Tôhoku Math. Journ.
38 (1986), 15-27.

ASYMPTOTIC PERIODICITY OF THE ITERATES OF WEAKLY CONSTRICTIVE MARKOV OPERATORS

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(Received September 22, 1984)

Introduction. The asymptotic periodicity of the iterates of Markov operators has been studied in [3], [4] and [5]. It has been proved to hold for

(i) strongly constrictive Markov operators

(ii) weakly constrictive Frobenius-Perron operators.

This paper is devoted to the proof of the conjecture formulated (in the invited address at the International Congress of Mathematicians in 1983) by Lasota. We extend the results mentioned above to the case of an arbitrary weakly constrictive Markov operator P.

1. Asymptotic properties of P. Let (X, Σ, μ) be a σ-finite measure space. We shall deal with the spaces $L^p = L^p(X, \Sigma, \mu)$ and the norms $\|\cdot\|_p = \|\cdot\|_{L^p}$. By D we denote the set of densities on X, i.e., the set of all normalized nonnegative elements of L^1.

$$D = \{f \in L^1 : \|f\|_1 = 1,\ f \geqq 0\} \tag{1.1}$$

A linear operator $P: L^1 \to L^1$ is called a Markov operator if

$$P(D) \subset D .$$

DEFINITION 1.1. We say that P is strongly (resp. weakly) constrictive if there exists a strongly (resp. weakly) compact set $F \subset L^1$ such that

$$\lim_{n\to\infty} d(P^n f, F) = 0 , \qquad \text{for} \quad f \in D , \tag{1.2}$$

where $d(g, F)$ is the infimum of $\|g - f\|_1$ for $f \in F$.

REMARK 1.1. It is obvious that for any $g \in L^1$ the set

$$F_g = \{f \in L^1 : 0 \leqq f \leqq |g|\}$$

is weakly compact. For $g \in L^1$ we define the support of g by

$$\operatorname{supp}(g) = \{x \in X : g(x) \neq 0\} . \tag{1.3}$$

The following theorem is a generalization of the main results of [3] and [4].

THEOREM 1.1. *Let P be a weakly constrictive Markov operator. Then there exists a sequence of densities $\{g_i\}$ $(i = 1, \cdots, r)$ with mutually disjoint supports and a sequence of linear functionals $\{\lambda_i\} \subset L^{1*}$ such that*

$$\lim_{n\to\infty} \left\| P^n\left(f - \sum_{i=1}^{r} \lambda_i(f) g_i\right)\right\|_1 = 0 \qquad \text{for} \quad f \in L^1 \tag{1.4}$$

and

$$P(g_i) = g_{\alpha(i)} \qquad \text{for} \quad i = 1, \cdots, r \, , \tag{1.5}$$

where α is a permutation of the integers $1, \cdots, r$.

From the above theorem it follows that the n-th power P^n of P can be written in the form

$$P^n f = \sum_{i=1}^{r} \lambda_i(f) \cdot g_{\alpha^n(i)} + R_n f \qquad \text{for} \quad f \in L^1 \, , \tag{1.6}$$

where α^n denotes the n-th iterates of the permutation α and the remainder $R_n(f)$ converges strongly to zero as $n \to \infty$. Thus every sequence $\{P^n f\}_{n \in N}$ is asymptotically periodic with a period which does not exceed $r!$.

We shall prove Theorem 1.1 under the additional assumptions

$$\mu(X) < \infty \, , \quad P1_X = 1_X \, , \tag{1.7}$$

which can be released (using ergodic theorem) in the same way as in [3] and [4].

2. Comments and applications. Let P be a Markov operator. Below we present a new criterion for the asymptotic stability of the sequence $\{P^n\}_{n=1}^{\infty}$ based on Theorem 1.1.

