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SOME REMARKS ON MEAN-VALUES OF SUBHARMONIC FUNCTIONS

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0. Introduction and notation. This paper is in two loosely related parts: the first part gives conditions for a nonnegative continuous function or its logarithm to be subharmonic, and the second includes a Fejér-Riesz type theorem for subharmonic functions.

The open ball, the closed ball, and the sphere of centre x and radius r in $\boldsymbol{R}^n$ $(n \geqq 2)$ are denoted by $B(x, r)$, $\bar{B}(x, r)$ and $S(x, r)$. We denote n-dimensional Lebesgue measure by ω and $(n-1)$-dimensional surface area measure on $S(x, r)$ by σ, and we write $\Omega(r)$ for the volume of $B(x, r)$ and $\Sigma(r)$ for the surface-area of $S(x, r)$. If a function f, defined at least on $\bar{B}(x, r)$, is ω-integrable on $B(x, r)$ and σ-integrable on $S(x, r)$, we define means as follows:

$$A(f, x, r) = (\Omega(r))^{-1}\int_{B(x,r)} f d\omega$$

and

$$M(f, x, r) = (\Sigma(r))^{-1}\int_{S(x,r)} f d\sigma .$$

Throughout the paper G will be a nonempty open subset of $\boldsymbol{R}^n$. Recall that a function is hypoharmonic in G if and only if in each connected component of G it is either subharmonic or identically $-\infty$. We shall say that a function is PL if its logarithm is hypoharmonic in G.

1. Mean value conditions for subharmonicity.

1.1. The following results are well-known.

THEOREM A. *Let $u: G \to \boldsymbol{R}$ be continuous in G. Then u is subharmonic in G if and only if*

$$A(u, x, r) \leqq M(u, x, r)$$

whenever $\bar{B}(x, r) \subset G$.

THEOREM B. *Let $u: G \to [0, \infty)$ be continuous in $G \subset \boldsymbol{R}^2$. Then u is* PL *if and only if*

$$A(u^2, x, r) \leqq (M(u, x, r))^2$$

whenever $\bar{B}(x, r) \subset G$.

See, for example, Radó [6, §3.25] for a proof of Theorem A in the case $n = 2$ and [6, §3.26] for a proof of Theorem B.

Mochizuki [5] proved that the "if" part of Theorem B continues to hold when $G \subset \boldsymbol{R}^n$ for $n \geqq 3$. By refining Mochizuki's method of proof, we give the following improvement of his result. We refer to [5] for references to the related literature.

THEOREM 1. *Let* $u: G \to [0, \infty)$ *be continuous in* G. *If*

$$A(u^{(n+2)/n}, x, r) \leqq (M(u, x, r))^{(n+2)/n} \tag{1}$$

whenever $\bar{B}(x, r) \subset G$, *then* u *is* PL.

The converse is true in the case $n = 2$ *and false in the case* $n \geqq 3$.

We note that

(i) In the case $n = 2$ Theorem 1 is Theorem B.

(ii) The continuity hypothesis cannot be replaced by upper semicontinuity. (Consider, for example, the characteristic function of a one-point set.)

(iii) In the case $n \geqq 3$ the hypothesis (1) is weaker than Mochizuki's hypothesis in which $(n + 2)/n$ is replaced by 2; this follows from Hölder's inequality.

The method used to prove Theorem 1 is also used to prove sufficiency in the following criterion for the subharmonicity of a nonnegative continuous function.

THEOREM 2. *Let* $u: G \to [0, \infty)$ *be continuous in* G. *Then* u *is subharmonic in* G *if and only if*

$$(n + 2)A(u^2, x, r) \leqq nM(u^2, x, r) + 2(M(u, x, r))^2 \tag{2}$$

whenever $\bar{B}(x, r) \subset G$. *Indeed, for* (2) *to hold, it suffices that* u *is nonnegative and subharmonic* (*but not necessarily continuous*) *in* G.

The sufficiency part of Theorem 2 has the following analogue for the case where u is required to be PL.

THEOREM 3. *Let* $u: G \to [0, \infty)$ *be continuous in* G. *If*

$$(n + 2)A(u^2, x, r) \leqq (n - 2)M(u^2, x, r) + 4(M(u, x, r))^2 \tag{3}$$

whenever $\bar{B}(x, r) \subset G$, *then* u *is* PL.

