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ROTATIONAL SURFACES IN A PSEUDO-RIEMANNIAN 3-SPHERE

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1. Introduction. K. Akutagawa has recently shown the following interesting result.

THEOREM A. *Let $S_1^{n+1}(c)$ be a pseudo-Riemannian $(n+1)$-sphere of signature $(1, n)$ and of constant positive sectional curvature c. Let M be a complete, space-like hypersurface with constant mean curvature h in $S_1^{n+1}(c)$. If*

(i) $|h| \leqq c^{1/2}$ *when* $n = 2$,

(ii) $|h| < (2/n)[(n-1)c]^{1/2}$ *when* $n \geqq 3$,

then M is totally umbilical.

In this paper, we shall show in case $n = 2$ that the estimate in Theorem A is sharp. In fact, for each constant $h > 1$ we shall construct some families of complete, space-like, rotational surfaces in S_1^3 $(:= S_1^3(1))$ with constant mean curvature h, none of which are umbilical.

(Added on March 6, 1985). K. Akutagawa has kindly sent us his preprint [1] in which he proves the above Theorem A and also independently shows that the estimate in (i) is sharp.

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2. Statement of results. All the surfaces in the following Theorems 1, 2 and 3 except those in Theorem 3 (iii) turn out not to be umbilical by Proposition 1 in Section 3 and (4.7) in Section 4. We refer the readers to Section 3 for the terminology.

THEOREM 1. (Spherical rotational, space-like surfaces). *Let h be a constant, $h > 1$.*

(i) *For each constant $a > (h^2-1)^{1/2}/2$, we define the function $u(s)$ by*

$$u(s) = [ah + \{a^2 - (h^2-1)/4\}^{1/2} \cosh 2(h^2-1)^{1/2}s]/2(h^2-1) ,$$

$s \in R$, and the functions $\phi(s)$, $x_1(s)$, $x_3(s)$ and $x_4(s)$ by

$$\phi(s) = \int_0^s [4u(\sigma)^2 + u'(\sigma)^2 - 1/4]^{1/2}(2u(\sigma) - 1/2)^{-1}(2u(\sigma) + 1/2)^{-1/2} d\sigma ,$$
$$x_1(s) = (2u(s) + 1/2)^{1/2} ,$$
$$x_3(s) = (2u(s) - 1/2)^{1/2} \sinh \phi(s) ,$$
$$x_4(s) = (2u(s) - 1/2)^{1/2} \cosh \phi(s) , \qquad s \in R .$$

Then the one-to-one analytic mapping $f: R \times S^1 \to S_1^3$,

$$f(s, t) = x_1(s)(\cos te_1 + \sin te_2) + x_3(s)e_3 + x_4(s)e_4 , \tag{2.1}$$

defines a complete, space-like surface with constant mean curvature h *in* S_1^3, *where* S^1 *is the unit circle in* R^2 *and* $\{e_k\}$ *is a basis of* L^4 *satisfying* $\langle x, y\rangle = x_1y_1 + \cdots + x_3y_3 - x_4y_4$ *for* $x = \sum_k x_ke_k$ *and* $y = \sum_k y_ke_k$.

(ii) *We define the function* $u(s)$ *by*

$$u(s) = \exp[2(h^2 - 1)^{1/2}s] + h/4(h^2 - 1)^{1/2} , \qquad s \in R ,$$

and the functions $\phi(s)$, $x_1(s)$, $x_3(s)$ *and* $x_4(s)$ *as in* (i). *Then the one-to-one analytic mapping* $f: R \times S^1 \to S_1^3$ *given in* (2.1), *defines a complete, space-like surface with constant mean curvature* h *in* S_1^3.

THEOREM 2. (Hyperbolic rotational, space-like surfaces). *Let* h *be a constant,* $h > 1$.