DEFINITION 2.1. The sequence $\{P^n\}$ is *asymptotically stable* if there exists a density f_0 such that

$$\lim_{n\to\infty} \| P^n f - f_0 \|_1 = 0 \qquad \text{for every} \quad f \in D \, . \tag{2.1}$$

DEFINITION 2.2. A set A with positive measure is called a *lower set* for P if

$$P^n f(x) > 0 \qquad \text{for every} \quad x \in A, \, f \in D, \, n \geqq n_0(f) \, . \tag{2.2}$$

LEMMA 2.1 (See [5]). *Suppose that P is strongly constrictive and has a lower set. Then $\{P^n\}_{n=1}^{\infty}$ is asymptotically stable.*

EXAMPLE 2.1. Let $X = [0, \infty)$ and μ be the Lebesgue measure. Let

$$Pf = \int_0^\infty K(x, y) f(y) dy \, , \tag{2.3}$$

where K is a stochastic kernel. Suppose that K satisfied the conditions

(2.4) $$\int_0^r \sup_{0\leqq y<\infty} K(x, y)dx < \infty \qquad \text{for all} \quad r \in R$$

and

(2.5) $$\int_0^\infty x\cdot K(x, y)dx \leqq \gamma\cdot y + \delta\,, \qquad \text{for} \quad y \geqq 0$$

for some constants γ and δ, $\gamma < 1$.

We show that P is weakly constrictive.

Applying the same arguments as in [6] we obtain form (2.5) that

(2.6) $$E_n(f) = \int_0^\infty x\cdot P^n f(x)dx \leqq \frac{\delta}{1-\gamma} + 1 = M$$

for sufficiently large $n \geqq n_0(f)$. Hence

(2.7) $$\mu\{(x: P^n f(x) \geqq a)\} < \frac{M}{a}$$

for every $a > 0$ and $n \geqq n_0(f)$. It is obvious that the set

(2.8) $$F = \bigcap_{a>0} \left\{ g \in L^1 \colon \mu\{x;\, g(x) > a\} < \frac{M}{a},\ g(x) \leqq \sup_{0\leqq y<\infty} K(x, y) \right\}$$

is weakly compact and that

(2.9) $$\lim_{n\to\infty} d(P^n f, F) = 0\,.$$

As an example we consider the kernel

(2.10) $$K(x, y) = \begin{cases} 0 & \text{if} \quad x \leqq y/2 \\ (4x/c)\cdot\exp\left[-\dfrac{2x^2}{c} + \dfrac{y^2}{2c}\right] & \text{if} \quad x > y/2 \end{cases}$$

where c is a positive constant.

The corresponding operator P is given by

(2.11) $$Pf(x) = (4x/c)\cdot\exp\{-2x^2/c\}\cdot\int_0^{2x} \exp(y^2/2c)\cdot f(y)dy\,.$$

This operator was used in [6] for modelling the cell division cycle in a population of cells. Pf means the density of distribution of mitogen level after one cell cycle if the initial density was f. It is easy to check that the conditions (2.4) and (2.5) are satisfied. Hence P is weakly constrictive. Moreover, P has a lower set $A = Y - \{\varnothing\}$. Therefore P is asymptotically stable. The same result was obtained in [6] in a more complicated way.

3. Construction of a limiting set Q. In the sequel we suppose that

P is a weakly constrictive Markov operator and that the conditions (1.7) are satisfied.

DEFINITION 3.1. A set $B \in \Sigma$ will be called a *nice set* if $P^n(1_B)$ is a characteristic function for each positive integer n. The characteristic function of a nice set B is called a *nice function*. We denote by Q the linear subspace of L^1 spanned by nice functions.

We shall utilize the following results obtained in [4].

LEMMA 3.1. (i) *There exists a real number $\delta > 0$ such that $\mu(B) > \delta$ for every nice set B with $\mu(B) > 0$.*

(ii) *The system C of nice sets is a finite algebra with atoms $X_1, \cdots, X_r$, where $r \leqq \delta^{-1}$.*

(iii) *There exists a permutation α of the set $\{1, \cdots, r\}$ such that*

$$P^n(X_i) = X_{\alpha^n(i)} \quad \text{for} \quad i \in \{1, \cdots, r\} . \tag{3.1}$$

(iv) *There exists an integer $n_0 \leqq r!$ such that*

$$P^{n_0}(f) = f \quad \text{for} \quad f \in Q . \tag{3.2}$$

(v) $Q \subset L^\infty$, $\dim(Q) = r$.

(vi) *If f_1 and f_2 are nonnegative elements of L^1 with the same supports, then Pf_1 and Pf_2 have the same supports.*

Utilizing the fact that a Markov operator is positive and applying the Riesz convexity theorem we obtain the following.