We note that

(i) In the case $n = 2$ Theorem 3 is simply the sufficiency part of

Theorem B, and in this case, by Theorem B, the converse of Theorem 3 is true. Whether the converse remains true when $n \geqq 3$ is an open question.

(ii) Again, the continuity hypothesis cannot be replaced by upper semi-continuity.

(iii) Hypothesis (3) is again weaker than the corresponding hypothesis in [5] when $n \geqq 3$, since the Cauchy-Schwarz inequality implies that $(M(u, x, r))^2 \leqq M(u^2, x, r)$.

The proofs of the following theorems are very similar to the proofs of Theorem 1, 2 and 3 and are therefore omitted.

THEOREM 4. *Let $u: G \to [0, \infty)$ be continuous in G. If $0 < p \leqq 1$ and*

$$A(u^{(n+2-2p)/n}, x, r) \leqq (M(u, x, r))^{(n+2-2p)/n}$$

whenever $\bar{B}(x, r) \subset G$, then u^p is subharmonic in G.

THEOREM 5. *Let $u: G \to [0, \infty)$ be continuous in G. If $0 < p \leqq 1$ and*

$$(n + 2)A(u^2, x, r) \leqq (n + 2p - 2)M(u^2, x, r) + (4 - 2p)(M(u, x, r))^2 ,$$

then u^p is subharmonic in G.

Recall that if u is PL, then u^p is subharmonic in G for each positive p and that if u^p is subharmonic in G for some $p \in (0, 1)$, then u is subharmonic in G. Thus Theorem 4 bridges the gap between Theorem A and Theorem 1; similarly, Theorem 5 links Theorems 2 and 3.

1.2. We prove here Theorems 1, 2 (sufficiency part) and 3 in the special case where $u \in \mathscr{C}^2(G)$ and $u > 0$ in G.

The proofs depend upon Pizzetti's formula, a general form of which is as follows: if $f \in \mathscr{C}^{2k}(G)$, where k is a positive integer and if $x \in G$, then, as $r \to 0+$,

$$(4) \qquad M(f, x, r) = \sum_{j=0}^{k} \{2^j j!\, n(n + 2) \cdots (n + 2j - 2)\}^{-1} \Delta^j f(x) r^{2j} + o(r^{2k}) ,$$

where Δ^j is the j^{th} iterated n-dimensional Laplace operator (Δ^0 is the identity operator, and $\Delta^j = \Delta\Delta^{j-1}$ for $j = 1, 2, \cdots$, where Δ is the ordinary n-dimensional Laplacian). This formula is given, with a smaller error term, for the case where $f \in \mathscr{C}^\infty(G)$ in [2, page 30]. The proof in [2] is readily adapted to give (4) when $f \in \mathscr{C}^{2k}(G)$. Since

$$A(f, x, r) = nr^{-n}\int_0^r t^{n-1} M(f, x, t)dt ,$$

we deduce from (4) that

$$A(f, x, r) = \sum_{j=0}^{k} \{2^j j!(n+2)(n+4)\cdots(n+2j)\}^{-1}\Delta^j f(x) r^{2j} + o(r^{2k}) . \tag{5}$$

In the case $k = 1$, which is all we need in this subsection

$$M(f, x, r) = f(x) + (2n)^{-1}\Delta f(x) r^2 + o(r^2) \tag{6}$$

and

$$A(f, x, r) = f(x) + (2n+4)^{-1}\Delta f(x) r^2 + o(r^2) . \tag{7}$$

Now suppose that $u \in \mathscr{C}^2(G)$ and $u > 0$ in G. If $q > 0$, then

$$\Delta u^q = q u^{q-1}\Delta u + q(q-1)u^{q-2}|\nabla u|^2 \tag{8}$$

and

$$\Delta(\log u) = u^{-1}\Delta u - u^{-2}|\nabla u|^2 , \tag{9}$$

where $|\nabla u|^2$ is the sum of the squares of the first partial derivatives of u.

