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STABLE MINIMAL SUBMANIFOLDS IN COMPACT RANK ONE SYMMETRIC SPACES

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Introduction. A compact submanifold M (without boundary) immersed in a Riemannian manifold $\bar{M}$ is called *minimal* if the first variation of its volume vanishes for every deformation of M in $\bar{M}$. Clearly, if the volume of M is a local minimum among all immersions, M is a minimal submanifold of $\bar{M}$. But the volume of a minimal submanifold is not always a local minimum. Nowadays we know a large number of examples of minimal submanifolds (e.g. totally geodesic submanifolds, complex submanifolds of Kaehler manifolds and extremal orbits of compact transformation groups, etc.). It is an important problem to know whether a given minimal submanifold has a local minimum volume or not.

We say that a compact minimal submanifold M in $\bar{M}$ is *stable* if the second variation of its volume is nonnegative for every deformation of M in $\bar{M}$. Clearly, if M has a local minimum volume, then it is stable. The class of stable minimal submanifolds is much smaller than the class of general minimal submanifolds. The existence of a stable minimal submanifold is closely related to the topological and Riemannian structures of the ambient manifold. In fact, Simons [13] and Lawson-Simons [9] proved the following remarkable theorems.

THEOREM A. *No p-dimensional compact minimal submanifold immersed in the Euclidean sphere S^n is stable for each p with $1 \leqq p \leqq n-1$.*

THEOREM B. *Let M be a p-dimensional compact minimal submanifold immersed in the complex projective space $P^n(\boldsymbol{C})$ with the Fubini-Study metric. Then M is stable if and only if $p = 2l$ for some integer $l \geqq 1$ and M is a complex submanifold in the sense that each tangent space of M is invariant under the complex structure of $P^n(\boldsymbol{C})$.*

The purpose of this paper is to complete the classification of compact stable minimal submanifolds in all compact rank one symmetric spaces (the sphere S^n, the real projective space $P^n(\boldsymbol{R})$, the complex projective space $P^n(\boldsymbol{C})$, the quaternionic projective space $P^n(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$). We will prove the following theorems.

THEOREM C. *Let M be a p-dimensional compact minimal submanifold immersed in the real projective space $P^n(\boldsymbol{R})$ with the standard metric. Then M is stable if and only if M is a real projective subspace $P^p(\boldsymbol{R})$ of $P^n(\boldsymbol{R})$.*

THEOREM D. *Le M be a p-dimensional compact minimal submanifold immersed in the quaternionic projective space $P^n(\boldsymbol{H})$ with the standard metric. Then M is stable if and only if $p = 4l$ for some integer $l \geqq 1$ and M is a quaternionic projective subspace $P^l(\boldsymbol{H})$ of $P^n(\boldsymbol{H})$.*

THEOREM E. *Let M be a p-dimensional compact minimal submanifold immersed in the Cayley projective plane $P^2(\boldsymbol{Cay})$ with the standard metric. Then M is stable if and only if $p = 8$ and M is a Cayley projective line $P^1(\boldsymbol{Cay}) = S^8$ of $P^2(\boldsymbol{Cay})$.*

From these results we see that for a compact rank one symmetric space $\bar{M}$ except $P^n(\boldsymbol{C})$, every compact stable minimal submanifold represents a basis of the homology group (with coefficients in $\boldsymbol{Z}$ or $\boldsymbol{Z}_2$ according as $\bar{M}$ is simply connected or not), and vice versa. For compact symmetric spaces of rank greater than one we cannot expect such a relationship between stable minimal submanifolds and homology (cf. Chen, Leung and Nagano [3]). Takeuchi [15] showed that there are many noncomplex compact stable minimal submanifolds in compact Hermitian symmetric spaces of rank greater than one.