LEMMA 3.2. (i) *The operator P preserves mean values, i.e.,*

$$EPf = Ef\left(=\int f d\mu\right) \quad \text{for} \quad f \in L^1 . \tag{3.3}$$

(ii) *Let $1 \leqq p \leqq \infty$. The subspace L^p is P-invariant and*

$$\| Pf \|_p \leqq \|f\|_p \quad \text{for} \quad f \in L^p . \tag{3.4}$$

This enables us to consider P as an operator on L^2 with the dual $U = P^*$.

THEOREM 3.1. *There exists a symmetric operator A on L^2 such that*

$$\lim_{n\to\infty} \| Af - U^n P^n f \|_2 = 0 \quad \text{and} \quad (Af, f) = \lim_{n\to\infty} \| P^n f \|_2^2 \quad \text{for every } f \in L^2 . \tag{3.5}$$

Moreover, the following set equality holds:

$$Q = \mathrm{Ker}(I - A) . \tag{3.6}$$

PROOF. The existence of A and validity of (3.5) are direct consequences of the fact that P is a contraction on L^2 (cf. [9]). We have

$$I \geqq U^n P^n \geqq U^{n+1} P^{n+1} \geqq A \geqq 0 \quad \text{for} \quad n \in N . \tag{3.7}$$

We show that

$$\begin{aligned} \operatorname{Ker}(I - A) &= \{f \in L^2 : (Af, f) = \|f\|_2^2\} \\ &= \{f \in L^2 : \forall\, n \in N \ \|P^n f\|_2 = \|f\|_2\} . \end{aligned} \tag{3.8}$$

The last of these set equalities follows from (3.5). Now we prove the first one. Let $(Af, f) = \|f\|_2^2$. Then $\|(I - A)^{1/2} f\|_2^2 = 0$, hence $f \in \operatorname{Ker}(I - A)$. The converse inclusion is obvious.

Now we prove the inclusion $Q \subset \operatorname{Ker}(I - A)$. It suffices to prove that $1_{X_i} \in \operatorname{Ker}(I - A)$ for any nice set X_i, $i \in \{1, \cdots, r\}$. From (3.1) we obtain

$$\|P^n 1_{X_i}\|_2^2 = \|1_{X_{\alpha^n(i)}}\|_2^2 = \|1_{X_{\alpha^n(i)}}\|_1 = \|1_{X_i}\|_1 = \|1_{X_i}\|_2^2 .$$

Finally we prove that $\operatorname{Ker}(I - A) \subset Q$. We have

$$P^n 1_X = 1_X$$

hence $1_X \in \operatorname{Ker}(I - A)$ according to (3.8). Therefore $f \in \operatorname{Ker}(I - A)$ implies $f - c \in \operatorname{Ker}(I - A)$ for $c \in R$. Consider $f \in \operatorname{Ker}(I - A)$, $f = f^+ - f^-$. We have

$$\begin{aligned} \|f\|_2^2 = \|Pf\|_2^2 &= \|Pf^+\|_2^2 + \|Pf^-\|_2^2 - 2(Pf^+, Pf^-) \\ &\leqq \|f^+\|_2^2 + \|f^-\|_2^2 - 2(Pf^+, Pf^-) = \|f\|_2^2 - 2\int Pf^+ \cdot Pf^- d\mu . \end{aligned}$$

However $Pf^+ \geqq 0$, $Pf^- \geqq 0$. Hence $Pf^+ \cdot Pf^- = 0$, and Pf^+ and Pf^- have disjoint supports.

Using the same arguments we obtain that the functions $P^n(f - c)^+$ and $P^n(f - c)^-$ have disjoint supports for any $n \in N$ and $c \in R$.

Suppose that $c \in R$ is such that $\mu(f^{-1}(c)) = 0$. Put

$$h_1 = 1_{f^{-1}(-\infty, c)},\ h_2 = 1 - h_1 .$$

We have

$$\operatorname{supp}(h_1) = \operatorname{supp}((f - c)^-),\ \operatorname{supp}(h_2) = \operatorname{supp}((f - c)^+) .$$

According to Lemma 3.1 (vi) the functions $P^n h_1$ and $P^n h_2$ have disjoint supports for any fixed $n \in N$. However $P^n h_1 + P^n h_2 = 1$. Therefore, $P^n h_1$ is a characteristic function and $f^{-1}(0, c)$ is a nice set. Suppose that $\mu(f^{-1}(c)) > 0$. There exists a sequence $c_i \nearrow c$ such that $f^{-1}(-\infty, c) = \bigcup_{i=1}^{\infty} f^{-1}(-\infty, c_i)$ and $\mu(f^{-1}(c_i)) = 0$.