To prove Theorem 1 for such a function u, we have by (6), (7), (8) and (9),

$$\begin{aligned}
&(M(u, x, r))^{(n+2)/n} - A(u^{(n+2)/n}, x, r) \\
&\quad = \{u(x) + (2n)^{-1}\Delta u(x) r^2 + o(r^2)\}^{(n+2)/n} \\
&\qquad\qquad - \{(u(x))^{(n+2)/n} + (2n+4)^{-1}\Delta u^{(n+2)/n}(x) r^2 + o(r^2)\} \\
&\quad = (u(x))^{(n+2)/n} + \frac{1}{2}(n+2)n^{-2}(u(x))^{2/n}\Delta u(x) r^2 \\
&\qquad - \left\{(u(x))^{(n+2)/n} + \frac{1}{2}n^{-1}(u(x))^{2/n}\Delta u(x) r^2 + n^{-2}(u(x))^{(2-n)/n}|\nabla u(x)|^2 r^2\right\} + o(r^2) \\
&\quad = n^{-2}(u(x))^{(n+2)/n}\{(u(x))^{-1}\Delta u(x) - (u(x))^{-2}|\nabla u(x)|^2 + o(1)\} r^2 \\
&\quad = n^{-2}(u(x))^{(n+2)/n}\{\Delta \log u(x) + o(1)\} r^2 ,
\end{aligned}$$

so that

$$\Delta \log u(x) = n^2(u(x))^{-(n+2)/n} \lim_{r\to 0+} r^{-2}\{(M(u, x, r))^{(n+2)/n} - A(u^{(n+2)/n}, x, r)\} . \tag{10}$$

Hence u is PL if and only if the limit on the right-hand side of (10) is nonnegative for each $x \in G$.

To prove the sufficiency of (2) in Theorem 2, we have by (6), (7) and (8),

$$\begin{aligned}
&nM(u^2, x, r) + 2(M(u, x, r))^2 - (n+2)A(u^2, x, r) \\
&\quad = n\{(u(x))^2 + (2n)^{-1}\Delta u^2(x) r^2 + o(r^2)\} + 2\{u(x) + (2n)^{-1}\Delta u(x) r^2 + o(r^2)\}^2 \\
&\qquad\qquad - (n+2)\{(u(x))^2 + (2n+4)^{-1}\Delta u^2(x) r^2 + o(r^2)\}
\end{aligned}$$

$$\begin{aligned}&= n\{(u(x))^2 + n^{-1}u(x)\Delta u(x)r^2 + n^{-1}|\nabla u(x)|^2r^2\} + 2\{(u(x))^2 + n^{-1}u(x)\Delta u(x)r^2\}\\ &\quad - (n+2)\{(u(x))^2 + (n+2)^{-1}u(x)\Delta u(x)r^2 + (n+2)^{-1}|\nabla u(x)|^2r^2\} + o(r^2)\\ &= 2n^{-1}u(x)\Delta u(x)r^2 + o(r^2)\ ,\end{aligned}$$

so that

$$\begin{aligned}\Delta u(x) = \frac{1}{2}n(u(x))^{-1}\lim_{r\to 0+} r^{-2}\{&nM(u^2, x, r)\\ &+ 2(M(u, x, r))^2 - (n+2)A(u^2, x, r)\}\ .\end{aligned} \tag{11}$$

Hence u is subharmonic in G if and only if the limit on the right-hand side of (11) is nonnegative for each $x \in G$.

To prove Theorem 3, we have, by a calculation similar to the above,

$$\begin{aligned}\Delta \log u(x) = \frac{1}{2}n(u(x))^{-2}\lim_{r\to 0+} r^{-2}\{&(n-2)M(u^2, x, r)\\ &+ 4(M(u, x, r))^2 - (n+2)A(u^2, x, r)\}\ ,\end{aligned} \tag{12}$$

so that u is PL if and only if the limit on the right-hand side of (12) is nonnegative for each $x \in G$.

1.3. Here we complete the proofs of Theorems 1, 2 (sufficiency) and 3.

Recall that $v: G \to [-\infty, \infty)$ is hypoharmonic in G if and only if v is upper semi-continuous in G and for each $x \in G$ there exists $r_0 > 0$ such that $v(x) \leqq M(v, x, r)$ whenever $0 < r < r_0$. Hence, to prove Theorems 1 and 3, it suffices to show that u is PL in the open set $G^* = \{x \in G: u(x) > 0\}$, for $\log u$ is continuous in G and trivially satisfies the mean-value inequality on spheres contained in G and centred in $G \setminus G^*$. Similarly, to prove sufficiency in Theorem 2, it is enough to prove that u is subharmonic in G^*. Hence, for the remainder of this subsection, we need consider only the case where $u > 0$ in G.