Lawson and Simons [9] generalized Theorems A and B to currents on S^n and $P^n(\boldsymbol{C})$. We generalize Theorems D and E to currents on $P^n(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$ (cf. Theorem 3.3). [9] and [13] carried out the proof by deforming a submanifold or a rectifiable current along conformal vector fields or holomorphic vector fields taking the average of the second variations. We deform a submanifold or a rectifiable current on a compact symmetric space along gradient vector fields of the first eigenfunctions for the Laplacian, and use the standard immersion of the compact symmetric space into the first eigenspace in order to compute the average of the second variations. Conformal vector fields of S^n, holomorphic vector fields of $P^n(\boldsymbol{C})$ and infinitesimal projective transformations of each projective space are gradient vector fields of the first eigenfunctions for the Laplacian. So our method is a generalization of that of Lawson and Simons. We will get the nonexistence of stable currents of certain degree on some compact rank two symmetric spaces (cf. Theorem 4.3).

In [9] the following was posed:

CONJECTURE. *Let M be a compact simply connected Riemannian manifold with the sectional curvature K satisfying $1/4 < K \leqq 1$. Then* *there exist no stable p-currents on M for* $1 \leqq p \leqq \dim(M) - 1$.

If the conjecture is true, then by virtue of the fundamental theorems on integral currents by Federer and Fleming (cf. [4], [8] or [9]), M is a homology sphere.

In the last section we show the nonexistence of stable currents on certain convex hypersurfaces of the Euclidean space as a partial answer to this conjecture.

THEOREM F. *Let* $\bar{M}$ *be an n-dimensional compact Riemannian manifold isometrically immersed in an* $(n+1)$*-dimensional Euclidean space* $\boldsymbol{E}^{n+1}$ *and suppose that every principal curvature* κ_i *of* $\bar{M}$ *satisfies* $\sqrt{\delta} \leqq \kappa_i \leqq 1$ $(i = 1, \cdots, n)$. *If* δ *satisfies* $1/2 < \delta \leqq 1$, *then there exist no stable p-currents on* $\bar{M}$ *for each p with* $1 \leqq p \leqq n - 1$.

Mori [10] showed the above consequence under the assumption that $\delta > n/(n+1)$. Theorem D was proved independently by M. Takeuchi.

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1. Second variational formulas.

1.1. Let $\psi: (M, g) \to (\bar{M}, \bar{g})$ be a minimal isometric immersion of a p-dimensional compact Riemannian manifold (M, g). We denote by $C^\infty(\psi^* T(\bar{M}))$ the space of all C^∞-vector fields along ψ. For any V in $C^\infty(\psi^* T(\bar{M}))$ let $\{\psi_t\}$ be a C^∞-one-parameter family of immersions of M into $\bar{M}$ with $\psi_0 = \psi$ and with the variation vector field $(d/dt)\psi_t(x)|_{t=0} = V_x$ $(x \in M)$. We put $\mathscr{V}(t) = \mathrm{Vol}(M, \psi_t^* \bar{g})$. We denote by $Q_M(V)$ the second derivative of $\mathscr{V}(t)$ at $t = 0$. From the classical second variational formula $Q_M(V)$ is described as follows (cf. Simons [13, p. 73]):

$$Q_M(V) = \int_M \langle -\Delta^\perp(V^N) - \tilde{A}(V^N) + \tilde{R}(V^N), V^N \rangle dv .$$

Here dv denotes the Riemannian measure of (M, g) and V^N the normal component of $V: V_x = V_x^N + \psi_* V_x^T$ $(x \in M)$. $\Delta^\perp = \mathrm{Tr}_g(\nabla^\perp)^2$ is the Laplacian on the normal bundle $N(M)$ of ψ. $\tilde{A}, \tilde{R} \in C^\infty(\mathrm{End}\, N(M))$ are defined by

$$\langle \tilde{A}(u), v \rangle = \mathrm{Tr}_g(A_u A_v) ,$$

$$\langle \tilde{R}(u), v \rangle = \sum_{i=1}^p \langle \bar{R}(e_i, u)e_i, v \rangle$$

