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CONTINUITY OF CERTAIN DIFFERENTIALS ON FINITELY AUGMENTED TEICHMÜLLER SPACES AND VARIATIONAL FORMULAS OF SCHIFFER-SPENCER'S TYPE

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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1. Introduction and statement of results. For a Riemann surface R^*, the finitely augmented Teichmüller space $\hat{T}(R^*)$ of R^* is the set of all marked Riemann surfaces R with at most a finite number of nodes such that there is a marking-preserving deformation of R^* to R, and $\hat{T}(R^*)$ is equipped with the conformal topology. (For the details, see [7, §1, 1°)].) Here we recall some of definitions.

For two given points R_1 and R_2 in $\hat{T}(R^*)$, a marking-preserving deformation $(f; R_1, R_2)$ of R_1 to R_2 is a marking-preserving continuous surjection f from R_1 onto R_2 such that f^{-1} restricted to $R_2 - \bar{U}$ is quasiconformal for every neighborhood U of the set $N(R_2)$ on R_2, and that $f^{-1}(p)$ is either a node of R_1 or a simple closed curve on $R_1 - N(R_1)$ for every p in $N(R_2)$, where here and in the sequel, $N(R)$ means the set of all nodes of R. A one-parameter family $\{(f_t; R_t, R_0)\}_{t\in(0,1]}$ of marking-preserving deformations f_t of $R_t \in \hat{T}(R^*)$ to $R_0 \in \hat{T}(R^*)$ is called *admissible* if

$$\lim_{t\to 0} K(f_t^{-1}, R_0 - \bar{U}) = 1$$

for every neighborhood U of $N(R_0)$, where here and in the sequel $K(f, E)$ is the maximal dilatation of a quasiconformal mapping f on a Borel set E. Recall that R_t converges to R_0 in $\hat{T}(R^*)$ if and only if there is an admissible family $\{(f_t; R_t, R_0)\}$.

In [7, §3], certain continuity property of holomorphic and harmonic differentials on $\hat{T}(R^*)$ was investigated. In particular, we showed strongly metrical continuity of period reproducers on $\hat{T}(R^*)$. Namely, let $\sigma(c, R)$ be the period reproducer for a 1-cycle c on $R \in \hat{T}(R^*)$ in the space $\Gamma_h(R)$ of all square integrable (real) harmonic differentials on $R - N(R)$ (cf. [7, §1, 2°)]). Then we have the following:

THEOREM A ([7, Proposition 4]). *Let an admissible family* $\{(f_t; R_t, R_0)\}_{t\in(0,1]}$ *of marking-preserving deformations and a* 1*-cycle* d *on*

$R_0 - N(R_0)$ be given arbitrarily. Set

$$\theta(t, d) = \sigma(f_t^{-1}(d), R_t) + \sqrt{-1}\cdot {}^*\sigma(f_t^{-1}(d), R_t)$$

for every t. Then $\theta(t, d)$ converges to $\theta(0, d) = \sigma(d, R_0) + \sqrt{-1}\cdot {}^*\sigma(d, R_0)$ strongly metrically with respect to $\{f_t\}$, that is,

$$\lim_{t\to 0} \|\theta(t, d)\circ f_t^{-1} - \theta(0, d)\|_{R_0 - \bar{U}} = 0$$

for every neighborhood U of $N(R_0)$.

Here and in the sequel, $f(d)$ is the 1-cycle corresponding to d under the mapping f, $\theta\circ f$ is the pull-back of a differential θ by f, and $\|\omega\|_E$ is the Dirichlet norm of a differential ω on a Borel set E, namely, $\|\omega\|_E^2 = \iint_E \omega\wedge\overline{{}^*\omega}$.

REMARK 1. In the proof of [7, Proposition 4], we have actually shown that $\theta(f_n^{-1}(d), R_n)$ converges to $\theta(d, R_0)$ for every admissible sequence $\{(f_n; R_n, R_0)\}_{n=1}^{\infty}$ of marking-preserving deformations, which clearly implies Theorem A. The proof is valid also for admissible families (with continuous parameters).

Now in this paper, continuing the above investigation, we show strongly metrical continuity of Green's functions. Here we define a function $g(p, q; R)$ on $R - N(R)$ for every $R\in\hat{T}(R^*)$ and $q\in R - N(R)$ as follows: On every component of $R - N(R)$ not containing q, we set $g(\cdot, q; R)\equiv 0$, and on the component S containing q, we set $g(\cdot, q; R)$ to be usual Green's function on S with the pole q, or to be identically zero on S, according as whether S admits Green's functions or not. If we set

$$\phi(q; R) = \sqrt{-1}\cdot dg(\cdot, q; R) - {}^*dg(\cdot, q; R)\ ,$$

then we can show the following:

THEOREM 1. *Let an admissible family $\{(f_t; R_t, R_0)\}_{t\in(0,1]}$ of marking-preserving deformations and a point q in $R_0 - N(R_0)$ be given arbitrarily. Suppose that $g(p, q; R_0)\not\equiv 0$ and that*

(*) *there is a neighborhood V of q on $R_0 - N(R_0)$ such that f_t^{-1} is conformal on V for every t.*

Then $\phi(f_t^{-1}(q); R_t)$ converges to $\phi(q; R_0)$ strongly metrically with respect to $\{f_t\}$.

The proof will be given in the next section. Here we also state the following corollary of Theorem 1. (The proof of Corollary 1 is exactly the same as that of [7, Theorem 5], and hence omitted.)

COROLLARY 1. *Under the same assumption as in Theorem 1,*

$g(f_t^{-1}(p), f_t^{-1}(q); R_t)$ converges to $g(p, q; R_0)$ locally uniformly on components of $R_0 - N(R_0) - \{q\}$ admitting Green's functions.

Concerning the assumption $(*)$ on $\{f_t\}$ in Theorem 1, we know the following:

PROPOSITION 1. *If R_t converges to R_0 in $\hat{T}(R^*)$ as t tends to 0, then we can find an admissible family $\{(f_t; R_t, R_0)\}$ satisfying the assumption $(*)$ in Theorem 1.*

The proof will be given in Section 3 with more remarks on the conformal topology on $\hat{T}(R^*)$. (In particular, see Theorem 3.)