We deal first with Theorem 1. For each $\rho > 0$ let

$$G_\rho = \{x \in G: \mathrm{dist}(x, \partial G) > \rho\}\ .$$

If G_ρ is nonempty, define u_ρ in G_ρ by

$$u_\rho(x) = (A(u^q, x, \rho))^{1/q}\ , \tag{13}$$

where $q = (n+2)/n$. Suppose, for the moment, that $u \in \mathscr{C}^1(G)$. Then, as is well-known, $u_\rho \in \mathscr{C}^2(G)$ and $u_\rho \to u$ uniformly on compact subsets of G as $\rho \to 0+$, and hence $\log u_\rho \to \log u$ uniformly on such sets. Since subharmonicity is preserved by uniform limits and is, as was indicated above, a local property, it suffices to prove that $\log u_\rho$ is subharmonic in G_ρ. If we show that

$$A(u_\rho^q, x, r) \leqq (M(u_\rho, x, r))^q \tag{14}$$

whenever $\bar{B}(x, r) \subset G_\rho$, then since $u_\rho \in \mathscr{C}^2(G_\rho)$ and $u_\rho > 0$ in G_ρ, the subharmonicity of $\log u_\rho$ in G_ρ will follow from the case considered in § 1.2, and the theorem will be established in the case where $u \in \mathscr{C}^1(G)$. A repetition of this argument will then give the theorem in its generality, for if u is continuous in G, we have $u_\rho \in \mathscr{C}^1(G_\rho)$. Hence the theorem will be proved, provided we can show that (1) implies (14).

Suppose that $\bar{B}(x, r) \subset G_\rho$, and let O be the origin of $\boldsymbol{R}^n$. Then

$$\begin{aligned} A(u_\rho^q, x, r) &= (\Omega(r))^{-1}\int_{B(x,r)} (\Omega(\rho))^{-1}\int_{B(O,\rho)} (u(y+z))^q d\omega(z) d\omega(y) \\ &= (\Omega(\rho))^{-1}\int_{B(O,\rho)} (\Omega(r))^{-1}\int_{B(x,r)} (u(y+z))^q d\omega(y) d\omega(z) , \end{aligned}$$

the change of order of integration being justified, since $u > 0$ in G. Hence, by (1) and the integral form of Minkowski's inequality,

$$\begin{aligned} A(u_\rho^q, x, r) &\leqq (\Omega(\rho))^{-1}\int_{B(O,\rho)} \left\{(\Sigma(r))^{-1}\int_{S(x,r)} u(y+z) d\sigma(y)\right\}^q d\omega(z) \\ &\leqq \left\{(\Sigma(r))^{-1}\int_{S(x,r)} \left\{(\Omega(\rho))^{-1}\int_{B(O,\rho)} (u(y+z))^q d\omega(z)\right\}^{1/q} d\sigma(y)\right\}^q \\ &= \left\{(\Sigma(r))^{-1}\int_{S(x,r)} u_\rho(y) d\sigma(y)\right\}^q \\ &= (M(u_\rho, x, r))^q , \end{aligned}$$

and the proof is complete.

Next consider the sufficiency part of Theorem 2. Arguing as in the case of Theorem 1, we see that it suffices to show that the function u_ρ, defined in G_ρ by (13) with the power q replaced by 2, satisfies

$$(n+2)A(u_\rho^2, x, r) \leqq nM(u_\rho^2, x, r) + 2(M(u_\rho, x, r))^2$$

whenever $\bar{B}(x, r) \subset G_\rho$, provided that u satisfies (2). Using changes of order of integration, hypothesis (2) and the integral form of Minkowski's inequality, we find that if $\bar{B}(x, r) \subset G_\rho$, then

$$\begin{aligned} (n+2)A(u_\rho^2, x, r) &= (n+2)(\Omega(r))^{-1}\int_{B(x,r)} (\Omega(\rho))^{-1}\int_{B(O,\rho)} (u(y+z))^2 d\omega(z) d\omega(y) \\ &= (n+2)(\Omega(\rho))^{-1}\int_{B(O,\rho)} (\Omega(r))^{-1}\int_{B(x,r)} (u(y+z))^2 d\omega(y) d\omega(z) \\ &\leqq (\Omega(\rho))^{-1}\int_{B(O,\rho)} \left\{n(\Sigma(r))^{-1}\int_{S(x,r)} (u(y+z))^2 d\sigma(y)\right. \\ &\qquad \left. + 2\left((\Sigma(r))^{-1}\int_{S(x,r)} u(y+z) d\sigma(y)\right)^2\right\} d\omega(z) \end{aligned}$$