for $u, v \in N_x(M)$, where A is the shape operator of ψ, $\{e_i\}$ is an orthonormal basis of $T_x(M)$ and $\bar{R}$ is the curvature tensor of $(\bar{M}, \bar{g})$. Put $\mathscr{J} = -\Delta^\perp - \tilde{A} + \tilde{R}$. $\mathscr{J}$ is a self-adjoint strongly elliptic differential operator of order 2 on the space $C^\infty(N(M))$ of all C^∞-sections of $N(M)$, called the *Jacobi operator* of ψ. $Q_M(V)$ defines a quadratic form on $C^\infty(\psi^* T(\bar{M}))$. A minimal immersion ψ is called *stable* if $Q_M(V) \geqq 0$ for all V in $C^\infty(\psi^* T(\bar{M}))$. In this case, M is said to be a *stable minimal submanifold* of $\bar{M}$. We put $E_\lambda = \{V \in C^\infty(N(M)); \mathscr{J}(V) = \lambda V\}$. $\sum_{\lambda<0} \dim E_\lambda$ is called the *index* of ψ. $\dim E_0$ is called the *nullity* of ψ. A normal vector field V in E_0 is called a *Jacobi field* of ψ. We define a subspace P of $C^\infty(N(M))$ by

$$P = \{X^N; X \text{ is a Killing vector field on } \bar{M}\}$$

and call $\dim P$ the *Killing nullity* of ψ. It is known that $P \subset E_0$ (cf. Simons [13]). A minimal immersion ψ is stable if and only if the index of ψ is zero.

1.2. In subsequent sections we need the description of the curvature tensor for $P^n(\boldsymbol{C})$, $P^n(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$ (cf. Brown and Gray [2]). The curvature tensor R of the Fubini-Study metric on $P^n(\boldsymbol{C})$ with constant holomorphic sectional curvature c is given by

$$(1.1) \qquad \begin{aligned} R(X, Y)Z = (c/4)\{&\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX \\ &- \langle JX, Z\rangle JY + 2\langle X, JY\rangle JZ\} \end{aligned}$$

for any $X, Y, Z \in T_x(P^n(\boldsymbol{C}))$, where J is the complex structure of $P^n(\boldsymbol{C})$.

The curvature tensor R of the standard metric on $P^n(\boldsymbol{H})$ with the maximum c of the sectional curvatures is given by

$$(1.2) \qquad \begin{aligned} R(X, Y)Z = (c/4)\Big\{&\langle Y, Z\rangle X - \langle X, Z\rangle Y \\ &+ \sum_{i=1}^{3} (\langle J_i Y, Z\rangle J_i X - \langle J_i X, Z\rangle J_i Y) + 2\sum_{i=1}^{3} \langle X, J_i Y\rangle J_i Z\Big\} \end{aligned}$$

for any $X, Y, Z \in T_x(P^n(\boldsymbol{H}))$, where $\{J_1, J_2, J_3\}$ is a canonical local basis of the quaternionic Kaehler structure (cf. Ishihara [6]) of $P^n(\boldsymbol{H})$.

A $4l$-dimensional submanifold M immersed in a quaternionic Kaehler manifold $\bar{M}$ is called a *quaternionic submanifold* if each tangent space of M is invariant under J_i $(i = 1, 2, 3)$.

Let x be a point in $P^2(\boldsymbol{Cay})$. We can identify $T_x(P^2(\boldsymbol{Cay}))$ with $\boldsymbol{Cay} \oplus \boldsymbol{Cay}$ in a natural manner. Using the structure of the Cayley algebra, the curvature tensor R of the standard metric on $P^2(\boldsymbol{Cay})$ with the maximum c of the sectional curvatures is given by

$$(1.3) \qquad \begin{aligned} &R((x, y), (z, w))(u, v) \\ &= (c/4)(-4\langle x, u\rangle z + 4\langle z, u\rangle x + (uw)y^* - (uy)w^* + (xw - zy)v, \\ &\qquad x^*(zv) - z^*(xv) - 4\langle y, v\rangle w + 4\langle w, v\rangle y - u^*(xw - zy)) \end{aligned}$$