Nice sets form a finite algebra which is a σ-algebra as well. Therefore $f^{-1}(-\infty, c)$ is a nice set for every $c \in R$, which yields that $f \in Q$.

COROLLARY 3.1. *Let n_0 be as in Lemma* 3.1 (iv). *Then*

$$U^{n_0}f = f \qquad \text{for} \quad f \in Q \,. \tag{3.9}$$

PROOF. From (3.6) and (3.7) we get

$$0 \leqq \|(I - U^{n_0}P^{n_0})^{1/2}f\|_2^2 = (f, (I - U^{n_0}P^{n_0})f) \leqq (f, (I - A)f) = 0 \,.$$

Hence $(I - U^{n_0}P^{n_0})f = 0$ for $f \in Q$. Utilising (3.2) we get (3.9).

4. Asymptotic periodicity of P. Let $X_1, \cdots, X_r$ and n_0 have the same meaning as in Lemma 3.1. Put

$$R = P^{n_0} \tag{4.1}$$

and

$$L_i = \{f \cdot 1_{X_i} \colon f \in L^1\} \quad \text{for} \quad i = 1, \cdots, r \,. \tag{4.2}$$

First we present the weak version of Theorem 1.1.

THEOREM 4.1. (i) *The subspaces L_i are R-invariant.*

(ii) *For $f \in L_i$ the sequence $\{R^n f\}$ converges weakly to the function $\lambda_i \cdot 1_{X_i}$, where*

$$\lambda_i = \int f d\mu / \mu(X_i) \,. \tag{4.3}$$

(iii) *For any $f \in L^1$ the sequence*

$$P^n f - \sum_{i=1}^{r} \lambda_i 1_{X_{\alpha^n(i)}} \tag{4.4}$$

converges weakly to 0.

PROOF. (i) The functions 1_{X_i} are invariant under the Markov operator R. Therefore $Rf \in L_i$ for $f \in L_i \cap L^\infty$, which is dense in L_i.

(ii) Let $f \in L_i \cap L^2$. The sequence $\{R^n f\}$ is weakly sequentially compact (cf. [10]). There exists $g \in L_i \cap L^2$ and a subsequence $\{n_k\} \subset N$ such that $\{R^{n_k} f\}$ converges weakly to g. We have

$$\lim_{k\to\infty} (R^{n_k} f, g) = \|g\|_2^2 \,,$$

hence

$$\lim_{k\to\infty} \|R^{n_k} f\|_2^2 = \lim_{k\to\infty} \|R^{n_k} f - g\|_2^2 + \|g\|_2^2 \,. \tag{4.5}$$

For any $m \in N$ and $h \in L^2$ the sequence

$$(R^{n_k+m} f, h) = (R^{n_k} f, R^{*m} h)$$

converges to

$$(g, R^{*m}h) = (R^m g, h) ,$$

hence $\{R^{n_k+m}f\}$ converges weakly to $R^m g$ and

$$\lim_{k\to\infty} \| R^{n_k+m}f \|_2^2 = \lim_{k\to\infty} \| R^{n_k+m}f - R^m g \|_2^2 + \| R^m g \|_2^2$$
$$\leqq \lim_{k\to\infty} \| R^{n_k}f - g \|_2^2 + \| R^m g \|_2^2 = \lim_{k\to\infty} \|R^{n_k}f \|_2^2 - \| g \|_2^2 + \| R^m g \|_2^2 .$$

Using (3.5), (4.1), (3.4) and (3.8) we get

$$\| R^m g \|^2 = \| g \|^2 \qquad \text{for} \quad m \in N ,$$

hence

$$g \in Q \cap L_i .$$

Therefore g is constant on X_i, $g = \lambda \cdot 1_{X_i}$ for some $\lambda \in R$. We have

$$\lambda \cdot \mu(X_i) = (g, 1_{X_i}) = \lim_{k\to\infty}(R^{n_k}f, 1_{X_i}) = \lim_{k\to\infty}(f, R^{*n_k}1_{X_i}) = (f, 1_{X_i}) = \int_{X_i} f d\mu$$

because of (3.9) and (4.1).