(i) *For each constant* $a > (h^2 - 1)^{1/2}/2$, *we define* $u(s)$ *as in Theorem* 1 (i), *and* $\phi(s)$, $x_1(s)$, $x_3(s)$ *and* $x_4(s)$ *by*

$$\phi(s) = \int_0^s [4u(\sigma)^2 + u'(\sigma)^2 - 1/4]^{1/2}(2u(\sigma) + 1/2)^{-1}(2u(\sigma) - 1/2)^{-1/2} d\sigma ,$$
$$x_1(s) = (2u(s) - 1/2)^{1/2} ,$$
$$x_3(s) = (2u(s) + 1/2)^{1/2} \cos \phi(s) ,$$
$$x_4(s) = (2u(s) + 1/2)^{1/2} \sin \phi(s) , \qquad s \in R .$$

Then the analytic mapping $f: R \times R \to S_1^3$

$$f(s, t) = x_1(s)(\cosh te_1 + \sinh te_2) + x_3(s)e_3 + x_4(s)e_4 , \tag{2.2}$$

defines a complete, space-like (immersed) surface with constant mean curvature h *in* S_1^3, *where* $\{e_k\}$ *is a basis of* L^4 *satisfying* $\langle x, y\rangle = -x_1y_1 + x_2y_2 + \cdots + x_4y_4$ *for* $x = \sum_k x_ke_k$ *and* $y = \sum_k y_ke_k$.

(ii) *We define* $u(s)$ *as in Theorem* 1 (ii), *and* $\phi(s)$, $x_1(s)$, $x_3(s)$ *and* $x_4(s)$ *as in* (i). *Then the one-to-one analytic mapping* $f: R \times R \to S_1^3$ *given in* (2.2), *defines a complete, space-like surface with constant mean curvature* h *in* S_1^3.

THEOREM 3. (Parabolic rotational, space-like surfaces).

(i) *Let* h *be a constant,* $h > 1$. *For each positive constant* a, *we* *define* $u(s)$ *by*

$$u(s) = [ah + a\cosh 2(h^2-1)^{1/2}s]/2(h^2-1)\ , \qquad s \in R\ ,$$

and $x_1(s)$, $x_4(s)$ *and* $x_3(s)$ *by*

$$x_1(s) = (2u(s))^{1/2}\ ,$$
$$x_4(s) = x_1(s)\int_0^s [x_1(\sigma)^2 + x_1'(\sigma)^2]^{1/2}/x_1(\sigma)^2 d\sigma\ ,$$
$$x_3(s) = (-x_4(s)^2 + 1)/2x_1(s)\ , \qquad s \in R\ .$$

Then the one-to-one analytic mapping $f\colon R\times R \to S_1^3$

$$f(s,t) = x_1(s)(e_1 + te_2) - \left[\frac{1}{2}t^2 x_1(s) - x_3(s)\right]e_3 + x_4(s)e_4\ , \tag{2.3}$$

defines a complete, space-like surface with constant mean curvature h *in* S_1^3, *where* $\{e_k\}$ *is a basis of* L^4 *satisfying* $\langle x, y\rangle = x_1y_3 + x_2y_2 + x_3y_1 + x_4y_4$ *for* $x = \sum_k x_ke_k$ *and* $y = \sum_k y_ke_k$.

(ii) *Let* h *be a constant,* $h > 1$. *We define* $u(s)$ *by*

$$u(s) = \exp[2(h^2-1)^{1/2}s]\ , \qquad s \in R\ ,$$

and $x_1(s)$, $x_4(s)$ *and* $x_3(s)$ *as in* (i). *Then the one-to-one analytic mapping* $f\colon R\times R \to S_1^3$ *given in* (2.3), *defines a complete, space-like surface with constant mean curvature* h *in* S_1^3.

(iii) *For each positive constant* a, *the one-to-one analytic mapping* $f\colon R\times R \to S_1^3$,

$$f(s,t) = a[e_1 + te_2] - [at^2/2 + (s^2-1)/2a]e_3 + se_4\ ,$$

defines a complete, space-like surface with constant mean curvature one in S_1^3, *where* $\{e_k\}$ *is a basis of* L^4 *as in* (i).

