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DIRICHLET SETS AND SOME UNIQUENESS THEOREMS FOR WALSH SERIES

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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1. Introduction. Concerning the problem of uniqueness for Walsh series, we have already proved that *a Dirichlet set is always a U-set in the ordinary sense* and that *a subset of the dyadic group is a Dirichlet set if and only if it is a subset of some closed subgroup with Haar measure zero* [6].

In this paper, we shall generalize the above results and prove some uniqueness theorems for Dirichlet sets.

Fine [2] defined the *dyadic field*, $\mathscr{F}$, which is the set of all $0-1$ sequences $(\cdots, t_{-1}, t_0, t_1, \cdots)$ with $t_i = 0$ or 1 and $\lim_{n\to-\infty} t_n = 0$. For convenience, when $x = (\cdots, t_i, \cdots)$ satisfies $t_i = 0$ for sufficiently large i, we shall identify x with $(\sum_{k=-\infty}^{\infty} t_k/2^k)$. For example, $(\cdots, 0, \overset{0}{\cup}{1}, 1, 0, \cdots) = 3$ and $(\cdots, 0, 0, \overset{1}{\cup}{1}, 0, \cdots) = 1/2$. Define two operations: the addition denoted by $+$ and the product denoted by $\cdot$. When $x = (\cdots, t_i, \cdots)$ and $y = (\cdots, u_j, \cdots)$ are arbitrary elements of $\mathscr{F}$, the addition is defined by

$$x + y = (\cdots, |t_i - u_i|, \cdots) .$$

The product is defined by $x \cdot y = (\cdots, v_k, \cdots)$ where

$$v_k \equiv \sum_{i+j=k} t_i u_j \pmod 2 .$$

The distance between x and y is given by

$$\sum_{k=-\infty}^{\infty} |t_k - u_k|/2^k .$$

Hence $\mathscr{F}$ becomes a metric space. Moreover it is easy to see that $\mathscr{F}$ becomes a locally compact totally disconnected abelian group (see Rudin [4]).

The *dyadic group*, $\mathscr{G}$, is the subgroup of $\mathscr{F}$ in which each ele-

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ment has the form $(\cdots, 0, t_1, t_2, \cdots)$. For convenience, we shall identify $(\cdots, 0, t_1, t_2, \cdots)$ with $(t_1, t_2, \cdots)$. $\mathscr{G}$ is a compact totally disconnected abelian group. For details of the dyadic group we shall refer the reader to Fine [1].

We shall introduce the concept of dyadic intervals. I_n^p denotes a *dyadic interval of rank* n which is a set of all elements $(\cdots, t_k, \cdots) \in \mathscr{F}$ such that

$$\sum_{k=-\infty}^{n} t_k/2^k = p/2^n ,$$

for $n = 0, \pm 1, \pm 2, \cdots$ and $p = 0, 1, \cdots$. $I_n(x)$ denotes the dyadic interval of rank n which contains x. It is easy to see that I_0^0 coincides with the dyadic group $\mathscr{G}$ and any dyadic interval is closed and open in $\mathscr{F}$.

Let $\Gamma = \{0, 1, 2, \cdots\}$ be a subgroup of $\mathscr{F}$. The character functions of $\mathscr{G}$ are called *Walsh functions*, $\{w_n(x)\}_{n \in \Gamma}$, which are defined by the equation

$$w_n(x) = (-1)^{\sum_{i+j=1} t_i n_j} ,$$

where $x = (t_1, t_2, \cdots) \in \mathscr{G}$ and $n = (\cdots, n_{-2}, n_{-1}, 0, \cdots) \in \Gamma$. Then Γ is the dual group of $\mathscr{G}$.

We shall introduce the dyadic measures (see [5]). A real valued set function m on the dyadic intervals is said to be a *dyadic measure* if it satisfies the following additivity

$$m(I_n^p) = m(I_{n+1}^{2p}) + m(I_{n+1}^{2p+1}) ,$$

for $n = 0, 1, \cdots$ and $p = 0, 1, \cdots, 2^n - 1$. When $f(x)$ is an integrable function, set

$$m_f(I_n^p) = \int_{I_n^p} f(x)dx ,$$

for each dyadic interval I_n^p. Then m_f becomes a dyadic measure. If m is a Radon measure on $\mathscr{G}$, then it is a dyadic measure which satisfies

$$\sup_n \left(\sum_{p=0}^{2^n-1} |m(I_n^p)| \right) < \infty .$$

Conversely, if a dyadic measure m satisfies the above condition, then there exists a Radon measure m^* such that $m^*(I_n^p) = m(I_n^p)$ for each dyadic interval I_n^p. We shall identify m^* with m.

Let

$$\mu \equiv \sum_{k=0}^{\infty} \hat{\mu}(k) w_k(x) \equiv \sum_{k \in \Gamma} \hat{\mu}(k) w_k(x)$$

be an arbitrary Walsh series. The quantity

$$m_\mu(I_n^p) = \lim_{N\to\infty} \int_{I_n^p} \sum_{k=0}^{N} \hat{\mu}(k) w_k(x)dx = \sum_{k=0}^{2^n-1} \hat{\mu}(k) \int_{I_n^p} w_k(x)dx$$
$$= (1/2^n) \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(p/2^n)$$

is determined for each dyadic interval I_n^p. Moreover we have

$$m_\mu(I_n(x)) = (1/2^n) \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) .$$

The set function m_μ becomes a dyadic measure and

$$\hat{\mu}(k) = \sum_{p=0}^{2^n-1} m_\mu(I_n^p) w_k(p/2^n) = \int_{\mathscr{G}} w_k(x) m_\mu(dx) , \tag{1.1}$$

for $0 \leqq k < 2^n$ and $n = 0, 1, \cdots$, where $\int_{I_n^p} m_\mu(dx) = m_\mu(I_n^p)$. We shall call $\hat{\mu}(k)$ the *k-th Walsh Fourier coefficient of* μ.