$$= n(\Sigma(r))^{-1}\int_{S(x,r)}(\Omega(\rho))^{-1}\int_{B(O,\rho)}(u(y+z))^2 d\omega(z)d\sigma(y)$$
$$+ 2(\Omega(\rho))^{-1}\int_{B(O,\rho)}\left((\Sigma(r))^{-1}\int_{S(x,r)}u(y+z)d\sigma(y)\right)^2 d\omega(z)$$
$$\leqq n(\Sigma(r))^{-1}\int_{S(x,r)}(u_\rho(y))^2 d\sigma(y)$$
$$+ 2\left\{(\Sigma(r))^{-1}\int_{S(x,r)}\left((\Omega(\rho))^{-1}\int_{B(O,\rho)}(u(y+z))^2 d\omega(z)\right)^{1/2} d\sigma(y)\right\}^2$$
$$= nM(u_\rho^2, x, r) + 2(M(u_\rho, x, r))^2 .$$

The proof of Theorem 3 is similar.

1.4. In proving the necessity of condition (2) in Theorem 2, we suppose without loss of generality that $x = O$. Let $u: G \to [0, \infty)$ be subharmonic (but not necessarily continuous) in G. Suppose that $\bar{B}(O, r) \subset G$, and let I denote the Poisson integral in $B(O, r)$ of the restriction of u to $S(O, r)$. Then I is a harmonic majorant of u in $B(O, r)$ and the function u^*, equal to I in $B(O, r)$ and equal to u in $G \diagdown B(O, r)$ is subharmonic in G ([4, p. 69]). As is well known, we can write $I = \sum_{j=0}^{\infty} a_j H_j$ in $B(O, r)$, where H_j is a homogeneous harmonic polynomial of degree j in $\boldsymbol{R}^n$ such that for each $\rho > 0$

$$M(H_j H_k, O, \rho) = \begin{cases} \rho^{2j} & (j = k) \\ 0 & (j \neq k) \end{cases}$$

and the series is locally uniformly convergent in $B(O, r)$. Hence, if $0 < \rho < r$, then

$$A(u^{*2}, O, \rho) = n\rho^{-n}\int_0^\rho t^{n-1}M(u^{*2}, O, t)dt = n\rho^{-n}\sum_{j=0}^{\infty} a_j^2 \int_0^\rho t^{2j+n-1}dt$$
$$= n\sum_{j=0}^{\infty}(2j+n)^{-1}a_j^2\rho^{2j} .$$

Hence, since $M(u^*, O, \rho) = u^*(O) = a_0$, we have

$$nM(u^{*2}, O, \rho) + 2(M(u^*, O, \rho))^2 - (n+2)A(u^{*2}, O, \rho) \tag{15}$$
$$= n\sum_{j=1}^{\infty}(1-(n+2)(n+2j)^{-1})a_j^2\rho^{2j} \geqq 0 .$$

Since u^* and u^{*2} are subharmonic in G, the means on the left-hand side of (15) are continuous functions of ρ on $(0, r]$. Hence the expression on the left-hand side of (15) remains nonnegative when $\rho = r$. Using this result and the fact that $u \leqq u^*$ in $\bar{B}(O, r)$ with equality on $S(O, r)$, we have

$$(n+2)A(u^2, O, r) \leqq (n+2)A(u^{*2}, O, r) \leqq nM(u^{*2}, O, r) + 2(M(u^*, O, r))^2$$
$$= nM(u^2, O, r) + 2(M(u, O, r))^2 .$$

1.5. In the case $n = 2$ the converse of Theorem 1 follows from Theorem B. We give here an example to show that the converse is false in the case $n \geqq 3$. The Euclidean norm of $x \in \boldsymbol{R}^n$ is denoted by $\|x\|$.

EXAMPLE. *If u is defined in $\boldsymbol{R}^n \setminus \{O\}$, where $n \geqq 3$, by*

$$u(x) = \exp(n\|x\|^{2-n}) ,$$

then u is PL in $\boldsymbol{R}^n \setminus \{O\}$, but there exist points y of $\boldsymbol{R}^n \setminus \{O\}$ such that

$$A(u^{(n+2)/n}, y, r) > (M(u, y, r))^{(n+2)/n}$$

for all sufficiently small r.