3. Preliminaries. In this section, we shall recall umbilical surfaces and rotational, space-like surfaces in the pseudo-Riemannian 3-sphere S_1^3 of signature (1, 2) and of constant sectional curvature one (see [4], [5]). We denote by L^4 the space of 4-tuples $x = (x_1, \cdots, x_4)$ with Lorentzian metric $\langle\ ,\ \rangle = -(dx_1)^2 + (dx_2)^2 + \cdots + (dx_4)^2$, and consider the pseudo-Riemannian 3-sphere $S_1^3(c)$ of signature (1, 2) and of constant positive sectional curvature c as a hypersurface of L^4, namely,

$$S_1^3(c) = \{x \in L^4;\ \langle x, x\rangle = 1/c\}\ .$$

First, we note (cf. [2]) that umbilical, space-like surfaces in S_1^3 are given by the intersection of S_1^3 with affine 3-spaces of L^4. Up to isometries of S_1^3, they are represented explicitly as follows: for each constant a, $0 \leqq a < 1$, the isometric embedding $f\colon S^2(1-a^2) \to S_1^3$, $f(x, y, z) =$

$(a/(1-a^2)^{1/2}, x, y, z)$, of the Euclidean 2-sphere $S^2(1-a^2)$ of constant Gaussian curvature $1-a^2$ into S^3_1, defines an umbilical, space-like surface $M(a)$ with constant mean curvature a; for each constant $a>1$, the isometric embedding $f\colon H^2(1-a^2)\to S^3_1$, $f(x, y, z)=(x, y, z, a/(a^2-1)^{1/2})$, of the hyperbolic 2-plane $H^2(1-a^2)$ of constant Gaussian curvature $1-a^2$ into S^3_1, defines an umbilical, space-like surface $M(a)$ in S^3_1 with constant mean curvature a; and finally, for each positive constant b, the isometric embedding $f\colon R^2\to S^3_1$, $f(x, y)=be_1+xe_2-((x^2+y^2-1)/2b)e_3+ye_4$, of the Euclidean 2-plane R^2 into S^3_1, defines an umbilical space-like surface $N(b)$ in S^3_1 with constant mean curvature 1, where $\{e_k\}$ is a basis of L^4 defined by $e_1=(1/\sqrt{2}, 0, 1/\sqrt{2}, 0)$, $e_2=(0, 1, 0, 0)$, $e_3=(-1/\sqrt{2}, 0, 1/\sqrt{2}, 0)$ and $e_4=(0, 0, 0, 1)$.

Next, we recall some properties of rotational, space-like surfaces in S^3_1 (cf. [5]). We denote by P^k, $1\leqq k\leqq 3$, a k-subspace of L^4 passing through the origin, and by $O(P^2)$ the largest subgroup of the identity component of the Lorentzian group $O(1, 3)$ which leaves P^2 pointwise fixed. We note that $O(1, 3)$ is the group of all isometries of S^3_1 (see [6]).

DEFINITION. Choose P^2 and $P^3\supset P^2$, and let C be a regular space-like C^2-curve in $S^3_1\cap(P^3-P^2)$. The orbit of C under the action of $O(P^2)$ is called a rotational, space-like surface M in S^3_1 generated by C around P^2. The surface M is said to be spherical (resp. hyperbolic, resp. parabolic) if the restriction $\langle\ ,\ \rangle|P^2$ is a Lorentzian metric (resp. a Riemannian metric, resp. a degenerate quadratic form).

We now write down the parametrization of the rotational surface explicitly (cf. [3]). It is easily seen that we can choose a basis $\{e_k\}$ of L^4 satisfying the following conditions:

(1) P^2 is the plane generated by e_3 and e_4;

(2) P^3 is the 3-subspace generated by e_1 and P^2;

(3) for two vectors $x=\sum_k x_ke_k$ and $y=\sum_k y_ke_k$, we have

$$\langle x, y\rangle=\begin{cases} x_1y_1+\cdots+x_3y_3-x_4y_4 & \text{(spherical case)}\,,\\ -x_1y_1+x_2y_2+\cdots+x_4y_4 & \text{(hyperbolic case)}\,,\\ x_1y_3+x_2y_2+x_3y_1+x_4y_4 & \text{(parabolic case)}\,.\end{cases}$$