for any (x, y), (z, w), $(u, v) \in T_x(P^2(\boldsymbol{Cay})) = \boldsymbol{Cay} \oplus \boldsymbol{Cay}$, where a^* denotes the conjugate of a as a Cayley number and $\langle a, b \rangle = (a^*b + b^*a)/2$ for $a, b \in \boldsymbol{Cay}$. $\boldsymbol{Cay} \oplus \{0\} = \{(x, 0); x \in \boldsymbol{Cay}\}$ is an 8-dimensional Lie triple system of the symmetric space $P^2(\boldsymbol{Cay})$. The complete totally geodesic submanifold generated by $\boldsymbol{Cay} \oplus \{0\}$ is the Cayley projective line $P^1(\boldsymbol{Cay})$ of $P^2(\boldsymbol{Cay})$. $P^1(\boldsymbol{Cay})$ is isometric to an 8-dimensional sphere S^8 with constant sectional curvature c. If an 8-dimensional subspace of a tangent space of $P^2(\boldsymbol{Cay})$ is congruent to the Lie triple system $\boldsymbol{Cay} \oplus \{0\}$ by an isometry of $P^2(\boldsymbol{Cay})$, then we call it *of Cayley type*. An 8-dimensional submanifold M immersed in $P^2(\boldsymbol{Cay})$ is called a *Cayley submanifold* if each tangent space of M is of Cayley type.

1.3. Here we give examples of compact stable minimal submanifolds in projective spaces.

PROPOSITION 1.1. *Let $\bar{M}$ be a projective space and M a projective subspace of $\bar{M}$ or a compact complex submanifold in $\bar{M} = P^n(\boldsymbol{C})$. Then the index, the nullity and the Killing nullity of M are given as in Table 1, where* CCS *means an l-dimensional compact complex submanifold.*

TABLE 1.

$\bar{M}$	M	index	nullity	Killing nullity
(1) $P^n(\boldsymbol{R})$	$P^l(\boldsymbol{R})$	0	$(l+1)(n-l)$	$(l+1)(n-l)$
(2) $P^n(\boldsymbol{C})$	CCS	0	$\geqq 2(l+1)(n-l)$	$\geqq 2(l+1)(n-l)$
	$P^l(\boldsymbol{C})$	0	$2(l+1)(n-l)$	$2(l+1)(n-l)$
(3) $P^n(\boldsymbol{H})$	$P^l(\boldsymbol{H})$	0	$4(l+1)(n-l)$	$4(l+1)(n-l)$
(4) $P^2(\boldsymbol{Cay})$	$P^1(\boldsymbol{Cay})$	0	16	16

REMARK. (1) The results in the case of $P^n(\boldsymbol{C})$ are contained in Simons [13] and Kimura [7]. The index and nullity for the other cases can be computed by means of Hopf fibrations and the method of Chen, Leung and Nagano [3]. The Killing nullity is determined in a way similar to Simons [13, p. 87]. Here we omit the detail of the proof.

(2) It is interesting to study the nullity of minimal submanifolds. Simons [13] asked when a Jacobi field on a minimal submanifold arises from a one-parameter family of minimal submanifolds. By the above result the nullity of the projective subspace of the projective space coincides with the Killing nullity. Hence in this case any Jacobi field arises from a one-parameter family of minimal submanifolds. It seems that the nullity coincides with the Killing nullity for fairly many compact totally geodesic submanifolds in compact symmetric spaces.