It is well-known that $\log u$ is harmonic in $\boldsymbol{R}^n \setminus \{O\}$, so u is PL there.

To estimate the means in this example, we use the case $k = 2$ of (4) and (5):

$$M(f, x, r) = f(x) + (2n)^{-1}\Delta f(x)r^2 + \{8n(n+2)\}^{-1}\Delta^2 f(x)r^4 + o(r^4) \tag{16}$$

and

$$A(f, x, r) = f(x) + (2n+4)^{-1}\Delta f(x)r^2 + \{8(n+2)(n+4)\}^{-1}\Delta^2 f(x)r^4 + o(r^4) . \tag{17}$$

For each $a > 0$, define v_a in $\boldsymbol{R}^n \setminus \{O\}$ by

$$v_a(x) = \exp(a\|x\|^{2-n}) .$$

Straightforward calculations give

$$\Delta v_a(x) = a^2(n-2)^2\|x\|^{2-2n}v_a(x) \tag{18}$$

and

$$\Delta^2 v_a(x) = a^2(n-2)^2\|x\|^{4-4n}\{a^2(n-2)^2 + 4a(n-1)(n-2)\|x\|^{n-2} + 2n(n-1)\|x\|^{2n-4}\}v_a(x) . \tag{19}$$

Now let y be a point of $\boldsymbol{R}^n$ such that

$$\|y\| = \{(2n-4)/n\}^{1/(n-2)} = \lambda , \quad \text{say} .$$

Since $u = v_n$, we can use (16), (18) and (19) to obtain after some simplification

$$(M(u, y, r))^{n+2} = \exp(n(n+2)\lambda^{2-n})\Big\{1 + \frac{1}{2}n(n+2)(n-2)^2\lambda^{2-2n}r^2$$
$$+ \frac{1}{8}n(n-2)^2\lambda^{4-4n}[n^2(n-2)^2 + 4n(n-1)(n-2)\lambda^{n-2}$$
$$+ 2n(n-1)\lambda^{2n-4} + n(n+1)(n+2)(n-2)^2]r^4 + o(r^4)\Big\} .$$

Similarly, since $u^{(n+2)/n} = v_{n+2}$, we obtain from (17), (18) and (19) that

$$(A(u^{(n+2)/n}, y, r))^n = \exp(n(n+2)\lambda^{2-n})\Big\{1 + \frac{1}{2}n(n+2)(n-2)^2\lambda^{2-2n}r^2$$
$$+ \frac{1}{8}n(n+2)(n-2)^2(n+4)^{-1}\lambda^{4-4n}[(n^2-4)^2+4(n+2)(n-1)(n-2)\lambda^{n-2}$$
$$+ 2n(n-1)\lambda^{2n-4} + (n+2)(n-1)(n-2)^2(n+4)]r^4 + o(r^4)\Big\} .$$

Subtracting and simplifying, we find that

$$(M(u, y, r))^{n+2} - (A(u^{(n+2)/n}, y, r))^n$$
$$= \frac{1}{2}\exp(n(n+2)\lambda^{2-n})n(n-2)^2(n+4)^{-1}\lambda^{4-4n}Kr^4 + o(r^4) , \tag{20}$$

where

$$K = 2(n-2)^2 - 4(n-1)(n-2)\lambda^{n-2} + n(n-1)\lambda^{2n-4} = 2(2-n)^3n^{-1} < 0 .$$

Hence for all sufficiently small r the left-hand side of (20) is negative, and the required conclusion follows.

2. Fejér-Riesz type inequalities.

2.1. In view of Theorem A, it is natural to ask whether, for any nonnegative subharmonic function in G, we have, for $p > 0$,

$$A(u^p, x, r) \leqq C(n, p)(M(u, x, r))^p \tag{21}$$

whenever $\bar{B}(x, r) \subset G$. Here and in the sequel we use $C(a, b, \cdots)$ to denote a positive constant, depending only on $a, b, \cdots$, not necessarily the same on any two occurrences. We infer from Theorem 4 that if $p > 1$, the inequality (21) fails in general with $C(n, p) = 1$. There is no loss of generality in supposing that $\bar{B}(O, 1) \subset G$ and considering (21) only in the case where $x = O$ and $r = 1$. A more general problem is then to find conditions on a measure μ on $B(O, 1)$ which are sufficient to ensure that

$$\int_{B(O,1)} u^p d\mu \leqq C(n, p, \mu)(M(u, O, 1))^p$$

for any nonnegative subharmonic function u in G.