Next, as an application of Theorems A and 1, we will derive variational formulas of Schiffer-Spencer's type.

Let R_0 be a Riemann surface with a single node p_0 and suppose that either

(i) $R_0 - \{p_0\}$ is connected, or

(ii) $R_0 - \{p_0\}$ consists of two components S_1 and S_2 both of which admit Green's functions.

And let p_1 and p_2 are punctures on $R_0' = R_0 - \{p_0\}$ corresponding to the node p_0. In the case (i), set $\bar{R}_0' = R_0' \cup \{p_1, p_2\}$, and

$$G(p; R_0', p_1, p_2) = g(p, p_1; \bar{R}_0') - g(p, p_2; \bar{R}_0') \quad \text{on} \quad R_0'$$

if R_0' admits Green's functions. If not, then let $G(p; R_0', p_1, p_2)$ be a harmonic function on R_0' defined in [6, p. 320] as $g(p; p_1, p_2)$. In the case (ii), we assume that p_j is a puncture of S_j, and set $\bar{S}_j = S_j \cup \{p_j\}$ $(j = 1, 2)$. Set

$$G(p; R_0', p_1, p_2) = g(p, p_1; \bar{S}_1) \quad \text{on} \quad S_1 , \quad \text{and}$$
$$G(p; R_0', p_1, p_2) = -g(p, p_2; \bar{S}_2) \quad \text{on} \quad S_2 .$$

In the sequel of this section, fix such R_0, p_1 and p_2 as above, and denote $G(p; R_0', p_1, p_2)$ simply by $G(p)$. Then there is an M_0 such that for every $M \geqq M_0$, $U_1(M) = \{p \in R_0'; G(p) > M\}$ and $U_2(M) = \{p \in R_0'; G(p) < -M\}$ are deleted neighborhoods of p_1 and p_2, respectively, which are conformally equivalent to $\{0 < |z| < 1\}$. Fix a local parameter $z_j = z_j(p)$ on $\bar{U}_j(M) = U_j(M) \cup \{p_j\}$ $(j = 1, 2)$ such that

$$|z_1(p)| \equiv \exp(-G(p)) \quad \text{and} \quad |z_2(p)| \equiv \exp(G(p)) .$$

Using this parameter, we write $\theta(d, R_0) = a_{0,d,j}(z_j)dz_j$ and $\phi(q; R_0) = b_{0,q,j}(z_j)dz_j$ $(j = 1, 2)$ on $\bar{U}_j(M_0)$ for every 1-cycle d and point q, respectively, on R_0'. For every positive t with $t < \exp(-M_0)$, set $R_t'' = R_0' - U_1(M_t) \cup U_2(M_t)$, where $M_t = \log(1/t)$, and identify two borders $C_{1,t} = \partial \bar{U}_1(M_t)$ and $C_{2,t} = \partial \bar{U}_2(M_t)$ of R_t'' by the mapping $(z_2)^{-1}(\eta \cdot t^2/z_1(p))$ with a fixed constant η such that $|\eta| = 1$. Then we have an ordinary Riemann surface

R_t such that $C_t = R_t - R_t''$, with the orientation induced from that of $C_{1,t}$, is a simple closed curve corresponding to p_0. With naturally induced markings, R_t converges to R_0 in the sense of the conformal topology (cf. the proof of Lemma 5 in Section 4).

This family $\{R_t\}$ gives variation by attaching a handle in the case (i), and variation by cutting a hole or connecting surfaces in the case (ii), according as whether one of S_1 and S_2 is conformally equivalent to $\{0 < |z| < 1\}$ or not (cf. [3, Ch. 7]). And we can show in a unified manner the following variational formulas of Schiffer-Spencer's type, whose proof will be given in Section 4.

THEOREM 2. (1) *Let d and d' be 1-cycles on R_0' which can be also considered ones on R_t'' for every t. Then*

$$\int_{d'} \sigma(d, R_t) - \int_{d'} \sigma(d, R_0) = (1/4\pi \cdot \log(1/t)) \cdot \int_d {}^*dG \cdot \int_{d'} {}^*dG$$
$$+ 2\pi t^2 \cdot \mathrm{Re}[\eta \cdot (a_{0,d,1}(0) \cdot a_{0,d',2}(0) + a_{0,d,2}(0) \cdot a_{0,d',1}(0))] + o(t^2)$$

as t tends to 0.

(2) *Let q be a point on R_0' and d be a 1-cycle on $R_0' - \{q\}$. Suppose that $g(p, q; R_0) \not\equiv 0$. Then*

$$\int_d {}^*dg(\cdot, q; R_t) - \int_d {}^*dg(\cdot, q; R_0) = (-1/2 \cdot \log(1/t)) \cdot G(p) \cdot \int_d {}^*dG$$
$$- 2\pi t^2 \cdot \mathrm{Re}[\eta \cdot (a_{0,d,1}(0) \cdot b_{0,q,2}(0) + a_{0,d,2}(0) \cdot b_{0,q,1}(0))] + o(t^2)$$

as t tends to 0.

(3) *Let q and q' be two distinct points on R_0'. Suppose that $g(p, q; R_0) \not\equiv 0$. Then*

$$g(q, q'; R_t) - g(q, q'; R_0) = (-1/2 \cdot \log(1/t)) \cdot G(q) \cdot G(q')$$
$$- t^2 \cdot \mathrm{Re}[\eta \cdot (b_{0,q,1}(0) \cdot b_{0,q',2}(0) + b_{0,q,2}(0) \cdot b_{0,q',1}(0))] + o(t^2)$$

as tends to 0.

Let d be a 1-cycle on R_0'. Then it is well-known as Accola's theorem that the extremal length $\lambda(0, d)$ (resp. $\lambda(t, d)$ $(t > 0)$) of the homology class of d on R_0' (resp. R_t) is equal to $\|\sigma(d, R_0)\|^2_{R_0'} = \int_d \sigma(d, R_0)$ $\left(\text{resp. } \|\sigma(d, R_t)\|^2_{R_t} = \int_d \sigma(d, R_t)\right)$. Hence Theorem 2, (1) gives the following:

COROLLARY 2.

$$\lambda(t, d) - \lambda(0, d) = (1/4\pi \cdot \log(1/t)) \cdot \left(\int_d {}^*dG\right)^2$$
$$+ 4\pi t^2 \cdot \mathrm{Re}[\eta \cdot a_{0,d,1}(0) \cdot a_{0,d,2}(0)] + o(t^2)$$

as t tends to 0.