1.4. Next we explain the variational formulas for rectifiable currents. We use the same notation as in Lawson and Simons [9]. See also [8] or [9] for detailed definitions. Let $(\bar{M}, \bar{g})$ be a compact n-dimensional Riemannian manifold and $\bar{\nabla}$ its Riemannian connection. We denote by $\bar{g}$ or $\langle\ ,\ \rangle$ the inner product of $\wedge^p T_x(\bar{M})$ induced by $\bar{g}$. Let $\mathscr{R}_p(\bar{M})$ be the set of all rectifiable p-currents on $\bar{M}$, where $0 \leqq p \leqq n$. For a current $\mathscr{S} \in \mathscr{R}_p(\bar{M})$, $\vec{\mathscr{S}}_x$ denotes the orientation of $\mathscr{S}$. It is an $\mathscr{H}^p$-measurable field on $\bar{M}$ of simple p-vectors of unit length which represent tangent planes of $\mathscr{S}$, where $\mathscr{H}^p$ is the Hausdorff p-measure on $\bar{M}$. For a vector field V on $\bar{M}$ we define an endomorphism $\mathscr{A}^V$ of $T_x(\bar{M})$ by $\mathscr{A}^V(X) = \bar{\nabla}_X V$ for $X \in T_x(\bar{M})$. This endomorphism can be extended to $\wedge^p T_x(\bar{M})$ uniquely as a derivation. At x in $\bar{M}$, we define also an endomorphism $\bar{\nabla}_{V,\cdot} V$ of $T_x(\bar{M})$ by

$$\bar{\nabla}_{V,X} V = \bar{\nabla}_V \bar{\nabla}_{\tilde{X}} V - \bar{\nabla}_{\bar{\nabla}_V \tilde{X}} V$$

for $X \in T_x(\bar{M})$, where $\tilde{X}$ is any extension of X to a local vector field. This is independent of an extension $\tilde{X}$, and also the endomorphism $\bar{\nabla}_{V,\cdot} V$ carries over to $\wedge^p T_x(\bar{M})$ uniquely as a derivation. Consider a current $\mathscr{S} \in \mathscr{R}_p(\bar{M})$ and a vector field V on $\bar{M}$. Let $\phi_t \colon \bar{M} \to \bar{M}$, $t \in \boldsymbol{R}$, be the flow generated by V. Then for each t we have a rectifiable current $\phi_t(\mathscr{S})$. Let $\boldsymbol{M}$ denote the mass of rectifiable currents which is defined as the norm of a linear functional on $C^\infty(\wedge^p(\bar{M}))$ which has the supremum topology. If $\mathscr{S}$ is an oriented C^1-submanifold with finite volume, then $\boldsymbol{M}(\mathscr{S})$ is just the volume of $\mathscr{S}$. Then,

$$\boldsymbol{M}(\phi_t \mathscr{S}) = \int_{\bar{M}} |\phi_{t*} \vec{\mathscr{S}}_x| d\|\mathscr{S}\| ,$$

where $|\phi_{t*}\vec{\mathscr{S}}_x| = ((\phi_t^* \bar{g})(\vec{\mathscr{S}}_x, \vec{\mathscr{S}}_x))^{1/2}$ and $\|\mathscr{S}\|$ is the total variation measure associated to $\mathscr{S}$ defined by means of the p-dimensional Hausdorff measure $\mathscr{H}^p$ on $\bar{M}$.

DEFINITION. A rectifiable p-current $\mathscr{S} \in \mathscr{R}_p(\bar{M})$ is called *stable* if, for each vector field V on $\bar{M}$, the following two conditions hold:

$$(d/dt)\boldsymbol{M}(\phi_t \mathscr{S})_{|t=0} = 0 ,$$

$$(d^2/dt^2)\boldsymbol{M}(\phi_t \mathscr{S})_{|t=0} \geqq 0 ,$$

where ϕ_t is the flow generated by V.