Conversely, for an arbitrary dyadic measure m, the quantity (1.1) is determined for each $k = 0, 1, \cdots$. It is easy to see that m_μ coincides with m.

2. Strong U-sets. When $\mathscr{A}$ is a certain class of Walsh series, a subset E of $\mathscr{G}$ is said to be a *U-set for $\mathscr{A}$*, if $\mu \in \mathscr{A}$ and

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0 \quad \text{everywhere except on} \quad E , \tag{2.1}$$

imply that $\hat{\mu}(k) = 0$ for all k. When E is not a U-set for $\mathscr{A}$, it is called an *M-set for $\mathscr{A}$*.

A subset E of $\mathscr{G}$ is said to be a *strong U-set*, if the equality

$$\sup_k |\hat{\mu}(k)| = \liminf_{k\to\infty} |\hat{\mu}(k)| \tag{2.2}$$

holds for any Walsh series μ which satisfies (2.1) everywhere except on E. It is easy to see that *a strong U-set is a U-set for the class of Walsh series μ such that $\hat{\mu}(k) = o(1)$ as $k \to \infty$.* The concept of strong U-set was introduced by Kahane [3] for trigonometric Fourier series of Radon measures.

A subset E of $\mathscr{G}$ is called a *Dirichlet set*, if it satisfies the following equation:

$$\liminf_{n\to\infty} \sup_{x\in E} |1 - w_n(x)| = 0 .$$

Kahane [3] proved that *a Dirichlet set defined on the unit circle is a strong U-set.* We shall prove the analogues of this result.

When Δ is a subgroup of Γ, let $H(\Delta)$ be the annihilator of Δ, that is,

$$H(\Delta) = \{x;\ w_k(x) = 1 \quad \text{for all} \quad k \in \Delta\} .$$

If $\Gamma = \Delta \dotplus \Delta^*$ (direct sum) where Δ and Δ^* are both infinite subgroups of Γ, then $H(\Delta)$ and $H(\Delta^*)$ are both closed subgroups of $\mathscr{G}$ with Haar measure zero and $H(\Delta) \dotplus H(\Delta^*)$ (direct sum) coincides with $\mathscr{G}$ (see Rudin [4]).

From the definition of Dirichlet sets, we can find a monotone increasing sequence of integers $(n_k;\ k = 1, 2, \cdots)$ such that $w_{n_k}(x) = 1$ for all $x \in E$ and all k. We shall generalize the definition of Dirichlet sets. When $\Theta = (\theta_k;\ k \in \Delta)$ is a sequence of elements of $\mathscr{G}$, set

$$K(\Theta) = \{x;\ w_k(x) = w_k(\theta_k) \quad k \in \Delta\} .$$

THEOREM 2.1. *A Walsh series μ satisfies* (2.1) *for $x \notin K(\Theta)$, if and only if*

$$\hat{\mu}(n \dotplus k) = \hat{\mu}(n) w_k(\theta_k) \tag{2.3}$$

for all $n \in \Gamma$ and $k \in \Delta$.

PROOF. We shall first consider the case $K(\Theta) \neq \varnothing$. Since

$$w_{i \dotplus j}(u) = w_i(u) w_j(u) = w_i(\theta_i) w_j(\theta_j) = w_{i \dotplus j}(\theta_{i \dotplus j})$$

for $u \in K(\Theta)$, i and $j \in \Delta$, the equation

$$w_{i \dotplus j}(\theta_{i \dotplus j}) = w_i(\theta_i) w_j(\theta_j) \tag{2.4}$$

holds for all i and $j \in \Delta$. On the other hand, we know already [6] that there exist a monotone increasing sequence of integers $(N_j;\ j = 1, 2, \cdots)$ and a sequence of integers $(n_j;\ j = 1, 2, \cdots)$ such that $N_1 < N_2 < \cdots$,

$$2^{N_j} \leqq n_j < 2^{N_j + 1} , \quad j = 1, 2, \cdots$$

and

$$\Delta = \{\varepsilon_1 n_1 \dotplus \varepsilon_2 n_2 \dotplus \cdots;\ \varepsilon_i = 0 \quad \text{or} \quad 1 \quad \text{and} \quad \varepsilon_n = 0 \quad \text{for sufficiently large} \quad n\} . \tag{2.5}$$

From (2.4) and the above just cited, Θ is generated by $(\theta_{n(j)};\ j = 1, 2, \cdots)$, where $n(j) = n_j$. Hence $K(\Theta)$ coincides with

$$\{x;\ w_{n(k)}(x) = w_{n(k)}(\theta_{n(k)}),\ k = 1, 2, \cdots\} .$$

By the same argument as that used in the proof of Theorem 2 in [6], we can find an element $v_0 \in H(\Delta^*)$ such that

$$K(\Theta) = v_0 \dotplus H(\Delta) \equiv \{v_0 \dotplus u;\ u \in H(\Delta)\} .$$

We write, as usual, $y \dotplus E = \{y \dotplus x;\ x \in E\}$.