The part (iii) is a direct consequence of (ii). We omit the detailed proof of it, because we do not need it in the proof of Theorem 1.1 which contains a stronger result.

Finally we present the proof of Theorem 1.1.

It suffices to prove that

$$\lim_{n\to\infty} \| R^n f - \lambda_i 1_{X_i} \|_1 = 0 \tag{4.6}$$

holds for every $i \in \{1, \cdots, r\}$ and $f \in L_i$. It is easy to show that (4.6) implies (1.4).

For $f \in L^1$ and $n \in N$ we can write $n = k \cdot n_0 + m$, where $0 \leqq m < n_0$. We have

$$\left\| P^n f - \sum_{i=1}^{r} \lambda_i 1_{X_{\alpha^n(i)}} \right\|_1 = \left\| P^m \left[\sum_{i=1}^{r} R^k(f \cdot 1_{X_i}) - \lambda_i 1_{X_i} \right] \right\|_1$$
$$\leqq \sum_{i=1}^{r} \left\| R^k(f \cdot 1_{X_i}) - \lambda_i \cdot 1_{X_i} \right\|_1 .$$

Let F be the weakly compact subset of L^1 mentioned in Definition 1.1. It is easy to check that for any $i \in \{1, \cdots, r\}$ the sets

$$F_i = \{f \cdot 1_{X_i} \colon f \in F\}$$

are weakly compact and the restriction of R on $L_i = L^1(X_i)$ is weakly constrictive.

For the sake of simplicity we shall omit the index i. Hence we shall restrict our attention to the case $r = n_0 = 1$. Moreover, we can suppose that $\mu(X) = 1$.

We obtain from Theorem 4.1 (ii) that for any $f \in D$ the sequence $\{R^n f\}$ converges weakly to 1_X. Our aim is to prove that this convergence is strong.

We shall utilize the following simple notions and results. For $f \in L^1$, such that $f \geqq 0$, $\|f\|_1 > 0$ we put

$$\nu(f) = f/\|f\|_1 . \tag{4.7}$$

DEFINITION 4.1. Let $0 \leqq \rho \in R$, $m \in N$. Nonnegative L^1-functions $f_1, \cdots, f_m$ are ρ-orthogonal if there exist nonnegative L^1-functions $h_1, \cdots, h_m$ with disjoint supports such that

$$\|f_i - h_i\|_1 \leqq \rho \qquad \text{for} \quad i = 1, \cdots, m . \tag{4.8}$$

PROPOSITION 4.1. (i) *Let* $f_1, \cdots, f_m$ be *ρ-orthogonal and* $\|f_i\|_1 \geqq e_0 > 0$ *for* $i = 1, \cdots, n$. *Then*

$$\nu(f_1), \cdots, \nu(f_m)$$

are ρ_1-orthogonal with $\rho_1 = \rho/e_0$.

(ii) *Let* $f_{1,1}, \cdots, f_{i,m_1}$ *be ρ_1-orthogonal and* $f_{2,1}, \cdots, f_{2,m_2}$ *be ρ_2-orthogonal. Then* $m_1 \cdot m_2$ *functions* $f_{1,i_1} \wedge f_{2,i_2}$, *where* $i_1 \in \{1, \cdots, m_1\}$, $i_2 \in \{1, \cdots, m_2\}$, *are* $\rho_1 + \rho_2$-*orthogonal.*

PROOF. (i) *Let* $h_1, \cdots, h_m$ be as in Definition 4.1. The functions $h'_i = h_i/\|f_i\|_1$ for $i = 1, \cdots, m$ have disjoint supports. We have

$$\|\nu(f_i) - h'_i\|_1 = \|f_i - h_i\|_1/\|f_i\|_1 \leqq \rho/e_0 .$$

(ii) Let

$$h_{1,1}, \cdots, h_{1,n_1} \quad \text{and} \quad h_{2,1}, \cdots, h_{2,n_2}$$

be two groups of nonnegative L^1-functions with disjoint supports corresponding to the functions $f_{j,i}$, for $j = 1, 2$, $i = 1, \cdots, n_j$ according to (4.8).