For the remainder of this section we simplify the notation by writing $B = B(O, 1)$ and $S = S(O, 1)$. Let K denote the Poisson kernel of B; then K is defined on $B \times S$ by

$$K(x, y) = (\Sigma(1))^{-1}(1 - \|x\|^2)\|x - y\|^{-n} .$$

We write $u \in \mathscr{S}$ if u is nonnegative and subharmonic in a domain containing $\bar{B}$, and we put $M(u) = M(u, O, 1)$.

THEOREM 6. *Suppose that $p \geqq 1$ and that μ is a measure on B such that*

$$\sup_{y \in S} \int_B (K(x, y))^p d\mu(x) = I < \infty .$$

If $u \in \mathscr{S}$, then

$$\int_B u^p d\mu \leqq I(\Sigma(1)M(u))^p .$$

The result fails if $0 < p < 1$.

COROLLARY. *Suppose that $p \geqq 1$ and that μ is a measure on B such that*

$$\sup_{y \in S} \int_B \|x - y\|^{(1-n)p} d\mu(x) = J < \infty .$$

If $u \in \mathscr{S}$, then

$$\int_B u^p d\mu \leqq J(2M(u))^p .$$

By taking μ to be n-dimensional Lebesgue measure, we obtain from the Corollary the following result in the case where $1 \leqq p < n/(n-1)$.

THEOREM 7. *If $0 < p < n/(n-1)$ and $u \in \mathscr{S}$, then*

$$A(u^p, O, 1) \leqq C(n, p)(M(u))^p .$$

The result fails if $p \geqq n/(n-1)$.

By modifying the proof of Theorem 6, we shall also prove the following theorem, which is analogous to the Fejér-Riesz theorem (see, e.g., [3, p. 46]); the analogy is most apparent when the measure μ is concentrated on B intersected with a proper subspace of $\boldsymbol{R}^n$.

THEOREM 8. *Suppose that $p > 0$ and that μ is a measure on B such that*

$$\int_B (1 - \|x\|)^{(1-n)p} d\mu(x) = L < \infty .$$

If $u \in \mathscr{S}$, then

$$\int_B u^p d\mu \leqq L(2M(u))^p .$$

2.2. Our proof of Theorem 6 is a generalization of the proof of [1, Theorem 2].

Let h be the Poisson integral in B of u restricted to S. Then

$$h(x) = \int_S K(x, y)u(y)d\sigma(y) \qquad (x \in B)$$

and h is a harmonic majorant of u in B (in fact, the least such majorant). Hence, if $p \geqq 1$,

$$\begin{aligned}\int_B u^p d\mu \leqq \int_B h^p d\mu &= \int_B \left(\int_S K(x, y)u(y)d\sigma(y)\right)^p d\mu(x)\\ &\leqq (\Sigma(1)M(u))^{p-1}\int_B\int_S (K(x, y))^p u(y)d\sigma(y)d\mu(x)\\ &= (\Sigma(1)M(u))^{p-1}\int_S\int_B (K(x, y))^p u(y)d\mu(x)d\sigma(y)\\ &\leqq (\Sigma(1)M(u))^{p-1}\int_S Iu(y)d\sigma(y) = I(\Sigma(1)M(u))^p .\end{aligned}$$

The second of the above inequalities follows from Hölder's inequality. The change of order of integration is justified, since the integrand is nonnegative.

2.3. We now suppose that $0 < p < 1$ and show that the result fails. Let $y_{(1)} = (1, 0, \cdots, 0)$ and let $w = K(\cdot, y_{(1)})$. Then it is easy to see that w^p is a potential in B and hence that $M(w^p, O, r) \to 0$ as $r \to 1-$. For each positive integer m, let μ_m be the measure supported by $S(O, m/(m+1))$ which is proportional to surface-area measure and is such that $\mu_m(B)M(w^p, O, m/(m+1)) = 1$. Then, by the symmetry of μ_m,

$$\sup_{y \in S} \int_B (K(x, y))^p d\mu_m(x) = \int_B w^p d\mu_m = 1 .$$

Hence, if the theorem held with $0 < p < 1$, then, taking $u \equiv 1$, we would have $\mu_m(B) \leqq (\Sigma(1))^p$ for all m, contradicting the fact that $\mu_m(B) \to \infty$.