To prove the necessity, we need the following:

LEMMA 2.2. *If a Walsh series ν satisfies*

$$\liminf_{n\to\infty}\left|\sum_{k=0}^{2^n-1}\hat{\nu}(k)w_k(x)\right| = 0 \quad \textit{everywhere}\ ,$$

then $\hat{\nu}(k) = 0$ *for all* k.

PROOF OF LEMMA 2.2. It is easy to see that the inequality

$$|m_\nu(I_0^0)| \leqq |m_\nu(I_1^0)| + |m_\nu(I_1^1)| \leqq 2|m_\nu(I_1^{p(1)})|\ ,$$

holds, where $|m_\nu(I_1^{p(1)})| = \max\{|m_\nu(I_1^p)|;\ p = 0, 1\}$. In the same way we have

$$\begin{aligned}|\hat{\nu}(0)| = |m_\nu(I_0^0)| &\leqq 2|m_\nu(I_1^{p(1)})| \leqq 2^2|m_\nu(I_2^{p(2)})| \\ &\leqq \cdots \leqq 2^n|m_\nu(I_n^{p(n)})| \leqq \cdots\ ,\end{aligned}$$

where $I_n^{p(n)}$ is a subset of $I_{n-1}^{p(n-1)}$ which satisfies

$$|m_\nu(I_n^{p(n)})| = \max\{|m_\nu(I_n^p)|;\ p = 2p_{n-1}, 2p_{n-1}+1\}$$

for all n. Since each $I_n^{p(n)}$ is a closed set, we can set $\bigcap_{n=1}^\infty I_n^{p(n)} = \{z\}$. Then we have $I_n^{p(n)} = I_n(z)$ for all n. From the above inequality, we obtain

$$|\hat{\nu}(0)| \equiv |m_\nu(I_0^0)| \leqq \cdots \leqq 2^n|m_\nu(I_n(z))|\ .$$

By

$$2^n m_\nu(I_n(z)) = \sum_{k=0}^{2^n-1}\hat{\nu}(k)w_k(z)$$

and by assumption, we have

$$\begin{aligned}|\hat{\nu}(0)| = |m_\nu(I_0^0)| &\leqq \cdots \leqq \liminf_{n\to\infty}|2^n m_\nu(I_n(z))| \\ &= \liminf_{n\to\infty}\left|\sum_{k=0}^{2^n-1}\hat{\nu}(k)w_k(z)\right| = 0\ .\end{aligned}$$

Thus $\hat{\nu}(0) = 0$. A similar argument shows $\hat{\nu}(k) = 0$ for all k. The proof is complete.

We shall prove the necessity. By Lemma 2.2, $m_\mu(I_n^p) = 0$ for each dyadic interval I_n^p such that $I_n^p \cap K(\Theta) = \varnothing$. Let $\{I_N^{p(i)};\ i = 1, 2, \cdots, s\}$ be the family of the dyadic intervals of rank N such that $I_N^{p(i)} \cap K(\Theta) \neq \varnothing$ and let x_i, for each i, be an element of $I_N^{p(i)} \cap K(\Theta)$. When $0 \leqq n \dotplus k < 2^N$, $k \in \varDelta$ and $n \in \Gamma$, the $(n \dotplus k)$-th Walsh Fourier coefficient $\hat{\mu}(n \dotplus k)$ of μ satisfies the following equation:

$$\hat{\mu}(n \dotplus k) = \sum_{p=0}^{2^N-1} m_\mu(I_N^p)w_{n\dotplus k}(p/2^N) = \sum_{I_N^p \cap K(\Theta)\neq\varnothing} m_\mu(I_N^p)w_{n\dotplus k}(p/2^N)$$

$$= \sum_{i=1}^{s} m_\mu(I_N(x_i))w_n(x_i)w_k(x_i) = w_k(\theta_k)\sum_{i=1}^{s} m_\mu(I_N(x_i))w_n(x_i)$$

$$= w_k(\theta_k)\sum_{p=0}^{2^N-1} m_\mu(I_N^p)w_n(p/2^n) = w_k(\theta_k)\hat{\mu}(n) .$$

Next we shall prove the sufficiency. Set

$$\Delta_n = \Delta \cap \{0, 1, \cdots, 2^n - 1\} \quad\text{and}\quad \Delta_n^* = \Delta^* \cap \{0, 1, \cdots, 2^n - 1\}$$

for all n. For each element x of $\mathscr{G}$, there exists a unique pair (u, v) such that $u \in H(\Delta)$, $v \in H(\Delta^*)$ and $x = u \dot{+} v$. Therefore, we have

$$\sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) = \sum_i^{(n)}\sum_j^{(n)*} \hat{\mu}(i \dot{+} j)w_{i\dot{+}j}(u \dot{+} v) = \sum_i^{(n)}\sum_j^{(n)*} \hat{\mu}(j)w_i(\theta_i)w_i(v)w_j(u)$$

$$= \left\{\sum_i^{(n)} w_i(v_0)w_i(v)\right\}\left\{\sum_j^{(n)*} \hat{\mu}(j)w_j(u)\right\}$$

$$= \left\{\sum_i^{(n)} w_i(v_0 \dot{+} v)\right\}\{1/\# \Delta_n^*\}\left\{\sum_j^{(n)*} 1\right\}\left\{\sum_j^{(n)*} \hat{\mu}(j)w_j(u)\right\}$$