REMARK 2. We can consider more general kinds of variation by pinching a loop, and can show certain variational formulas with the same leading terms, some of which will appear in the forthcoming paper [9].

Finally, the author would like to express his hearty thanks to the referee for valuable advice and various helpful suggestions.

2. Proof of Theorem 1. First set $\phi(t, q) = \phi(f_t^{-1}(q); R_t)$ for every positive t and $\phi(0, q) = \phi(q; R_0)$. We may assume without loss of generality that V is relatively compact in $R_0 - N(R_0)$. Also note that $\phi(t, q)\circ f_t^{-1} - \phi(0, q)$ is holomorphic on V for every t, which can be seen from the assumption $(*)$. Let U be a neighborhood of $N(R_0)$ in R_0 such that $U \cap V$ is empty and each component of $U - N(R_0)$ is conformally equivalent to $\{0 < |z| < 1\}$. Let $e(p)$ be a smooth function on R_0 such that de has a compact support in $U - N(R_0)$, $e(p) \equiv 1$ on $R_0 - U$ and $e(p) \equiv 0$ in a neighborhood of $N(R_0)$. Set $g_t(p) = g(p, f_t^{-1}(q); R_t)$, $\omega_t = \operatorname{Im}\phi(t, q)\circ f_t^{-1}$ $(= d(g_t \circ f_t^{-1}))$ and

$$F(\omega_t) = e\cdot\omega_t + g_t\circ f_t^{-1}\cdot de(= d(e\cdot g_t\circ f_t^{-1}))$$

on $R_0 - \{q\} \cup N(R_0)$. If we can show that

(1) $\limsup_{t\to 0}\|\phi(t, q)\|_{R_t - V_t}$ is finite, and

(2) $F(\omega_t) - \operatorname{Im}\phi(0, q)$ belongs to $\Gamma_{e0}(R_0)$ for every t,

where $V_t = f_t^{-1}(V)$ and $\Gamma_{e0}(R_0)$ is defined in [7, §1, 2°)], then we have the assertion by the same argument as in the proof of [7, Theorem 3].

Here the claim (2) is clear, for $F(\omega_t) - \operatorname{Im}\phi(0, q)$ is square integrable on $R_0 - N(R_0)$ and coincides with an element of $\Gamma_{e0}(R_0)$ outside V. To show the claim (1), we need the following lemma, which may be of independent interest.

LEMMA 1. *Let R and R' be two Riemann surfaces, both of which admit Green's functions. Fix a point q on R, and a real number M so large that the domain $D_M = \{p \in R; g(p, q; R) > M\}$ is simply connected and relatively compact in R. Then there is an absolute constant A_0 (depending neither on R, R', q nor on M) such that for every K-quasiconformal mapping f from D_M into R', we have*

$$\sup_{p\in R'-f(D_M)} g(p, f(q); R') \leqq 2\pi/\|\sigma(d, S')\|_{S'}^2\,,$$

where $S' = R' - f(\{p \in R; g(p, q; R) \geqq M + KA_0\})$ and d is the dividing cycle on S' corresponding to the relative boundary of S' on R'.

PROOF. Consider the harmonic function

$$u(p)=(1/2\pi)\cdot g(p, f(q); R')$$

on the ring domain $W=S'\cap f(D_M)=f(\{p\in R; M<g(p, q; R)<M+KA_0\})$. Then $u(p)$ satisfies the conditions for a height function stated in [8, §2], and the modulus of W is not less than $(KA_0/2\pi)/K=A_0/2\pi$ by [2, Theorem I.7.1]. Hence we can conclude by [8, Proposition 2] that, setting $A_0=2\pi B+1$ with the absolute constant B in [8, Proposition 2], we have

$$m/2\pi\equiv\inf_{p\in R'-S'} u(p)\geqq \sup_{p\in R'-f(D_M)} u(p)\ .$$

On the other hand, since $S'_m=\{p\in R'; g(p, f(q); R')<m\}$ is contained in S' and the moduli of S' and S'_m are equal to $1/\|\sigma(d, S')\|^2_{S'}$ by Accola's theorem and $m/2\pi$, respectively, we conclude that $1/\|\sigma(d, S')\|^2_{S'}\geqq m/2\pi$, which shows the assertion. q.e.d.

LEMMA 2. *In a neighborhood of* $t=0$ *in* $[0, 1]$,

$$M(t)=\sup_{p\in R_t-V_t} g_t(p)$$

is bounded. And the claim (1) *holds.*

PROOF. Fix M so large as in Lemma 1 with the given q on $R=R_0$. Then we may assume without loss of generality that f_t^{-1} is K-quasiconformal on D_M for every t with a suitable finite K. Let A_0 be as in Lemma 1, and apply Lemma 1 to $R'=R_t$ and $f=f_t^{-1}$ for arbitrarily fixed t. Then denoting by S'_t and σ_t the surface and the reproducer corresponding to S' and $\sigma(d, S')$ in Lemma 1, respectively, we have

$$M(t)\leqq 2\pi/\|\sigma_t\|^2_{S'_t}\ .$$

On the other hand, we can regard $\{(f_t|_{S'_t}; S'_t, S'_0)\}_{t\in(0,1]}$ as an admissible family of marking-preserving deformations of S'_t to S'_0 $(=f_t(S_t))=R_0-\{p\in R_0-N(R_0); g(p, q; R_0)\geqq M+KA_0\}$ with naturally induced markings. Hence by Theorem A, σ_t converges to σ_0 strongly metrically. In particular, for every compact set E in $S'_0-N(S'_0)$, we have

$$\|\sigma_0\|^2_E=\lim_{t\to 0}\|\sigma_t\circ f_t^{-1}\|^2_E\leqq \liminf_{t\to 0} K(f_t^{-1}, E)\cdot\|\sigma_t\|^2_{f_t^{-1}(E)}\leqq \liminf_{t\to 0}\|\sigma_t\|^2_{S'_t}\ .$$

Since E is arbitrary, we conclude that

$$\limsup_{t\to 0} 1/\|\sigma_t\|^2_{S'_t}\leqq 1/\|\sigma_0\|^2_{S'_0}=1/\|\sigma(d, S'_0)\|^2_{S'_0}$$

which shows the first assertion.