Let $x_1=x_1(s)$, $x_3=x_3(s)$ and $x_4=x_4(s)$, $s\in J$, be the equations of the curve C which is parametrized by arc length and whose domain of definition J is an open interval of the set R of real numbers. Then we see that for each fixed $s\in J$, the intersection $U(s)$ of S^3_1 with the affine plane passing through $(0, 0, x_3(s), x_4(s))$ and parallel to the plane generated by e_1 and e_2 is a circle (resp. a hyperbola, resp. a parabola) in the spherical case (resp. hyperbolic case, resp. parabolic case), and we may give the following parametrization of the surface M (see [4]):

$$f(s, t) = x_1(s)\cos te_1 + x_1(s)\sin te_2 + x_3(s)e_3 + x_4(s)e_4 , \quad s \in J , \quad t \in S^1 , \text{ the unit circle in } R^2, \quad \text{(spherical case)}, \tag{3.1}$$

$$f(s, t) = x_1(s)\cosh te_1 + x_1(s)\sinh te_2 + x_3(s)e_3 + x_4(s)e_4 , \quad s \in J , \quad t \in R , \quad \text{(hyperbolic case)} , \tag{3.2}$$

$$f(s, t) = x_1(s)e_1 + tx_1(s)e_2 + \left(\frac{-1}{2}t^2x_1(s) + x_3(s)\right)e_3 + x_4(s)e_4 , \quad s \in J , \quad t \in R , \quad \text{(parabolic case)} . \tag{3.3}$$

From the parametrization, we see that the first fundamental form I of the C^2-mapping f is

$$I = ds^2 + x_1(s)^2dt^2 \qquad \text{in each case} , \tag{3.4}$$

and the following relations hold on J:

$$x_1^2 + x_3^2 - x_4^2 = 1 , \quad x_1'^2 + x_3'^2 - x_4'^2 = 1 \qquad \text{(spherical case)} , \tag{3.5}$$

$$-x_1^2 + x_3^2 + x_4^2 = 1 , \quad -x_1'^2 + x_3'^2 + x_4'^2 = 1 \qquad \text{(hyperbolic case)} , \tag{3.6}$$

$$2x_1x_3 + x_4^2 = 1 , \quad 2x_1'x_3' + x_4'^2 = 1 \qquad \text{(parabolic case)} . \tag{3.7}$$

From (3.4)–(3.7) and the assumption that f is an immersion, we may assume that on the interval J,

$$\begin{aligned} &x_1(s) > 1 \qquad \text{(spherical case)} \\ &x_1(s) > 0 \qquad \text{(hyperbolic or parabolic case)} . \end{aligned} \tag{3.8}$$

It is convenient to use the notation M_δ, $\delta = 1, 0$ or -1, to denote a rotational, space-like surface in S_1^3, where $\delta = 1$ (resp. $\delta = 0$, resp. $\delta = -1$) means M_δ is a spherical (resp. parabolic, resp. hyperbolic) surface. After a long calculation we can show the following result (cf. [3]).

PROPOSITION 1. *Let M_δ be a rotational, space-like surface in S_1^3 defined by the mapping f. Then the directions of the parameters t and s are principal directions, the principal curvature along the coordinate t (resp. s) being given by $(x_1^2 + x_1'^2 - \delta)^{1/2}/x_1$ (resp. $(x_1'' + x_1)/(x_1^2 + x_1'^2 - \delta)^{1/2}$).*

4. **Rotational, space-like surfaces in S_1^3 with constant mean curvature.** From Proposition 1 and (3.8) it can be shown that the mapping f is an immersion with constant mean curvature h if and only if, on the interval J, the following relations hold.