$$= \{1/\# \Delta_n^*\}\left\{\sum_i^{(n)} w_i(v_0 \dot{+} v)\right\}\left\{\sum_j^{(n)*} w_j(v_0 \dot{+} v)\right\}\left\{\sum_j^{(n)*} \hat{\mu}(j)w_j(u)\right\}$$

$$= \{1/\# \Delta_n^*\}\left\{\sum_{i=0}^{2^n-1} w_i(v_0 \dot{+} v)\right\}\left\{\sum_j^{(n)*} \hat{\mu}(j)w_j(u)\right\} ,$$

where $\sum_i^{(n)}$ and $\sum_j^{(n)*}$ denote summations over all $i \in \Delta_n$ and $j \in \Delta_n^*$, respectively, and $\# S$ is the cardinality of S. If $x \notin K(\Theta)$, then $v \neq v_0$. For sufficiently large n, the expression inside the first bracket is zero. Hence (2.1) holds for all $x \notin K(\Theta)$.

When $K(\Theta) = \varnothing$, by Lemma 2.2, we have $\hat{\mu}(n) = 0$ for all n. Then (2.3) holds.

We shall prove the converse. Since $K(\Theta) = \varnothing$, there exist a pair of integers i and j of Δ which do not satisfy (2.4). On the other hand, for each n,

$$\hat{\mu}(n \dot{+} i \dot{+} j) = \hat{\mu}(n)w_{i\dot{+}j}(\theta_{i\dot{+}j}) = \hat{\mu}(n \dot{+} i)w_j(\theta_j) = \hat{\mu}(n)w_i(\theta_i)w_j(\theta_j) .$$

Hence we have $\hat{\mu}(n) = 0$ for all n. Therefore, if a Walsh series μ satisfies (2.3), then (2.1) holds. Theorem 2.1 is proved.

COROLLARY 2.3. *When Δ and Δ^* are infinite subgroups of Γ and $\Delta \dot{+} \Delta^* = \Gamma$ (direct sum), a Walsh series μ satisfies* (2.1) *for $x \notin H(\Delta)$ if and only if*

$$\hat{\mu}(n \dot{+} k) = \hat{\mu}(n) \tag{2.6}$$

for each $k \in \Delta$ and $n \in \Gamma$.

Corollary 2.3 is a generalization of Theorem 2.7.1 in [4] for the dyadic group.

COROLLARY 2.4. *The set $K(\Theta)$ which is defined in Theorem 2.1 is a strong U-set.*

PROOF. It is obvious that

$$\limsup_{k\to\infty} |\hat{\mu}(k)| \leqq \sup_k |\hat{\mu}(k)| .$$

On the other hand, since $|w_k(\theta)| = 1$ for all $\theta \in \mathscr{G}$,

$$\begin{aligned}\sup_n |\hat{\mu}(n)| &= \sup_i \sup_j |\hat{\mu}(i+j)| = \sup_i \sup_j |\hat{\mu}(j)w_i(\theta_i)| = \sup_j |\hat{\mu}(j)| \\ &= \lim_{\substack{i\to\infty \\ i\in\Delta}} \sup_j |\hat{\mu}(j)| = \lim_{\substack{i\to\infty \\ i\in\Delta}} \sup_j |\hat{\mu}(i+j)| \\ &\leqq \limsup_{k\to\infty} |\hat{\mu}(k)| ,\end{aligned}$$

where $\sup_i$ and $\sup_j$ mean the upper limits in the regions Δ and Δ^*, respectively. Therefore, the equality (2.2) holds.

The proof is complete.

3. Uniqueness theorems for some classes of Walsh series. It is known [7] that *any perfect set of Haar measure zero is an M-set for the class of Walsh series μ such that*

$$(1/2^n)\sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 = \sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|^2 = o(1) \quad as \quad n \to \infty .$$

Now we can prove the following:

THEOREM 3.1. *If $(\varepsilon_n;\ n = 1, 2, \cdots)$ is a sequence of positive numbers such that $\varepsilon_n \downarrow 0$ and $1/\varepsilon_n = o(n)$ as $n \to \infty$, then there exists a Dirichlet set which is an M-set for the class of Walsh series μ such that*

$$(1/2^n)\sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 = o(\varepsilon_n) \quad as \quad n \to \infty .$$

To prove Theorem 3.1, we need the following:

LEMMA 3.2. *When $(\gamma_n = \gamma(n);\ n = 0, 1, \cdots)$ is a monotone increasing sequence of positive integers such that $\gamma_n = \gamma_{n+1}$ or $\gamma_{n+1} = 2\gamma_n$, $\gamma_0 = 1$ and $\gamma_n = o(2^n)$ as $n \to \infty$, there exists a Dirichlet set H which satisfies*

$$\sharp\{p;\ I_n^p \cap H \neq \varnothing\} = \gamma_n \quad for\ all \quad n . \tag{3.1}$$

PROOF. Set $E_0 = \mathscr{G} \equiv I_0^0$. Set $E_1 = I_1^0$ if $\gamma_1 = 1$, while $E_1 = I_1^0 \cup I_1^1 = E_0$ if $\gamma_1 = 2$. In general, when

$$E_n = I_n^0 \cup I_n^{i_1} \cup \cdots \cup I_n^{i_{\gamma(n)}} ,$$

we shall define E_{n+1} as follows:

$$E_{n+1} = \begin{cases} I^0_{n+1} \cup I^{2i_1}_{n+1} \cup \cdots \cup I^{2i_{\gamma(n)}}_{n+1}, & \text{if} \quad \gamma_{n+1} = \gamma_n , \\ E_n , & \text{if} \quad \gamma_{n+1} = 2\gamma_n . \end{cases}$$

It is obvious that E_n is a subgroup of the dyadic group. Set $H = \bigcap_{n=0}^{\infty} E_n$. Then H is a Dirichlet set which satisfies (3.1). The proof is complete.