The second assertion follows from the inequality

$$\|\phi(t, q)\|^2_{R_t-V_t}=2\cdot\|dg_t\|^2_{R_t-V_t}\leqq 4\pi\cdot M(t)\ .$$

q.e.d.

3. The conformal topology. Let an admissible family $\{(f_t; R_t, R_0)\}_{t\in(0,1]}$ be given arbitrarily. Then for every t and every part S of R_0, i.e., a component of $R_0 - N(R_0)$, we can construct a Riemann surface, say S_t, from the subsurface $f_t^{-1}(S)$ of R_t by attaching a once punctured disc to each border that corresponds to a node of R_0. Such a surface S_t is not determined uniquely, but for any choice of S_t, we have the following:

PROPOSITION 2. *Fix a part S_0 of R_0 and a surface S_t as above for every t. Let P be a finite set of punctures of S_0 containing all those corresponding to the nodes of R_0, and let a neighborhood U of P in $S_0 \cup P$ be given. Then there is a family $\{g_t\}_{t\in(0,1]}$ of quasiconformal mappings g_t from S_0 onto S_t such that*

(i) $f_t^{-1} \equiv g_t$ *on* $S_0 - U$ *for every* t,

(ii) $\lim_{t\to 0} K(g_t, S_0) = 1$, *and*

(iii) *there is a positive t_0 and a neighborhood V of P in $S_0 \cup P$ such that g_t is conformal on V for every $t < t_0$.*

In particular, with naturally induced markings, S_t converges to S_0 in the sense of the Teichmüller topology.

PROOF. Take another neighborhood U' of P in $S_0 \cup P$ such that $\bar{U}' \subset U$, that the relative boundary of U' in $S_0 \cup P$ consists of simple closed curves, and that each component of $U' - P$ is a once punctured disc. By the same argument as in the proof of [6, Lemma 1], we can find a (P-)weakly admissible family $\{h_t\}_{t\in(0,T]}$ of K-quasiconformal mappings from S_0 onto S_t with a suitable finite K and positive $T < 1$ such that $h_t \equiv f_t^{-1}$ on $S_0 - U'$. By the same argument as in the proof of [6, Theorem 1 and Lemma 4], we can construct a desired family. q.e.d.

PROOF OF PROPOSITION 1. Regard $\{(f_t|_{R_t - \{f_t^{-1}(q)\}}; R_t - \{f_t^{-1}(q)\}, R_0 - \{q\})\}$ as an admissible family. Let S_0 be the part of $R_0 - \{q\}$ containing q. Let P consist of q and all punctures of S_0 corresponding to nodes of R_0, and U' be a neighborhood of P as in the proof of Proposition 2. Then by Proposition 2, there is a family $\{g_t\}$ satisfying the conditions (i), (ii) and (iii) in Proposition 2 with $U = U'$. Replacing f_t^{-1} by g_t only on the component of U' containing q for every t, we have a desired family.

q.e.d.

Now, in connection with Proposition 2, we can show certain necessary and sufficient condition for a family or sequence of points on $\hat{T}(R^*)$ be convergent. Here for the sake of simplicity, we restrict ourselves to the case as stated in Theorem 3 below. (The general case will be treated in [9].)

First, arbitrarily fix a simple closed curve c on R^* such that $\sigma(c, R^*) \not\equiv 0$. Recall then that $\sigma(c, R) \not\equiv 0$ for every R in $T(R^*)$. Let $W(R)$ be the characteristic ring domain of $\theta(c, R)$ for c on R (see, for example, [6, §2]), and $m(R)$ be the modulus of $W(R)$ for every R in $T(R^*)$. Here we set $m(R) = 0$, when $W(R)$ is empty. Then we know following:

LEMMA 3. *The modulus $m(R)$ is continuous on $T(R^*)$.*

PROOF. Let R_t converge to R_0 on $T(R^*)$ as t tends to 0. If $m(R_0) > 0$, then $m(R_t)$ converges to $m(R_0)$ by [6, Theorem 5]. If $m(R_0) = 0$, then again by [6, Theorem 5] we see that $a(R_t)^2 \cdot m(R_t)$ converges to 0, where $a(R_t) = \int_c \sigma(c, R_t)$. Since $a(R_t)$ converges to $a(R_0) = \|\sigma(c, R_0)\|^2_{R_0} > 0$ by [6, Proposition 4 and Corollary 3], we conclude the assertion. q.e.d.

Next set

$$S_c = \{R \in T(R^*);\ W(R) \neq \emptyset,\ \text{i.e.,}\ m(R) > 0\}\ , \quad \text{and}$$
$$\partial_c T(R^*) = \{R \in \hat{T}(R^*);\ N(R)\ \text{consists of a single node}\ p(R)\ \text{corresponding to}\ c\}\ .$$

Then for every $R \in S_c$, we can construct one (or a pair of) Riemann surface(s) $R^\#$ with two distinguished punctures $p_1(R)$ and $p_2(R)$, uniquely determined from R, as follows:

For every $R \in S_c$, let H_R be a conformal mapping from $W(R)$ onto $\{r(R) < |z| < 1/r(R)\}$ with $r(R) = \exp(-\pi \cdot m(R))$, and $C(R)$ be the simple closed curve $H_R^{-1}(\{|z| = 1\})$ on R with the same orientation as that of c. Using this H_R, attach domains $\{0 < |z| < 1\}$ and $\{1 < |z| < +\infty\}$ to the border of $R - C(R)$ corresponding to $C(R)$. Then we have one (or a pair of) surface(s) $R^\#$ with two distinguished punctures $p_1(R)$ and $p_2(R)$, where we denote by $p_1(R)$ the puncture corresponding to the border of $R - C(R)$ having the same orientation as that of c.