The first and second variational formulas for the mass of rectifiable currents were obtained by Lawson and Simons [9] as follows:

PROPOSITION 1.2. *Let $\bar{M}$ be a compact Riemannian manifold and V*

a vector field on $\bar{M}$ with associated flow ϕ_t. Then for any rectifiable p-current $\mathscr{S} \in \mathscr{R}_p(\bar{M})$,

$$(d^k/dt^k)\boldsymbol{M}(\phi_t\mathscr{S})|_{t=0} = \int_{\bar{M}} (d^k/dt^k)|\phi_{t*}\vec{\mathscr{S}}_x|_{|t=0} d\|\mathscr{S}\| \quad (k = 1, 2)\ ,$$

where for a simple unit p-vector $\xi \in \wedge^p(\bar{M})$

$$(1.4)\qquad (d/dt)|\phi_{t*}\xi|_{|t=0} = \langle \mathscr{A}^V(\xi), \xi\rangle\ ,$$

$$(1.5)\qquad (d^2/dt^2)|\phi_{t*}\xi|_{|t=0} = -\langle \mathscr{A}^V(\xi), \xi\rangle^2 + \langle \mathscr{A}^V\mathscr{A}^V(\xi), \xi\rangle + |\mathscr{A}^V(\xi)|^2 + \langle \bar{\nabla}_{V,\xi} V, \xi\rangle\ .$$

REMARK. In the special case where $V = \operatorname{grad} f$ for some $f \in C^\infty(\bar{M})$, $\mathscr{A}^V$ is symmetric and (1.5) is simplified as

$$(1.6)\qquad (d^2/dt^2)|\phi_{t*}\xi|_{|t=0} = -\langle \mathscr{A}^V(\xi), \xi\rangle^2 + 2|\mathscr{A}^V(\xi)|^2 + \langle \bar{\nabla}_{V,\xi} V, \xi\rangle\ .$$

For future reference we shall write the right-hand side of (1.6) at $x \in \bar{M}$ in terms of tangent vectors at x. Let $\{e_1, \cdots, e_p, n_1, \cdots, n_q\}$ be an orthonormal basis of $T_x(\bar{M})$ and $\xi = e_1\wedge \cdots \wedge e_p$. Then

$$(1.7)\qquad -\langle \mathscr{A}^V(\xi), \xi\rangle^2 + 2|\mathscr{A}^V(\xi)|^2 + \langle \bar{\nabla}_{V,\xi} V, \xi\rangle = \left\{\sum_{j=1}^{p} \langle \mathscr{A}^V(e_j), e_j\rangle\right\}^2 + 2\sum_{j=1}^{p}\sum_{k=1}^{q} \langle \mathscr{A}^V(e_j), n_k\rangle^2 + \sum_{j=1}^{p} \langle \bar{\nabla}_{V,e_j} V, e_j\rangle\ ,$$

where $|\mathscr{A}^V(\xi)|$ denotes the length of the p-vector $\mathscr{A}^V(\xi)$.

To a simple p-vector $\xi \in \wedge^p T_x(\bar{M})$, $x \in \bar{M}$, we can associate a quadratic form Q_ξ on the space $\mathfrak{X}(\bar{M})$ of all C^∞-vector fields on $\bar{M}$ as follows; for $V \in \mathfrak{X}(\bar{M})$ with associated flow ϕ_t, define $Q_\xi(V) = (d^2/dt^2)|\phi_{t*}\xi|_{|t=0}$. We associate to each $\mathscr{S} \in \mathscr{R}_p(\bar{M})$ a quadratic form $Q_{\mathscr{S}}$ on $\mathfrak{X}(\bar{M})$ as follows; for $V \in \mathfrak{X}(\bar{M})$ with associated flow ϕ_t, define

$$Q_{\mathscr{S}}(V) = (d^2/dt^2)\boldsymbol{M}(\phi_t\mathscr{S})_{|t=0} = \int_{\bar{M}} Q_{\vec{\mathscr{S}}_x}(V) d\|\mathscr{S}\|\ .$$

REMARK. If a p-dimensional compact oriented minimal submanifold M of $\bar{M}$ is stable in the sense of 1.1., then M is stable as a current. But in general the converse is not true.

1.5. We define a class of rectifiable currents on a quaternionic Kaehler manifold $\bar{M}$ and $P^2(\boldsymbol{Cay})$, respectively.

