)(z) = e^{-i(n-2)t/2}\varphi(e^{-it}z)
$$

(see [8, p. 177]). Then

$$
V_t\varphi_m = e^{-i\{m+(n-2)/2\}t}\varphi_m
$$

for any $\varphi_m \in P_m(M)$, and $\{V_t\}$ is strongly continuous in t. The infinitesimal generator of $\{V_t\}$ is given by $X_r - i(n-2)/2$. Now, let $U_t = \mathscr{F}^{-1}\circ V_t \circ \mathscr{F}$ be the operator corresponding to V_t under the unitary isomorphism $\mathscr{F}$. Then, for any $f \in L^2(S^{n-1})$ and $x' \in S^{n-1}$, we have from Theorem 2.11

$$
\begin{aligned}
(U_t f)(x') &= \operatorname*{Lim}_{\sigma\to\infty} \int_{M(\sigma)} e^{x'\cdot\bar z}e^{-i(n-2)t/2}\int_{S^{n-1}} e^{x\cdot\exp(-it)z}f(x)dS(x)d\mu_n(z) \\
&= \operatorname*{Lim}_{\sigma\to\infty}\int_{S^{n-1}} U^{(\sigma)}(t, x', x)f(x)dS(x) ,
\end{aligned}
$$

where

$$
U^{(\sigma)}(t, x', x) = e^{-i(n-2)t/2}\int_{M(\sigma)} e^{x'\cdot\bar z+\exp(-it)x\cdot z}d\mu_n(z)
$$

(cf. [1, (3.10a)]). Since $U_t f_m = e^{-i\{m+(n-2)/2\}t} f_m$ for any $f_m \in H_m(S^{n-1})$, we have $U_t = \exp[-i\{\Delta + (n-2)^2/4\}^{1/2} t]$, where Δ is the Laplace-Beltrami operator on S^{n-1} (see [8, p. 177]). Thus, we have the following:

THEOREM 3.1. *The one-parameter group of unitary transformations, $U_t = \exp[-i\{\Delta + (n-2)^2/4\}^{1/2} t]$, on $L^2(S^{n-1})$ generated by the operator $-i\{\Delta + (n-2)^2/4\}^{1/2}$ is represented by*

$$(U_t f)(x') = \operatorname*{Lim}_{\sigma\to\infty} \int_{S^{n-1}} U^{(\sigma)}(t, x', x) f(x) dS(x) .$$

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