Here note that, with naturally induced markings, the above $R^\#$ can be regarded as a point in $T(R'_c)$, where R_c is any point in $\partial_c T(R^*)$ and $R'_c = R_c - N(R_c)$ with naturally induced marking. Note also that the differential $\theta(R) = -2\pi \cdot \theta(c, R)/\|\sigma(c, R)\|^2_R$ restricted to $R - C(R)$ can be extended to a holomorphic differential $\phi(R^\#)$, which should be equal to

$$\sqrt{-1} \cdot dG(\cdot\,; R^\#, p_1(R), p_2(R)) - {}^*dG(\cdot\,; R^\#, p_1(R), p_2(R))\ .$$

(If c is a dividing curve, then each component of $R^\#$ admits Green's functions, which can be seen by the assumption that $\sigma(c, R) \not\equiv 0$.)

Now we can show the following:

THEOREM 3. *In* $\hat{T}(R^*)$, $R_t \in T(R^*)$, *converges to* $R_0 \in \partial_c T(R^*)$ *as* t *tends to* 0, *if and only if*

(i) $\lim_{t\to 0} m(R_t) = +\infty$, *and*

(ii) $(R_t)^\sharp$ *converges to* $R_0' = R_0 - N(R_0)$ *in* $T(R_c')$ *as* t *tends to* 0.

Here note that, if the condition (i) in Theorem 3 holds, then $W(R_t)$ should not be empty and $(R_t)^\sharp$ can be defined for every t sufficiently small.

To prove Theorem 3, we first recall the following facts.

PROPOSITION B ([8, Theorem]). *There is an absolute constant* A *such that for every* $R \in T(R^*)$, *we have*

$$m(R) \leqq 1/\lambda(R) \leqq m(R) + A\,,$$

where $\lambda(R)$ *is the extremal length of the free homotopy class of* c *on* R.

PROPOSITION C ([6, Propositions 5 and 6]). *Suppose that a given family* $\{R_t\}_{t\in(0,1]}$ *in* $T(R^*)$ *and point* $R_0 \in \partial_c T(R^*)$ *satisfy the conditions* (i) *and* (ii) *in Theorem* 3. *Then* $\phi((R_t)^\sharp)$ *converges to*

$$\phi(R_0') = \sqrt{-1}\cdot dG(\cdot\,; R_0', p_{1,0}, p_{2,0}) - {}^*dG(\cdot\,; R_0', p_{1,0}, p_{2,0})$$

metrically (cf. [6, Definition 2]), *where* $p_{1,0}$ *and* $p_{2,0}$ *are two punctures of* R_0' *corresponding to* $N(R_0)$ *with a suitable order.*

PROOF OF THEOREM 3. First suppose that the given $\{R_t\}_{t\in[0,1]}$ satisfies the conditions (i) and (ii). Then by (ii), there is a family $\{g_t\}_{t\in(0,1]}$ of marking-preserving quasiconformal mappings g_t of R_0' to $(R_t)^\sharp$ such that $\lim_{t\to 0} K(g_t; R_0') = 1$. And by Proposition C, $\phi((R_t)^\sharp)$ converges to $\phi(R_0')$ metrically. Hence by (i) and [6, Lemma 7], we see that for every neighborhood U of $p(R_0)$ on R_0 there is a positive t_0 (<1) such that

$$g_t^{-1}((R_t)^\sharp - (R_t - C_t)) \subset U\,, \quad \text{i.e.}$$
$$g_t^{-1}(R_t - C_t) \supset R_0 - U$$

for every $t < t_0$. Hence we can easily construct an admissible family $\{(f_t; R_t, R_0)\}_{t\in(0,1]}$ by deforming $\{g_t\}$. (See also [4, Lemma 3]).

Next suppose that $R_t \in T(R^*)$ converges to $R_0 \in \partial_c T(R^*)$ in $\hat{T}(R^*)$. Then it is well-known that the extremal length $\lambda(R_t)$ of the free homotopy class of c on R_t converges to 0 as t tends to 0. Hence we conclude (i) by Proposition B.

Now take any admissible family $\{(f_t; R_t, R_0)\}_{t\in(0,1]}$, which exists by assumption. Let M_0 be as in Section 1 with $G(p) = G(p; R_0', p_{1,0}, p_{2,0})$. Fix $M_1 > M_0$ arbitrarily, and set $W = \{p \in R_0';\ M_1 < |G(p)| < M_1 + 4\pi B + 1\}$ with the absolute constant B in [8, Proposition 2]. Then we may assume without loss of generality that f_t^{-1} is 2-quasiconformal on W, and hence the modulus of each component of $f_t^{-1}(W)$ is greater than B for every t. Fix t arbitrarily. Then since we can take $u(p) = -\|\sigma(C_t, R_t)\|_{R_t}^{-2} \cdot \int^p *\sigma(C_t, R_t)$ as a height function on each component of $f_t^{-1}(W)$, we see by [8, Proposition 2] that each component of $f_t^{-1}(\{p \in R_0'; |G(p)| = M_1 + 4\pi B + 1\})$ is contained in $W(R_t)$. In particular, $f_t^{-1}(p(R_0))$ is contained in $W(R_t)$.

On the other hand, from the construction, we can regard $W(R)$ as a neighborhood of $C_{j,t}$ in $(R_t)^\#$ $(j = 1, 2)$, hence f_t^{-1} can be regarded as a homeomorphism from R_0' into $(R_t)^\#$ such that each component of $(R_t)^\# - \overline{f_t^{-1}(R_0')}$ is conformally equivalent to a punctured disc. Thus we can apply Proposition 2, and conclude that $(R_t)^\#$ converges to R_0' in the sense of the Teichmüller topology, that is, (ii) holds. q.e.d.

REMARK 3. In [4, Introduction], the author asked whether the fine topology and the conformal one on ${}_c\hat{T}_g$ are coincident. Theorem 3 gives the affirmative answer to this question.