$$x_1x_1'' + x_1'^2 + 2x_1^2 - \delta = 2hx_1(x_1^2 + x_1'^2 - \delta)^{1/2} , \qquad \text{in each case} , \tag{4.1}$$

$$x_1^2 + x_1'^2 - \delta > 0 , \qquad \text{in each case} , \tag{4.2}$$

$$(4.3)\qquad x_3 = (x_1^2 - 1)^{1/2}\sinh\phi(s)\ ,\quad x_4 = (x_1^2 - 1)^{1/2}\cosh\phi(s)\ ,$$
$$\phi(s) = \int_0^s (x_1^2 + x_1'^2 - 1)^{1/2}(x_1^2 - 1)^{-1}d\sigma\ ,\quad \text{and}$$
$$x_1 > 1\ ,\qquad (\text{spherical case})\ ,$$

$$(4.4)\qquad x_3 = (x_1^2 + 1)^{1/2}\cos\phi(s)\ ,\quad x_4 = (x_1^2 + 1)^{1/2}\sin\phi(s)\ ,$$
$$\phi(s) = \int_0^s (x_1^2 + x_1'^2 + 1)^{1/2}(x_1^2 + 1)^{-1}d\sigma\ ,\ \text{and}$$
$$x_1 > 0\ ,\qquad (\text{hyperbolic case})$$

$$(4.5)\qquad x_3 = (-x_4^2 + 1)/2x_1\ ,\quad x_4 = x_1\int_0^s (x_1^2 + x_1'^2)^{1/2}x_1^{-2}d\sigma\ ,\quad \text{and}$$
$$x_1 > 0\ ,\qquad (\text{parabolic case})\ .$$

We now try to solve the equation (4.1) explicitly under the conditions (4.2) and

$$(4.6)\qquad \begin{array}{ll} x_1 > 0 & \text{in cases } \delta = 0, -1\ ,\quad \text{and} \\ x_1 > 1 & \text{in case } \ \delta = 1\ . \end{array}$$

Defining $u(s)$ by

$$(4.7)\qquad u(s) = x_1(s)^2/2 - \delta/4\ ,$$

we can easily show (cf. [5]) that (4.1) with the conditions (4.2) and (4.6) is equivalent to

$$(4.8)\qquad u'^2 = 4(h^2 - 1)u^2 - 4ahu + a^2 + \delta^2/4$$

with the conditions

$$(4.9)\qquad -a + 2hu > 0\ ,\qquad a : \text{constant}\ ,$$

and

$$(4.10)\qquad u > \delta^2/4 \qquad \text{for each } \delta\ ,$$

provided the subset of J, consisting of zero points of the first derivative of the solution $u(s)$ of (4.8), is discrete. This restriction, however, will turn out to be satisfied as we solve (4.8) explicitly. From that point on, our argument is almost the same as that in [5], and we only give an outline.

We first consider (4.8) in the case where $|h| > 1$. There are three subcases: the constant $A := a^2 - \delta^2(h^2 - 1)/4$ is positive, zero and negative. When A is positive after replacing the parameter s by the new one $s + c$ for a suitable constant c, we have an explicit form of the solution $u = u(s)$ of (4.8):

(4.11) $\quad u(s) = [ah + \{a^2 - \delta^2(h^2 - 1)/4\}^{1/2} \cosh 2(h^2 - 1)^{1/2}s]/2(h^2 - 1)$.

From (4.11) it follows that J, the domain of definition of $u(s)$, can be extended to R, and we denote the extended function by the same symbol $u(s)$. Then we see that for the extended function $u(s)$, the conditions (4.9) and (4.10) are equivalent to

(4.12) $$a > |\delta|(h^2 - 1)^{1/2}/2 \quad \text{for} \quad h > 1 ,$$

and that there are no solutions with domain $J = R$ of (4.8) with (4.9) for $h < -1$.

Putting (4.11) with (4.12) into (4.7) with $x_1 > 0$ and (4.3), (4.4), (4.5) the functions $u(s)$, $x_1(s)$, $x_3(s)$, $x_4(s)$ and $\phi(s)$ are determined in the following form.