PROOF OF THEOREM 3.1. Let $(\gamma_n; n = 0, 1, \cdots)$ be a sequence of positive integers satisfying the assumptions of Lemma 3.2 and $1/\gamma(n) = o(\varepsilon_n)$ as $n \to \infty$. When I_n is a dyadic interval of rank n, let m_μ be the positive Radon measure which is defined by the following equation:

$$m_\mu(I_n) = \begin{cases} 1/\gamma(n) , & \text{if} \quad I_n \cap H \neq \varnothing , \\ 0 , & \text{if} \quad I_n \cap H = \varnothing . \end{cases}$$

Therefore, we have

$$\begin{aligned} (1/2^n) \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 &= \sum_{p=0}^{2^n-1} |m_\mu(I^p_n)|^2 \\ &= \sum_{I^p_n \cap H \neq \varnothing} |m_\mu(I^p_n)|^2 = \sharp\{p: I^p_n \cap H \neq \varnothing\} 1/\gamma(n)^2 \\ &= 1/\gamma(n) = o(\varepsilon_n) \quad \text{as} \quad n \to \infty . \end{aligned}$$

It is obvious that $\hat{\mu}(0) = 1$ and μ satisfies (2.1) except on H. Hence the proof of Theorem 3.1 is complete.

We can easily modify the argument used to prove Theorem 3.1 and obtain the following result as well.

COROLLARY 3.3. *For each $\alpha > 0$, any perfect set of Haar measure zero is an M-set for the class of Walsh series μ such that*

$$\sum_{p=0}^{2^n-1} |m_\mu(I^p_n)|^{1+\alpha} = o(1) \quad \textit{as} \quad n \to \infty .$$

Moreover, if $(\varepsilon_n; n = 0, 1, \cdots)$ is a sequence of positive numbers introduced in Theorem 3.1, then there exists a Dirichlet set which is an M-set for the class of Walsh series μ such that

$$\sum_{p=0}^{2^n-1} |m_\mu(I^p_n)|^{1+\alpha} = o(\varepsilon_n) \quad \textit{as} \quad n \to \infty .$$

THEOREM 3.4. *Let ψ be a non-negative function defined on $[0, \infty)$ and satisfying $\psi(0+) = 0$. Any Dirichlet set is a U-set for the class of Walsh series μ such that*

$$\sum_{k=2^n}^{2^{n+1}-1} \psi(|\hat{\mu}(k)|) = O(1) \quad \textit{as} \quad n \to \infty .$$

PROOF. Let E be a Dirichlet set. There exists a closed subgroup $H(\Delta)$ of Haar measure zero such that $H(\Delta)\supset E$ where Δ has the form (2.5). Let a Walsh series μ satisfy (2.1) everywhere except on E. By Corollary 2.2 we obtain

$$\sum_{k=2^{N(s)}}^{2^{N(s)+1}-1} \psi(|\hat{\mu}(k)|) = \sum_{k=0}^{2^{N(s)}-1} \psi(|\hat{\mu}(n_s + k)|) = \sum_{i}^{(N(s))} \sum_{j}^{(N(s))^*} \psi(|\hat{\mu}(n_s + i + j)|)$$
$$= \sum_{i}^{(N(s))} \sum_{j}^{(N(s))^*} \psi(|\hat{\mu}(j)|) = \# \Delta_{N(s)} \cdot \sum_{j}^{(N(s))^*} \psi(|\hat{\mu}(j)|) ,$$

where $N(s) = N_s$. By assumption we have

$$\sum_{j}^{(N(s))^*} \psi(|\hat{\mu}(j)|) = O(1/\# \Delta_{N(s)}) = o(1) \quad \text{as} \quad s \to \infty .$$

It follows that

$$\psi(|\hat{\mu}(j)|) = 0 \quad \text{for all} \quad j \in \Delta^* .$$

In particular, we have $\hat{\mu}(j) = 0$ for all $j \in \Delta^*$. Consequently, from (2.6) we can prove that $\hat{\mu}(k) = 0$ for all k. The proof is complete.

An easy computation gives the following corollary.

COROLLARY 3.5. *Any Dirichlet set is a U-set for the class of Walsh series μ such that*

$$\sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}(k)|^2 = O(1) \quad as \quad n \to \infty .$$

COROLLARY 3.6. *Any Dirichlet set is a U-set for the class of Walsh series μ such that*

$$\sum_{k=2^n}^{2^{n+1}-1} \hat{\mu}(k) w_k(x) = O(1) \quad uniformly\ in \quad x \quad as \quad n \to \infty .$$

PROOF. An easy computation shows that

$$\sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}(k)|^2 = 2^n \sum_{p=0}^{2^n-1} \left| (1/2^n) \sum_{k=2^n}^{2^n-1} \hat{\mu}(k) w_k(p/2^n) \right|^2$$
$$= 2^n \sum_{p=0}^{2^n-1} O(1/4^n) = O(1) \quad \text{as} \quad n \to \infty .$$

Hence the class of Walsh series mentioned in this corollary is included in the class mentioned in Corollary 3.5. Then the conclusion follows from Corollary 3.5.