The $m_1 \cdot m_2$ functions

$$h_{1,i_1} \wedge h_{2,i_2}\colon i_1 = 1, \cdots, m_1;\ i_2 = 1, \cdots, m_2$$

have disjoint supports. Utilizing the inequality

$$|x \wedge z - y \wedge z| \leqq |x - y|$$

which holds for any real numbers x, y, z we get

$$\|f_{1,i_1} \wedge f_{2,i_2} - h_{1,i_1} \wedge h_{2,i_2}\|_1 \leqq \|f_{1,i_1} \wedge f_{2,i_2} - f_{1,i_1} \wedge h_{2,i_2}\|_1 + \|f_{1,i_1} \wedge h_{2,i_2} - h_{1,i_1} \wedge h_{2,i_2}\|_1 \leqq \|f_{2,i_2} - h_{2,i_2}\|_1 + \|f_{1,i_1} - h_{1,i_1}\|_1 \leqq \rho_1 + \rho_2 .$$

PROPOSITION 4.2. *Let F be a weakly compact subset of L^1. Let* $\varepsilon \in$

(0, 1) *be a given real number and* $\delta > 0$ *is such that*

$$\int_B g d\mu < \varepsilon \quad \textit{for} \quad g \in F \quad \textit{and} \quad \mu(B) < \delta \,. \tag{4.9}$$

Let ρ *and* κ *are positive real numbers such that*

$$\varepsilon + 2\rho + \kappa \leqq 1 \,. \tag{4.10}$$

Then the maximal number of ρ*-orthogonal densities contained in the set*

$$O_\kappa(F) = \{f : f \in L^1, d(g, F) < \kappa\} \tag{4.11}$$

is not greater than δ^{-1}.

PROOF. Let $m > \delta^{-1}$ and $f_1, \cdots, f_m$ be ρ-orthogonal densities contained in $O_\kappa(F)$. Let $g_1, \cdots, g_m$ be elements of F such that

$$\|f_i - g_i\|_1 \leqq \kappa \qquad \text{for} \quad i = 1, \cdots, m \,. \tag{4.12}$$

Let $h_1, \cdots, h_m$ be as in Definition 4.1. Let $\{B_i\}_{i=1}^m$ be disjoint supports of $\{h_i\}_{i=1}^m$. There exists $j \in \{1, \cdots, m\}$ such that

$$\mu(B_j) \leqq 1/m < \delta \,. \tag{4.13}$$

We have

$$\int_{B_j} h_j d\mu = \int_X h_j d\mu \geqq \int_X f_j d\mu - \|f_j - h_j\|_1 \geqq 1 - \rho$$

and

$$\int_{B_j} |h_j - g_j| d\mu \leqq \|h_j - g_j\|_1 \leqq \kappa + \rho \,.$$

Hence

$$\int_{B_j} g_j d\mu \geqq \int_{B_j} h_j d\mu - \int_{B_j} |h_j - g_j| d\mu \geqq 1 - 2\rho - \kappa \geqq \varepsilon \,,$$

which contradicts (4.9).

PROPOSITION 4.3. *Let* $f \in L^1$ *and* $\lambda = Ef$. *Then for any* $\rho > 0$ *there exists* N_ρ *such that for* $m \geqq N_\rho$ *and* $n \geqq 0$ *the functions*

$$R^n([R^m f - \lambda]^+) \quad \textit{and} \quad R^n([R^m f - \lambda]^-) \tag{4.14}$$

are ρ*-orthogonal.*

PROOF. The sequence $\|R^m f - \lambda\|_1$ is nonincreasing. Put

$$M_1 = \frac{1}{2} \cdot \lim_{m \to \infty} \|R^m f - \lambda\|_1 \,. \tag{4.15}$$

Let us denote

(4.16) $$d_{m,1} = (R^m f - \lambda)^+ , \quad d_{m,2} = (R^m f - \lambda)^-$$

We have

$$\| d_{m,1} \|_1 = \| d_{m,2} \|_1 = \frac{1}{2} \| R^m f - \lambda \|_1$$

because of

$$E(R^m f - \lambda) = Ed_{m,1} - Ed_{m,2} = 0 .$$

Let $\rho > 0$ be given. We can choose N_ρ so that for $m \geqq N_\rho$

(4.17) $$M_1 \leqq \frac{1}{2} \| R^m f - \lambda \|_1 = \| d_{m,1} \|_1 = \| d_{m,2} \|_1 \leqq M_1 + \rho .$$