PROPOSITION 3. *Let $\{(f_t; R_t, R_0)\}_{t \in (0,1]}$ be an admissible family of marking-preserving deformations f_t of $R_t \in T(R^*)$ to $R_0 \in \partial_c T(R^*)$. Then $\theta(R_t)$ converges to $\phi(R_0')$ strongly metrically with respect to $\{f_t\}$.*

PROOF. Fix a neighborhood U of $N(R_0)$ arbitrarily. Then in the proof of Theorem 3, we have actually shown (by Proposition 2) that there is a family $\{g_t\}$ of quasiconformal mapping g_t from R_0' onto $(R_t)^\#$ which satisfies three conditions in Proposition 2 with $S_0 = R_0'$, $S_t = (R_t)^\#$ and $P = \{p_{1,0}, p_{2,0}\}$. Then as in the proof of [6, Proposition 5], we can show that

$$\lim_{t\to 0} \|\phi((R_t)^\#) \circ g_t - \phi(R_0')\|_{R_0 - U} = 0 .$$

Since $g_t^{-1}(R_t - C(R_t))$ contains $R_0 - U$ for every t sufficiently small as is shown in the proof of Theorem 3, and since $\theta(R_t) \equiv \phi((R_t)^\#)$ on $R_t - C(R_t)$, we conclude that

$$\lim_{t\to 0} \|\theta(R_t) \circ f_t^{-1} - \phi(R_0')\|_{R_0 - U} = 0 .$$

Since U is arbitrary, we have the assertion. q.e.d.

4. **Proof of Theorem 2.** First we recall the following:

LEMMA 4. *Let D be a subsurface of a Riemann surface with compact smooth relative boundary ∂D, and h be a real smooth Dirichlet function in a neighborhood of $\bar{D} = D \cup \partial D$* (cf. [1, Abs. 7]) *which coincides, outside a compact neighborhood of ∂D in $\bar{D}$, with a Dirichlet potential on D.*

Then for every real smooth closed differential ω in a neighborhood *of $\bar{D}$ which is square integrable on D, it holds that*

$$\int_{\partial D} h \cdot \omega = -(dh, {}^*\omega)_D = \iint_D dh \wedge \omega .$$

PROOF. By assumption, we can find a smooth Dirichlet potential P on D such that $P \equiv 0$ in a neighborhood of ∂D and $P \equiv h$ outside a compact bordered subsurface $\bar{D}_1$ of $\bar{D}$ which is a neighborhood of ∂D in $\bar{D}$. Since $dP \in \Gamma_{e0}(D)$ and $-{}^*\omega \in {}^*\Gamma_c(D)$, it holds that $(dP, -{}^*\omega)_D = 0$, hence $(dh, -{}^*\omega)_D = (d(h-P), -{}^*\omega)_D$. By Green-Stokes' theorem, we conclude that

$$(dh, -{}^*\omega)_D = \iint_{D_1} d(h-P) \wedge \omega = \int_{\partial D_1} (h-P) \cdot \omega = \int_{\partial D} h \cdot \omega .$$

q.e.d.

From Theorems A and 1 we have the following:

LEMMA 5. *Fix a point q and a 1-cycle d on R_0'. Write $\theta(t, d) = a_{t,d,j}(z_j)dz_j$ and $\phi(t, q) = b_{t,q,j}(z_j)dz_j$ on $U_j(M_0) - \overline{U_j(M_t)}$ with the local parameter z_j for every t (>0) $(j = 1, 2)$, where $\phi(t, q)$ are as in Section 2. Then $a_{t,d,j}(z_j)$ and $b_{t,q,j}(z_j)$ converges to $a_{0,d,j}(z_j)$ and $b_{0,q,j}(z_j)$, respectively, locally uniformly on $U_j(M_0)$ as t tends to 0 $(j = 1, 2)$.*

PROOF. It is easy to construct a deformation f_t of R_t to R_0 such that f_t^{-1} is the identical mapping on, say, R_{2t}' regarded as a subsurface of both R_t' and R_0 for every t. Then with naturally induced markings, $\{(f_t; R_t, R_0)\}_{t \in (0, \exp(-M_0/2))}$ is an admissible family of marking-preserving deformations, which also satisfies the assumption in Theorem 1.

Now as for $\{a_{t,d,j}\}$, we have by Theorem A that

$$\lim_{t \to 0} \|\theta(t, d) \circ f_t^{-1} - \theta(0, d)\|_E^2 = \lim_{t \to 0} \iint_E |a_{t,d,j}(z_j) - a_{0,d,j}(z_j)|^2 \cdot |dz_j \wedge d\bar{z}_j| = 0$$

for every compact set E in $U_j(M_0)$, which implies the assertion, since $a_{t,d,j}$ is holomorphic on $U_j(M_0) - \overline{U_j(M_t)}$ for every t and each j.

Similarly we have the assertion for $\{b_{t,q,j}\}$ by using Theorem 1 instead of Theorem A. q.e.d.

PROOF OF THEOREM 2, (1). From additivity with respect to d and d' of each terms in the formula, it suffices to consider only the case where d and d' are smooth simple closed curves.

First we also assume that none of d and d' is degenerate, where we say that a simple closed curve d'' on R_0' is degenerate if $\sigma(d'', R_0) \equiv 0$. Then since ${}^*\sigma(d', R_0)$ $(\not\equiv 0$ on $R_0')$ is exact on every component of $R_0' - d'$, there is a harmonic function $h(p, d')$ on $R'_0 - d'$ uniquely determined by the following conditions: $h(p, d')$ coincides with a Dirichlet potential on $S - d'$ outside a compact neighborhood of d' in S, $dh(\cdot, d') \equiv {}^*\sigma(d', R_0)$ on R'_0, and $h(p, d') \equiv 0$ on $R'_0 - S$ (which may be empty), where S is the part of R_0 on which d' lies. Here we may further assume without loss of generality that d and d' are contained in $R'_0 - U_1(M_0) \cup U_2(M_0)$. Let $A(p, d')$ be a holomorphic function on $\bar{U}_1(M_0) \cup \bar{U}_2(M_0)$ such that $\operatorname{Im} A(p, d') \equiv h(p, d')$. Then we can show that

$$(1) \qquad -({}^*\sigma(d', R_0), {}^*\sigma(d, R_t))_{R''_t} = -\int_{d'} \sigma(d, R_t) + \int_{\partial R''_t} h(\cdot, d') \cdot \sigma(d, R_t) ,$$

where $\partial R''_t = C_{1,t} + C_{2,t}$.