2.4. The Corollary follows from the inequality

$$K(x, y) < (2/\Sigma(1))\,\|x - y\|^{1-n} \qquad (x \in B,\ y \in S) .$$

2.5. To prove Theorem 7 when $1 \leqq p < n/(n-1)$, note that for each $y \in S$,

$$\int_B \|x - y\|^{(1-n)p}dx \leqq \int_{B(y,2)} \|x - y\|^{(1-n)p}dx = C(n)\int_0^2 t^{(n-1)(1-p)}dt = C(n, p) ,$$

so that the required result follows if we take μ to be Lebesgue volume measure in the Corollary to Theorem 6.

If $0 < p < 1$, then, by Hölder's inequality and the result already established for $p = 1$,

$$A(u^p, O, 1) \leqq (A(u, O, 1))^p \leqq (C(n, 1))^p(M(u))^p .$$

2.6. Now suppose that $p \geqq n/(n-1)$. Let $y_{(\rho)} = (\rho, 0, \cdots, 0)$, where $\rho \geqq 1$, and let

$$K_\rho(x) = (\rho^2 - \|x\|^2)\|x - y_{(\rho)}\|^{-n} \qquad (x \in B(O, \rho)) .$$

Then K_ρ is positive and harmonic in $B(O, \rho)$ (since it is a positive multiple of the Poisson kernel of $B(O, \rho)$ with pole $y_{(\rho)}$). Hence if $\rho > 1$, then $K_\rho \in \mathscr{S}$ and

$$M(K_\rho) = K_\rho(O) = \rho^{2-n} . \tag{22}$$

Further, by considering a contraction mapping on $\boldsymbol{R}^n$, we find that

$$A(K_\rho^p, O, 1) = \rho^{p(2-n)} A(K_1^p, O, \rho^{-1}) . \tag{23}$$

Let Λ denote the Stolz cone

$$\{x \in B \colon \|x - y_{(1)}\| < 1 , \quad x_2^2 + \cdots + x_n^2 \leqq (1 - x_1)^2\} .$$

Then it is easy to see that

$$1 - \|x\|^2 \geqq (\sqrt{2} - 1)\|x - y_{(1)}\| \qquad (x \in \Lambda) ,$$

so that

$$K_1(x) \geqq (\sqrt{2} - 1)\|x - y_{(1)}\|^{1-n} \qquad (x \in \Lambda) .$$

Let $\Lambda_\rho = \{x \in \Lambda \colon \|x - y_{(1)}\| > (2\sqrt{2} + 2)(1 - \rho^{-1})\}$, where $1 < \rho < (2\sqrt{2} + 2) \times (2\sqrt{2} + 1)^{-1}$. Then, since $\Lambda_\rho \subset \Lambda \cap B(O, \rho^{-1})$, we have

$$\begin{aligned} A(K_1^p, O, \rho^{-1}) &\geqq (\Omega(\rho^{-1}))^{-1}(\sqrt{2} - 1)^p \int_{\Lambda \cap B(O, \rho^{-1})} \|x - y_{(1)}\|^{(1-n)p} dx \\ &\geqq C(n, p)\rho^n \int_{\Lambda_\rho} \|x - y_{(1)}\|^{(1-n)p} dx \\ &= C(n, p)\rho^n \int_{(2\sqrt{2}+2)(1-\rho^{-1})}^{1} t^{(n-1)(1-p)} dt \to \infty \quad \text{as} \quad \rho \to 1+ . \end{aligned}$$

It now follows from (22) and (23) that the theorem fails if $p \geqq n/(n - 1)$.

2.7. To prove Theorem 8, note first that, as in § 2.2,

$$\int_B u^p d\mu \leqq \int_B \Big(\int_S K(x, y) u(y) d\sigma(y) \Big)^p d\mu(x) .$$

Now for each $x \in B$ and each $y \in S$,

$$\frac{1}{2} \Sigma(1) K(x, y) \leqq \|x - y\|^{1-n} \leqq (1 - \|x\|)^{1-n} .$$

Hence

$$\int_B u^p d\mu \leqq (2/\Sigma(1))^p \int_B (1 - \|x\|)^{(1-n)p} (\Sigma(1) M(u))^p d\mu(x) = L(2M(u))^p .$$

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