(i) (Spherical case). For $h > 1$ and $a > (h^2 - 1)^{1/2}/2$ we have

(4.13) $$u(s) = [ah + \{a^2 - (h^2 - 1)/4\}^{1/2} \cosh 2(h^2 - 1)^{1/2}s]/2(h^2 - 1) ,$$

(4.14) $$\phi(s) = \int_0^s [4u(\sigma)^2 + u'(\sigma)^2 - 1/4]^{1/2}(2u(\sigma) - 1/2)^{-1}(2u(\sigma) + 1/2)^{-1/2}d\sigma ,$$

(4.15) $$x_1(s) = (2u(s) + 1/2)^{1/2} ,$$

(4.16) $$x_3(s) = (2u(s) - 1/2)^{1/2} \sinh \phi(s) ,$$

(4.17) $$x_4(s) = (2u(s) - 1/2)^{1/2} \cosh \phi(s) .$$

(ii) (Hyperbolic case). For $h > 1$ and $a > (h^2 - 1)^{1/2}/2$ we have $u(s)$ as in (i) and

(4.18) $$\phi(s) = \int_0^s [4u(\sigma)^2 + u'(\sigma)^2 - 1/4]^{1/2}(2u(\sigma) + 1/2)^{-1}(2u(\sigma) - 1/2)^{1/2}d\sigma ,$$

(4.19) $$x_1(s) = (2u(s) - 1/2)^{1/2} ,$$

(4.20) $$x_3(s) = (2u(s) + 1/2)^{1/2} \cos \phi(s) ,$$

(4.21) $$x_4(s) = (2u(s) + 1/2)^{1/2} \sin \phi(s) .$$

(iii) (Parabolic case). For $h > 1$ and $a > 0$ we have

(4.22) $$u(s) = [ah + a \cosh 2(h^2 - 1)^{1/2}s]/2(h^2 - 1) ,$$

(4.23) $$x_1(s) = (2u(s))^{1/2} ,$$

(4.24) $$x_4(s) = x_1(s)\int_0^s [x_1(\sigma)^2 + x_1'(\sigma)^2]^{1/2}/x_1(\sigma)^2 d\sigma ,$$

(4.25) $$x_3(s) = (-x_4(s)^2 + 1)/2x_1(s) .$$

When $A = 0$, after replacing the parameter s by the new one $s + c$, for a suitable constant c, we have an explicit solution $u = u(s)$ of (4.8) with the maximal domain of definition (i.e., $J = R$ in this case):

(4.26) $$u(s) = \exp(2(h^2 - 1)^{1/2}s) + h\delta^2/4(h^2 - 1)^{1/2} \quad \text{for} \quad h > 1 ,$$

which satisfies the conditions (4.9) and (4.10) automatically, and there are no solutions with maximal domain satisfying (4.9) and (4.10) for $h < -1$. Just as in the case where A is positive, we can also define the functions $x_1(s)$, $x_3(s)$ and $x_4(s)$ explicitly corresponding to the cases $\delta = 1, 0$ and -1.

When A is negative, it can be shown that there are no solutions of (4.8) with maximal domain satisfying (4.9) and (4.10).

Next, we consider (4.8) in the case where $|h| = 1$. There are two subcases: $a \neq 0$ and $a = 0$. When $a = 0$, after replacing the parameter s by the new one $s + c$ for a suitable constant c, we have an explicit form of the solution $u = u(s)$ of (4.8):

$$u(s) = \pm\delta s/2 + b\ , \qquad b\text{: constant}\ . \tag{4.27}$$

From (4.27) it follows that J, the domain of definition of $u(s)$, can be extended to R, and we denote the extended function by the same symbol $u(s)$. Then we see that for the extended function $u(s)$, the conditions (4.9) and (4.10) are equivalent to

$$u(s) = b\ , \qquad b\text{: a positive constant, for } \delta = 0,\ h = 1\ , \tag{4.28}$$

and that there are no solutions with maximal domain satisfying (4.9) and (4.10) for $h = -1$, or $\delta = \pm 1$. When $a \neq 0$ it can be easily shown that there are no solutions with maximal domain of (4.8) satisfying (4.9) and (4.10).

Finally, we consider (4.8) in the case where $|h| < 1$. In this case we see that there are no solutions with maximal domain of (4.8) satisfying (4.9) and (4.10).

Reversing the above argument and taking the completeness into consideration we see that our main results in Section 2 are true.

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