Indeed, set $D = R''_t - d'$, and regard D as the interior of at most three bordered Riemann surfaces $\bar{D}$ with compact total border $\partial D = \partial R''_t + d'_1 + d'_2$, where d' is regarded as two different components d'_1 and d'_2 of ∂D, and d'_1 has the same orientation as that of d'. Apply Lemma 4 to $h(p, d')$ and $\sigma(d, R_t)$ on each component of D. Then we have the equation (1) by noting that $h(p'_1, d') - h(p'_2, d') \equiv -1$ for every $p'_1 \in d'_1$ and $p'_2 \in d'_2$ corresponding to the same point of d'.

Using the same argument as above, we can show that

$$(2) \qquad -({}^*\sigma(d, R_t), {}^*\sigma(d', R_0))_{R''_t} = \int_{\partial R''_t} h_t(p, d) \cdot \sigma(d', R_0) - \int_{d} \sigma(d', R_0) ,$$

where $h_t(p, d)$ is the harmonic function associated with ${}^*\sigma(d, R_t)$ on R_t defined similarly to $h(p, d')$ with R_0, R'_0, S and $\sigma(d', R_0)$ in the definition of $h(p, d')$ replaced by R_t, R_t, R_t and $\sigma(d, R_t)$, respectively. Noting that $\int_d \sigma(d', R_0) = \int_{d'} \sigma(d, R_0)$ and $\int_{\partial R''_t} h_t(\cdot, d) \cdot \sigma(d', R_0) = \int_{\partial R''_t} h_t(\cdot, d) \times d(\operatorname{Re} A(\cdot, d)) = -\int_{\partial R''_t} \operatorname{Re} A(\cdot, d) \cdot dh_t(\cdot, d) = -\int_{\partial R''_t} \operatorname{Re} A(\cdot, d) \cdot {}^*\sigma(d, R_t)$, we conclude from (1) and (2) the relation

$$(3) \qquad \int_{d'} \sigma(d, R_t) - \int_{d'} \sigma(d, R_0) = \operatorname{Im} \int_{\partial R''_t} A(\cdot, d') \cdot \theta(t, d) .$$

Next considering $h(p, d')$ and ${}^*dG$ instead of $\sigma(d, R_t)$ on $R''_t - d'$, we can show similarly to (1) that

$$(4) \qquad 2\pi \cdot (h(p_1, d') - h(p_2, d')) - \int_{d'} {}^*dG = ({}^*\sigma(d', R_0), dG)_{R''_t} .$$

On the other hand, applying Lemma 4 to $G(p)$ and $-\sigma(d', R_0)$ on R''_t, we have

$$(5) \qquad (dG, {}^*\sigma(d', R_0))_{R''_t} = \int_{\partial R''_t} G \cdot (-\sigma(d', R_0)) = 0 ,$$

for $G(p) \equiv M_t$ and $\equiv -M_t$ on $C_{1,t}$ and $C_{2,t}$, respectively. Hence we conclude from (4) and (5) that

$$h(p_1, d') - h(p_2, d') = (1/2\pi) \cdot \int_{d'} {}^*dG . \tag{6}$$

Now let $\sum_{n=0}^{\infty} c_n z_1^n$ be the Taylor expansion of $A(p, d')$ in $\bar{U}_1(M_0)$. Then since $a_{t,d,2}(z_2)dz_2 \equiv a_{t,d,1}(z_1)dz_1$ on C_t, we have

$$\begin{aligned}\int_{C_{1,t}} A(\cdot, d') \cdot \theta(t, d) &= -\sum_{n=0}^{\infty} c_n \cdot \oint_{\{|z_1|=t\}} a_{t,d,1}(z_1) z_1^n dz_1 \\ &= c_0 \cdot \int_{C_{1,t}} \sigma(d, R_t) + \sum_{n=1}^{\infty} c_n \cdot \oint_{\{|z_2|=t\}} a_{t,d,2}(z_2)(\eta t^2/z_2)^n dz_2 .\end{aligned}$$

Fix a positive $t_0(<\exp(-M_0))$. Then for every $t < t_0$ sufficiently small, we have by Cauchy's theorem and Lemma 5 that

$$\begin{aligned}\sum_{n=1}^{\infty} c_n \cdot \oint_{\{|z_2|=t\}} a_{t,d,2}(z_2)(\eta t^2/z_2)^n dz_2 &= \sum_{n=1}^{\infty} c_n \cdot \oint_{\{|z_2|=t_0\}} a_{t,d,2}(z_2)(\eta t^2/z_2)^n dz_2 \\ &= \sum_{n=1}^{\infty} c_n \cdot \oint_{\{|z_2|=t_0\}} a_{0,d,2}(z_2)(\eta t^2/z_2)^n dz_2 + o(t^2) \\ &= 2\pi\sqrt{-1} \cdot \sum_{n=1}^{\infty} c_n \cdot e_{n-1} \cdot (\eta t^2)^n + o(t^2) = 2\pi\sqrt{-1} \cdot c_1 \cdot e_0 \cdot \eta t^2 + o(t^2) ,\end{aligned}$$

where $\sum_{n=0}^{\infty} e_n z_2^n$ is the Taylor expansion of $a_{0,d,2}(z_2)$ in $\bar{U}_2(M_0)$. Here $e_0 = a_{0,d,2}(0)$, $\operatorname{Im} c_0 = h(p_1, d')$ and $c_1 = a_{0,d',1}(0)$ by definition. Since

$$-(1/2M_t) \cdot dG|_{R_t''} = {}^*\sigma(C_t, R_t)|_{R_t''} ,$$

it holds that

$$\int_{C_{1,t}} \sigma(d, R_t) = \int_d \sigma(C_t, R_t) = (1/2M_t) \cdot \int_d {}^*dG .$$

Hence we have

$$\begin{aligned}\operatorname{Im} \int_{C_{1,t}} A(\cdot, d') \cdot \theta(t, d) = h(p_1, d') \cdot (1/2M_t) \cdot \int_d {}^*dG \\ + 2\pi t^2 \cdot \operatorname{Re}[\eta \cdot a_{0,d',1}(0) \cdot a_{0,d,2}(0)] + o(t^2) .\end{aligned}$$

Similarly we can show that

$$\begin{aligned}\operatorname{Im} \int_{C_{2,t}} A(\cdot, d') \cdot \theta(t, d) = h(p_2, d') \cdot (1/2M_t) \cdot \left(-\int_d {}^*dG\right) \\ + 2\pi t^2 \cdot \operatorname{Re}[\eta \cdot a_{0,d',2}(0) \cdot a_{0,d,1}(0)] + o(t^2) .\end{aligned}$$

Adding these two equations up and using the relation (6), we have the desired formula in the case where none of d and d' is degenerate.