THEOREM 3.7. *When $(\eta_n;\ n = 1, 2, \cdots)$ is a monotone increasing sequence of positive numbers tending to infinity, there exists a Dirichlet set which is an M-set for the class of Walsh series μ such that*

$$\sum_{k=2^n}^{2^{n+1}-1} \hat{\mu}(k) w_k(x) = o(\eta_n) \quad \textit{uniformly in } x \textit{ as } n \to \infty .$$

PROOF. Let $(\varepsilon_k; k = 1, 2, \cdots)$ be a monotone decreasing sequence of positive numbers tending to zero such that $\eta_k \varepsilon_k \uparrow \infty$ as $k \to \infty$. Let $(N(n); n = 1, 2, \cdots)$ be a monotone increasing sequence of positive integers tending to infinity and satisfying

$$2^{N(n+1)-n} < \eta_n \varepsilon_n \quad \text{and} \quad 2^{N(n+1)-n} \uparrow \infty \quad \text{as} \quad n \to \infty .$$

Let $\xi_n \in I^1_{N(n)}$ for each n and set

$$H(\Delta) = \{\alpha_1 \xi_1 + \alpha_2 \xi_2 + \cdots ; \alpha_i = 0 \quad \text{or} \quad 1\} .$$

We shall construct a positive dyadic measure m_μ. To begin with, set $m_\mu(I^0_0) \equiv m_\mu(\mathscr{G}) = 1$. Next, when $0 < N < N(1)$ and I_N is a dyadic interval of rank N, set

$$m_\mu(I_N) = \begin{cases} 1, & \text{if} \quad I_N \cap H(\Delta) \neq \emptyset , \\ 0, & \text{otherwise} . \end{cases}$$

In general, set

$$m_\mu(I_N) = \begin{cases} 1/\lambda(N), & \text{if} \quad I_N \cap H(\Delta) \neq \emptyset , \\ 0, & \text{otherwise} , \end{cases}$$

where $\lambda(N) = \# \{p; I^p_N \cap H(\Delta) \neq \emptyset\}$. When $N(n) \leqq N < N(n+1)$, we have

$$m_\mu(I_N(x)) = (1/2^N) \sum_{k=0}^{2^N-1} \hat{\mu}(k) w_k(x) = \begin{cases} 1/2^n, & \text{if} \quad x \in H(\Delta) , \\ 0, & \text{otherwise} . \end{cases}$$

It follows that

$$\sum_{k=2^N}^{2^{N+1}-1} \hat{\mu}(k) w_k(x) = 0 \quad \text{for} \quad N = N(1), N(2), \cdots ,$$

and for $N(n) < N < N(n+1)$

$$\sum_{k=2^N}^{2^{N+1}-1} \hat{\mu}(k) w_k(x) = \begin{cases} 1/2^n, & \text{if} \quad x \in H(\Delta) , \\ 0, & \text{otherwise} . \end{cases}$$

Therefore for $N(n) \leqq N < N(n+1)$, m_μ satisfies

$$\sum_{k=2^N}^{2^{N+1}-1} \hat{\mu}(k) w_k(x) = O(2^N/2^n) = O(2^{N(n+1)-n}) = O(\eta_n \varepsilon_n) = o(\eta_n) \quad \text{as} \quad n \to \infty .$$

The proof is complete.

4. Uniqueness problem for some special Walsh series. Throughout this chapter, Δ and Δ^* denote infinite subgroups of Γ such that $\Delta + \Delta^* = \Gamma$ (direct sum).

THEOREM 4.1. *If a Walsh series satisfies* $\hat{\mu}(k) = 0$ *for* $k \notin \Delta$ *and*

$$\liminf_{n\to\infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| = 0 \quad \textit{everywhere in} \quad H(\Delta^*) ,$$

then $\hat{\mu}(k) = 0$ *for all* k.

PROOF. For each element x of $\mathscr{G}$, set $x = u + v$ (uniquely) where $u \in H(\Delta)$ and $v \in H(\Delta^*)$. Then

$$\sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = \sum_i^{(n)} \hat{\mu}(i) w_i(x) = \sum_i^{(n)} \hat{\mu}(i) w_i(u + v) = \sum_i^{(n)} \hat{\mu}(i) w_i(v) ,$$

where $\sum_i^{(n)}$ is the summation over all $i \in \Delta_n$. By assumption, we obtain that

$$\liminf_{n\to\infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| = 0 \quad \text{everywhere} .$$

By Lemma 2.2 we have $\hat{\mu}(k) = 0$ for all k. The proof is complete.

THEOREM 4.2. *A Walsh series* μ *satisfies the assumption of Theorem* 4.1 *if and only if the dyadic measure* m_μ *satisfies*

$$m_\mu(y + I_n^p) = m_\mu(I_n^p) \tag{4.1}$$

for each dyadic interval I_n^p *and* $y \in H(\Delta)$.