For $n \geqq 0$ we have

$$(R^{m+n} f - \lambda) = R^n (R^m f - \lambda) \leqq R^n([R^m f - \lambda]^+) .$$

Hence

(4.18) $$d_{m+n,1} \leqq R^n(d_{m,1}) .$$

Similarly

(4.19) $$d_{m+n,2} \leqq R^n(d_{m,2}) .$$

Therefore

$$\| R^n(d_{m,i}) - d_{m+n,i} \|_1 = E(R^n d_{m,i} - d_{m+n,i}) = Ed_{m,i} - Ed_{m+n,i} \leqq \rho$$
$$\text{for} \quad i \in I_2 = \{1, 2\} .$$

Note that $d_{m+n,1}$ and $d_{m+n,2}$ have disjoint supports.

COROLLARY 4.1. *Let $\rho > 0$ and N_ρ be as in Proposition* 4.2. *For a given $m \geqq N_\rho$ and $i \in I_2$ put*

(4.20) $$h_i = d_{m,i} .$$

Let $s > 0$, $0 \leqq n_1 < n_2 < \cdots < n_s$. Then the 2^s functions given by

(4.21) $$g(i) = R^{n_1} h_{i_1} \wedge \cdots \wedge R^{n_s} h_{i_s} \quad \textit{for} \quad i = (i_1, \cdots, i_s) \in I_2^s$$

are ρ_1-orthogonal with $\rho_1 = s \cdot \rho$.

PROPOSITION 4.4. *Let $f_1, \cdots, f_s$ be nonnegative L^∞-functions such that*

(4.22) $$\| f_i \|_\infty \leqq M_0$$

for some positive constant M_0.

(i) *The following inequality holds*:

(4.23) $$E(f_1 \wedge f_2 \wedge \cdots \wedge f_s) \geqq E(f_1 \cdot f_2, \cdots, f_s)/M_0^{s-1} .$$

(ii) *The sequence* $\{E(R^n f_1 \wedge R^n f_2 \wedge \cdots \wedge R^n f_s)\}_{n=1}^{\infty}$ *is nondecreasing in* n.

PROOF. (i) We have

$$0 \leqq f_i/M_0 \leqq 1_X \qquad \text{for} \quad i = 1, \cdots, s .$$

Hence

$$(f_1/M_0)\cdot(f_2/M_0) \cdots (f_s/M_0) \leqq (f_1/M_0) \wedge (f_2/M_0) \wedge \cdots \wedge (f_s/M_0) = (f_1 \wedge \cdots \wedge f_s)/M_0 .$$

(ii) We have

$$R^m(R^n f_1 \wedge \cdots \wedge R^n f_s) \leqq R^{n+m} f_i \qquad \text{for} \quad i = 1, \cdots, s, m \geqq 1 .$$

Hence $R^m(R^n f_1 \wedge \cdots \wedge R^m f_s) \leqq R^{m+n} f_1 \wedge \cdots \wedge R^{n+m} f_s$. The rest of the proof follows from the fact that R preserves mean values.

PROPOSITION 4.5. *Let* $f_1, f_2 \in L^1$ *and* $\|f_1 - f_2\|_1 < \|f_1\|_1$. *Then the inequality*

$$\|\nu(f_1) - \nu(f_2)\|_1 \leqq 2\|f_1 - f_2\|_1/\|f_1\|_1 , \tag{4.24}$$

holds.

PROOF. $\|f_2\|_1 \geqq \|f_1\|_1 - \|f_1 - f_2\|_1 > 0$, and $\|f_1/\|f_1\|_1 - f_2/\|f_2\|_1\|_1 \leqq \|f_1 - f_2\|_1/\|f_1\|_1 + \|f_2\|_1 \cdot |(\|f_1\|_1^{-1} - \|f_2\|_1^{-1})| \leqq 2 \cdot \|f_1 - f_2\|_1/\|f_1\|_1$. Now we are able to finish the proof of Theorem 1.1 by proving (4.6). It is obvious that we can restrict our considerations to the space L^∞, which is dense in L^1. Let $f \in L^\infty$ and $M_0 = \|f\|^\infty > 0$. Let M_1 be given by (4.15). It is evident that (4.6) is equivalent to $M_1 = 0$. Let $M_1 > 0$. Let F be the weakly compact set mentioned in Definition 1.1. Let $\varepsilon = 1/4$ and δ be determined by (4.9). Take s so that