Finally, suppose that one of d and d' is degenerate. We consider only the case that d is degenerate, since the other case can be treated similarly. Then by [7, Lemma 4] we see that $\sigma(d, R_t) \equiv \varepsilon \cdot \sigma(C_t, R_t)$ for every t with a suitable ε in $\{1, 0, -1\}$. Since $a_{0,d,j} \equiv 0$ $(j = 1, 2)$ and since

$$\int_{d'} \sigma(C_t, R_t)\Big(=\int_{C_t} \sigma(d', R_t)\Big) = \int_{d'} (1/2M_t) \cdot {}^*dG = (1/4\pi M_t) \cdot \int_{C_{1,t}} {}^*dG \cdot \int_{d'} {}^*dG ,$$

we have the desired formula also in this case without the term $o(t^2)$.

q.e.d.

PROOF OF THEOREM 2, (2). Again it suffices to consider only the case where d is a smooth simple closed curve. First we assume that d is non-degenerate, and let $h(p, d)$ and $A(p, d)$ be associated with ${}^*\sigma(d, R_0)$ instead of ${}^*\sigma(d', R_0)$ as in the proof of Theorem 2, (1).

Fix $t > 0$ so small that R_t'' contains d and q, and fix N so large that $E_0 = \{p \in R_0'; g_0(p) \geqq N\}$ and $E_t = \{p \in R_t; g_t(p) \geqq N\}$ are simply connected and contained in $R_t'' - d$, where $g_t(p) = g(p, q; R_t)$ for every t $(\geqq 0)$.

Apply Lemma 4 to the cases: (7) $h(p, d)$ and ${}^*dg_0$ on $R_0' - d \cup E_0$, (8) $g_0(p)$ and $-\sigma(d, R_0)$ on $D_0 = R_0' - E_0$, (9) $h(p, d)$ and ${}^*dg_t$ on $R_t'' - d \cup E_t$, and (10) $g_t(p)$ and $-\sigma(d, R_0)$ on $D_t = R_t'' - E_t$, and we can show as before the following relations:

$$({}^*\sigma(d, R_0), dg_0)_{D_0} = -\int_d {}^*dg_0 + 2\pi \cdot h(q, d) \tag{7}$$

$$(dg_0, {}^*(d, R_0))_{D_0} = 0 \tag{8}$$

$$({}^*\sigma(d, R_0), dg_t)_{D_t} = \int_{\partial R_t''} h(\cdot, d) \cdot {}^*dg_t - \int_d {}^*dg_t + 2\pi \cdot h(q, d) \tag{9}$$

$$(dg_t, {}^*\sigma(d, R_0))_{D_t} = -\int_{\partial R_t''} g_t \cdot \sigma(d, R_0) = \int_{\partial R_t''} \operatorname{Re} A(\cdot, d) \cdot dg_t . \tag{10}$$

Hence we conclude that

$$\int_d {}^*dg_t - \int_d {}^*dg_0 = -\operatorname{Im} \int_{\partial R''} A(\cdot, d) \cdot \phi(t, q) . \tag{11}$$

On the other hand, we can show similarly as in the proof of Theorem 2, (1) that

$$\begin{aligned} -\operatorname{Im} \int_{\partial R_t''} A(\cdot, d) \cdot \phi(t, q) &= (h(p_1, d) - h(p_2, d)) \cdot \int_{C_{1,t}} {}^*dg_t \\ &\quad - 2\pi t^2 \cdot \operatorname{Re}[\eta \cdot (a_{0,d,1}(0) \cdot b_{0,q,2}(0) + a_{0,d,2}(0) \cdot b_{0,q,1}(0))] + o(t^2) . \end{aligned} \tag{12}$$

Hence we have the desired formula by (6) and the following equality:

$$\int_{C_{1,t}} {}^*dg_t = (-\pi/M_t) \cdot G(q) = (-1/2M_t) \cdot \Big(\int_{C_{1,t}} {}^*dG\Big) \cdot G(q) , \tag{13}$$

which we can show by applying Lemma 4 to $G(p)$ and $^*dg_t$ on D_t and then to $g_t(p)$ and $^*dG$ on D_t.

Finally, the case where d is degenerate can be treated, again as in the proof of Theorem 2, (1), by the relation (13). q.e.d.

PROOF OF THEOREM 2, (3). Set $g_{0'}(p)=g(p, q'; R_0)$. Fix $t>0$ so small that R_t'' contains q and q', and N so large that $E_0'=\{p\in R_0'; g_{0'}(p)\geqq N\}$ and E_t are mutually disjoint, simply connected and contained in R_t''. Apply Lemma 4 to $g_{0'}(p)$ and $^*dg_t$, and then to $g_t(p)$ and $^*dg_{0'}$, on $R_t''-E_0'\cup E_t$, we can show that

$$\int_{\partial R_t''} g_{0'}\cdot {}^*dg_t+2\pi g_{0'}(q)=\int_{\partial R_t''} g_t\cdot {}^*dg_{0'}+2\pi g_t(q')\ .$$

Since $g_{0'}(q)=g_0(q')$ and $\int_{\partial R_t''} g_t\cdot {}^*dg_{0'}=\int_{\partial R_t''}\operatorname{Re} A(\cdot, q')\cdot dg_t$, where $A(p, q')$ is a holomorphic function is a neighborhood of $\bar{U}_1(M_t)\cup\bar{U}_2(M_t)$ such that $\operatorname{Im} A(p, q')=g_{0'}(p)$, we have

$$g_t(q')-q_0(q')=(-1/2\pi)\cdot\operatorname{Im}\int_{\partial R_t''} A(\cdot, q')\cdot\phi(t, q)\ . \tag{14}$$

Hence, using (13) and the same argument as before, we can show the desired formula. q.e.d.

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