PROOF. It is already known that

$$m_\mu(I_n(x)) = (1/2^n) \sum_i^{(n)} \hat{\mu}(i) w_i(x) .$$

Since $y + I_n(x) = I_n(y + x)$, we have

$$\begin{aligned} m_\mu(y + I_n(x)) &= (1/2^n) \sum_i^{(n)} \hat{\mu}(i) w_i(y + x) = (1/2^n) \sum_i^{(n)} \hat{\mu}(i) w_i(x) w_i(y) \\ &= (1/2^n) \sum_i^{(n)} \hat{\mu}(i) w_i(x) = m_\mu(I_n(x)) . \end{aligned}$$

Next, we shall prove the converse. Since $\mathscr{G} = H(\Delta) + H(\Delta^*)$ (direct sum), for each $x \in \mathscr{G}$, we can write $x = u + v$, $u \in H(\Delta)$ and $v \in H(\Delta^*)$. Then, we have

$$\begin{aligned} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) &= \sum_i^{(n)} \sum_j^{(n)*} \hat{\mu}(i + j) w_{i+j}(u + v) = \sum_i^{(n)} \sum_j^{(n)*} \hat{\mu}(i + j) w_i(v) w_j(u) \\ &= 2^n m_\mu(I_n(x)) = 2^n m_\mu(I_n(y + x)) = \sum_i^{(n)} \sum_j^{(n)*} \hat{\mu}(i + j) w_i(v) w_j(u) w_j(y) . \end{aligned}$$

On the other hand, it is easy to see that $\{w_j(u);\ j \in \Delta^*\}$ forms the character functions of $H(\Delta^*)$, which is a topological group. Let dy_Δ denote the

Haar measure on $H(\Delta)$. We have

$$\sum_{k=0}^{2^n-1}\hat{\mu}(k)w_k(x) = \sum_i^{(n)}\sum_j^{(n)*}\hat{\mu}(i\dotplus j)w_i(v)w_j(u)\int_{H(\Delta)}w_j(y)dy_\Delta$$
$$= \sum_i^{(n)}\hat{\mu}(i\dotplus 0)w_i(v)w_0(u) = \sum_i^{(n)}\hat{\mu}(i)w_i(x) .$$

Thus, we have proved that $\hat{\mu}(k) = 0$ for $k \notin \Delta$. The proof is complete.

Now, we shall consider a uniqueness problem on $H(\Delta^*)$ for the class of Walsh series μ such that $\hat{\mu}(k) = 0$ for $k \notin \Delta$. A subset E of $H(\Delta^*)$ is called a *U-set on* $H(\Delta^*)$, if the following two conditions, $\hat{\mu}(k) = o(1)$ as $k \to \infty$ in Δ and

$$\lim_{n\to\infty}\sum_i^{(n)}\hat{\mu}(i)w_i(x) = 0 \quad \text{everywhere in} \quad H(\Delta^*) \quad \text{except on} \quad E ,$$

imply that $\hat{\mu}(k) = 0$ for all k.

THEOREM 4.3. *When a subset E of $H(\Delta^*)$ is closed, a set $E \dotplus H(\Delta)$ is a U-set in the ordinary sense if and only if E is a U-set on $H(\Delta^*)$.*

PROOF. Assume that E is a U-set on $H(\Delta^*)$. Then

$$\sum_{k=0}^{2^n-1}\hat{\mu}(k)w_k(x) = \sum_i^{(n)}\sum_i^{(n)*}\hat{\mu}(i\dotplus j)w_{i\dotplus j}(x) = \sum_i^{(n)}\sum_j^{(n)*}\hat{\mu}(i\dotplus j)w_j(u)w_i(v) ,$$

where $x = u \dotplus v$, $u \in H(\Delta)$ and $v \in H(\Delta^*)$. Since E is a closed set, $E^c \dotplus H(\Delta)$ is an open set where $E^c = H(\Delta^*)\setminus E$. Set

$$E^c \dotplus H(\Delta) = \bigcup_j I_{N(j)}^{p(j)} ,$$

where $N(1) < N(2) < \cdots$. Let I_N^p be one of the above dyadic intervals. By assumption, $m_\mu(I') = 0$ for each dyadic interval $I' \subset I_N^p$. Thus we deduce that

$$\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\hat{\mu}(k)w_k(x) = 0 \quad \text{uniformly in} \quad I_N^p .$$

Set $J = I_N^p \cap H(\Delta^*)$. Then $I_N^p = J \dotplus H(\Delta^*)$. Let du_Δ be the Haar measure on $H(\Delta)$. For each $v \in J$, we can prove that

$$\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\hat{\mu}(k)w_k(x) = 0 \quad \text{uniformly in} \quad v \dotplus H(\Delta) ,$$

since $v \dotplus H(\Delta) \subset J \dotplus H(\Delta) = I_N^p$. In particular,

$$\lim_{n\to\infty}\int_{H(\Delta)}\sum_{k=0}^{2^n-1}\hat{\mu}(k)w_k(u\dotplus v)du_\Delta$$
$$= \lim_{n\to\infty}\sum_i^{(n)}\sum_j^{(n)*}\hat{\mu}(i\dotplus j)w_i(v)\int_{H(\Delta)}w_j(u)du_\Delta = \lim_{n\to\infty}\sum_i^{(n)}\hat{\mu}(i)w_i(v) = 0 .$$

We have proved that

$$\lim_{n\to\infty} \sum_i^{(n)} \hat{\mu}(i) w_i(v) = 0 \quad \text{everywhere in} \quad H(\Delta^*) \quad \text{except on } E\,.$$

Since E is a U-set on $H(\Delta^*)$, it follows that $\hat{\mu}(k) = 0$ for all $k \in \Delta$. Similarly, if we consider for each $s \in \Delta^*$ the Walsh series

$$\sum_{k=0}^{\infty} \hat{\mu}(k) w_k(x) w_s(x) = \sum_{k=0}^{\infty} \hat{\mu}(k \dotplus s) w_k(x)\,,$$

then we can prove that $\hat{\mu}(k \dotplus s) = 0$ for all $k \in \Delta$. Since $s \in \Delta^*$, we obtain $\hat{\mu}(k) = 0$ for all k.