$$2^s > \delta^{-1} \tag{4.25}$$

and

$$\rho = 1/(4se_s) \tag{4.26}$$

where

$$e_s = M_1^s/(2M_0)^{s-1} . \tag{4.27}$$

Let N_ρ be as in Proposition 4.3 and $m \geqq N_\rho$. Let h_i, $i = 1, 2$, by given by (4.18) and (4.20). We show that there exist natural numbers $k_2 < k_3 < \cdots < k_s$ such that for any $n \geqq 0$ and $i \in I_2^s$ the nonnegative function

$$\bar{g}_n(i) = R^n h_{i_1} \wedge R^{n+k_2} h_{i_2} \wedge \cdots \wedge R^{n+k_s} h_{i_s} \tag{4.28}$$

satisfies the inequality

$$E\bar{g}_n(i) \geqq e_s \,. \tag{4.29}$$

According to Proposition 4.4 it suffices to prove that there exist $k_2 < k_3 < \cdots < k_s$ such that

$$E(h_{i_1} \cdot R^{k_1} h_{i_2} \cdot \cdots \cdot R^{k_s} h_{i_s}) \geqq M_1^s / 2^s \quad \text{for any} \quad i \in I_2^s \,. \tag{4.30}$$

But this is a direct consequence of Theorem 4.1 (ii) which yields that (under the assumption $\mu(X) = 1$)

$$\lim_{k \to \infty} E(f_1 \cdot R^k f_2) = Ef_1 \cdot Ef_2 \qquad \text{for} \quad f_1, f_2 \in L^2 \,. \tag{4.31}$$

According to Proposition 4.5 the 2^s sequences $\{E\bar{g}_n(\mathrm{i})\}$ are nondecreasing in n. Moreover, all of them are bounded from above by M_0. Hence they converge in n uniformly with respect to $i \in I_2^s$. Let $\kappa_1 = e_s/10$. There exists n_0 such for every $n \geqq n_0$ and $i \in I_2^s$ the inequality

$$E\bar{g}_n(i) - E\bar{g}_{n_0}(i) < \kappa_1 \tag{4.32}$$

holds. In the same way as in the proof of Proposition 4.5 we obtain that

$$\| \bar{g}_n(i) - R^{n-n_0} \bar{g}_{n_0}(i) \|_1 = E\bar{g}_n(i) - E\bar{g}_{n_0}(i) \,. \tag{4.33}$$

Finally, using Proposition 4.5 we get

$$\begin{aligned} \| \nu\bar{g}_n(i) - \nu[R^{n-n_0} \bar{g}_{n_0}(i)] \|_1 &= \| \nu\bar{g}_n(i) - R^{n-n_0} \nu[\bar{g}_{n_0}(i)] \|_1 \leqq 2\kappa_1/e_s \\ &= 1/50 \,. \end{aligned} \tag{4.34}$$

But for every $i \in I_2^s$ the sequence $R^{n-n_0}(\nu[\bar{g}_{n_0}(i)])$ converges to F because R is weakly constrictive. The number of considered sequences is finite, hence the above convergence is uniform with respect to i. Let $\kappa_2 = 1/20$. There exists n_1 such that

$$R^{n_1 - n_0}(\nu[\bar{g}_{n_0}(i)]) \in O_{\kappa_2}(F) \quad \text{for} \quad i \in I_2^s \,. \tag{4.35}$$

Combining (4.34) and (4.35) we obtain that for $\kappa = 1/4$ the neighbourhood

$$O_\kappa(F)$$

contains 2^s function $\{\bar{g}_{n_1}(i) \colon i \in I_2^s\}$. Moreover, these functions are ρ_1-orthogonal with $\rho_1 = 1/4$ according to (4.28), Proposition 4.3 and Corollary 4.1. But this contradicts Proposition 4.2. Hence we conclude

$$M_1 = 0 \qquad \text{for every} \quad f \in L^\infty \,,$$

which implies (4.1) and proves Theorem 1.1.

ACKNOWLEDGEMENT. The author is indebted to *A*. Lasota for helpful discussions and comments.

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