DEFINITION. A $4l$-current $\mathscr{S} \in \mathscr{R}_{4l}(\bar{M})$ is called a *quaternionic current* if $\|\mathscr{S}\|$-almost all tangent planes $\vec{\mathscr{S}}_x$ of $\mathscr{S}$ are invariant under J_i ($i = 1, 2, 3$), where $\{J_1, J_2, J_3\}$ is the canonical local basis of the quaternionic Kaehler structure of $\bar{M}$. An 8-current $\mathscr{S} \in \mathscr{R}_8(P^2(\boldsymbol{Cay}))$ is called a *Cayley current* if $\|\mathscr{S}\|$-almost all tangent planes of $\mathscr{S}$ are of Cayley type.

Recently Tasaki [17] introduced a calibration by the fundamental $4l$-form for the quaternionic Kaehler structure and obtained the following:

PROPOSITION 1.3. *Any closed, canonically oriented quaternionic current is a homologically mass minimizing current, in particular a stable current.*

2. Stable minimal submanifolds in $P^n(\boldsymbol{R})$. In this section we shall prove the following:

THEOREM 2.1. *If M is a p-dimensional compact stable minimal submanifold in the real projective space $P^n(\boldsymbol{R})$, then M is a p-dimensional real projective subspace $P^p(\boldsymbol{R})$ of $P^n(\boldsymbol{R})$.*

We may suppose that the standard metric on $P^n(\boldsymbol{R})$ has constant sectional curvature 1. Let S^n be the unit hypersphere in the $(n+1)$-dimensional Euclidean space $\boldsymbol{E}^{n+1}$ with the standard inner product $\langle\ ,\ \rangle$ and $\pi: S^n \to P^n(\boldsymbol{R})$ the natural isometric covering. We denote by σ the antipodal involution of S^n: $\sigma(x) = -x$ $(x \in S^n)$. Let M be a p-dimensional compact minimal submanifold immersed in $P^n(\boldsymbol{R})$ and we denote the immersion by ψ. Let M' be a connected component of the total space of the pull-back $\psi^{-1}S^n$ for the principal $\boldsymbol{Z}_2$-bundle $\pi: S^n \to P^n(\boldsymbol{R})$. We have a commutative diagram:

$$\begin{tikzcd} M' \arrow[r, "\psi'"] \arrow[d, "\pi_M"'] & S^n \subset \boldsymbol{E}^{n+1} \arrow[d, "\pi"'] \\ M \arrow[r, "\psi"] & P^n(\boldsymbol{R})\,. \end{tikzcd}$$

$\psi': M' \to S^n$ is also an immersion, and so we may define a Riemannian metric on M' in such a way that $\psi': M' \to S^n$ is an isometric immersion. Then $\pi_M: M' \to M$ is an isometric covering and $\psi': M' \to S^n$ is also minimal. We denote by A', B' and $\mathscr{J}'$ the shape operator, the second fundamental form and the Jacobi operator for ψ', respectively. We denote by $\mathscr{J}$ the Jacobi operator for ψ.

For any vector v in $\boldsymbol{E}^{n+1}$, we define a C^∞-function f_v on S^n by $f_v(x) = \langle x, v\rangle$ $(x \in S^n)$. f_v is the height function on S^n in the direction of v. Put $V = \operatorname{grad} f_v \in \mathfrak{X}(S^n)$, where $\mathfrak{X}(S^n)$ is the space of all C^∞-vector fields on S^n. V is a conformal vector field on S^n. Since $\sigma_*(V) = -V$, V does not project to any vector field on $P^n(\boldsymbol{R})$. Now let u and v be two arbitrary vectors of $\boldsymbol{E}^{n+1}$ and put $U = \operatorname{grad} f_u$ and $V = \operatorname{grad} f_v$. We consider a vector field $Z' = f_u V = f_u(\operatorname{grad} f_v)$ on S^n. Since $\sigma_*(Z') = Z'$, Z' projects to a vector field on $P^n(\boldsymbol{R})$, that is, there is a vector field Z on $P^n(\boldsymbol{R})$ such that $\pi_* Z'_x = Z_{\pi(x)}$ for $x \in S^n$. We shall compute $\mathscr{J}'(Z'^N)$, where Z'^N is the component of Z' normal to M'.