Conversely, assume that E is an M-set on $H(\Delta^*)$. There exists a Walsh series μ such that $\hat{\mu}(k) = 0$ for all $k \notin \Delta$,

$$\lim_{n\to\infty} \sum_i^{(n)} \hat{\mu}(i) w_i(v) = 0 \quad \text{everywhere in} \quad H(\Delta^*) \quad \text{except on} \quad E\,,$$

$\hat{\mu}(k) = o(1)$ for $k \in \Delta$ as $k \to \infty$ and $\hat{\mu}(0) \neq 0$. It is obvious that $w_k(u \dotplus v) = w_k(v)$ for each $u \in H(\Delta)$ and $k \in \Delta$. When $x = u \dotplus v$ for $u \in H(\Delta)$ and $v \in H(\Delta^*)$, then

$$\sum_i^{(n)} \hat{\mu}(i) w_i(x) = \sum_i^{(n)} \hat{\mu}(i) w_i(v)\,.$$

Hence we conclude that

$$\lim_{n\to\infty} \sum_i^{(n)} \hat{\mu}(i) w_i(x) = 0 \quad \text{everywhere except on} \quad E \dotplus H(\Delta)\,.$$

It follows that $E \dotplus H(\Delta)$ is an M-set in the ordinary sense. The proof is complete.

5. An extension of Dirichlet sets. It is already known that when $\Delta \dotplus \Delta^* = \Gamma$ (direct sum) for infinite subgroups Δ and Δ^* of Γ, the dyadic group $\mathscr{G}$ coincides with $H(\Delta) \dotplus H(\Delta^*)$ (direct sum), where $H(\Delta)$ and $H(\Delta^*)$ are both Dirichlet sets. When $T(u)$ is a function which is defined on $H(\Delta)$ and takes values in $H(\Delta^*)$, set

$$S_T = \{u \dotplus T(u);\, u \in H(\Delta)\}\,.$$

THEOREM 5.1. *If a continuous function T satisfies*

$$T(u \dotplus u') = T(u) \dotplus T(u')$$

for $u, u' \in H(\Delta)$, then S_T is a Dirichlet set.

PROOF. For x and $y \in S_T$, there exist u and $w \in H(\Delta)$ such that $x = u \dotplus T(u)$ and $y = w \dotplus T(w)$. Since

$$x \dot{+} y = (u \dot{+} w) \dot{+} (T(u) \dot{+} T(w)) = (u \dot{+} w) \dot{+} T(u \dot{+} w)\ ,$$

we have $x \dot{+} y \in S_T$. That is, S_T is a subgroup of the dyadic group. Let $(x_n;\ n = 1, 2, \cdots)$ be a sequence of elements of S_T such that $\lim_{n\to\infty} x_n = x = u \dot{+} v$ with $u \in H(\Delta)$ and $v \in H(\Delta^*)$. We shall prove that $x \in S_T$. Set $x_n = u_n \dot{+} T(u_n)$, where $u_n \in H(\Delta^*)$. It suffices to show that $\lim_{n\to\infty} u_n = u$.

By assumption, the sequence $(u_n;\ n = 1, 2, \cdots)$ is bounded. Hence there exist a subsequence $(u_{n(k)};\ k = 1, 2, \cdots)$ of this sequence and $u_0 \in H(\Delta)$ such that $\lim_{k\to\infty} u_{n(k)} = u_0$. Then we have

$$\lim_{k\to\infty} \{u_{n(k)} \dot{+} T(u_{n(k)})\} = u_0 \dot{+} T(u_0) = u \dot{+} v\ .$$

Hence $u_0 \dot{+} u = T(u_0) \dot{+} v$. It is obvious that $H(\Delta) \cap H(\Delta^*) = \{0\}$. Since $u_0 \dot{+} u \in H(\Delta)$ and $T(u_0) \dot{+} v \in H(\Delta^*)$, we see that $u_0 \dot{+} u = T(u_0) \dot{+} v = 0$, that is, $u_0 = u$ and $v = T(u_0) = T(u)$. We proved that $u \in H(\Delta)$ and $x = u \dot{+} T(u)$. The proof is complete.

References

[1] N. J. Fine, On Walsh functions, Trans. Amer. Math. Soc. 65 (1949), 372–414.

[2] N. J. Fine, The generalized Walsh functions, *ibid.* 69 (1950), 66–77.

[3] J. P. Kahane, A metric condition for a closed circular set to be a set of uniquness, J. Approximation theory 2 (1969), 233–246.

[4] W. Rudin, Fourier analysis on groups, Interscience Publishers, New York-London, 1962.

[5] W. R. Wade and K. Yoneda, Uniqueness and quasi-measures on the group of integers of a p-series field, Proc. Amer. Math. Soc. 84 (1982), 202–206.

[6] K. Yoneda, Perfect sets of uniqueness on the group 2, Canad. J. Math. 34 (1982), 759–764.

[7] K. Yoneda, Sets of multiplicity on the dyadic group, Acta Math. Acad. Sci. Hungar. 41 (1983), 195–200.

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