LEMMA 2.2 (cf. Simons [13, p. 85]). *If V^T and V^N are the components of V tangential and normal to M', respectively, then they satisfy*

$$\nabla'_X V^T = A'_{V^N}(X) - f_v(x)X , \tag{2.1}$$

$$\nabla'^{\perp}_X V^N = -B'(X, V^T) , \tag{2.2}$$

for any $X \in T_x(M')$, where ∇' and $\nabla'^{\perp}$ are the Riemannian connection and normal connection of M', respectively.

LEMMA 2.3. *If Z'^T and Z'^N are the components of Z' tangential and normal to M', respectively, then they satisfy*

$$\nabla'_X Z'^T = A'_{Z'^N}(X) + \langle X, u\rangle V^T - f_u(x)f_v(x)X , \tag{2.3}$$

$$\nabla'^{\perp}_X Z'^N = -B'(X, Z^T) + \langle X, u\rangle V^N , \tag{2.4}$$

for any $X \in T_x(M')$.

PROOF. Using (2.1) and (2.2), we have $\nabla'_X Z'^T = \nabla'_X(f_u V^T) = (Xf_u)V^T + f_u\nabla'_X V^T = \langle X, u\rangle V^T + A'_{Z'^N}(X) - f_u f_v X$, and $\nabla'^{\perp}_X Z'^N = \nabla'^{\perp}_X(f_u V^N) = (Xf_u)V^N + f_u\nabla'^{\perp}_X V^N = \langle X, u\rangle V^N - B'(X, Z'^T)$. q.e.d.

LEMMA 2.4. $\mathscr{J}'(Z'^N) = 2B'(U^T, V^T)$.

PROOF. Choose an orthonormal frame field $(e_1, \cdots, e_p)$ around a point x in M' such that $(\nabla' e_i)_x = 0$ $(1 \leqq i \leqq p)$. By (2.4) at x we have

$$\begin{aligned}
\Delta'^{\perp} Z'^N &= \sum_{i=1}^{p} \nabla'^{\perp}_{e_i}\nabla'^{\perp}_{e_i} Z'^N \\
&= \sum_{i=1}^{p} \nabla'^{\perp}_{e_i}(-B'(e_i, Z'^T) + \langle e_i, u\rangle V^N) \\
&= \sum_{i=1}^{p} (-(\nabla'^{*}_{e_i}B')(e_i, Z'^T) - B'(e_i, \nabla'_{e_i}Z'^T) + \langle \bar{\nabla}'_{e_i}e_i - \langle e_i, e_i\rangle x, u\rangle V^N \\
&\qquad + \langle e_i, u\rangle \nabla'^{\perp}_{e_i} V^N) ,
\end{aligned}$$

where $\bar{\nabla}'$ is the Riemannian connection of S^n and $\nabla'^{*}B'$ is defined by $(\nabla'^{*}_X B')(Y, Z) = \nabla'^{\perp}_X(B'(Y, Z)) - B'(\nabla'_X Y, Z) - B'(Y, \nabla'_X Z)$. By (2.2), (2.3) and the minimality of ψ', we have

$$\begin{aligned}
\Delta'^{\perp} Z'^N &= \sum_{i=1}^{p} (-(\nabla'^{*}_{e_i}B')(e_i, Z'^T) - B'(e_i, A'_{Z'^N}(e_i)) - \langle e_i, u\rangle B'(e_i, V^T) \\
&\qquad + f_u(x)f_v(x)B'(e_i, e_i) + \langle B'(e_i, e_i), u\rangle V^N - pf_u(x)V^N \\
&\qquad - \langle e_i, u\rangle B'(e_i, V^T)) \\
&= -\tilde{A}'(Z'^N) - 2B'(U^T, V^T) - pZ'^N .
\end{aligned}$$

Since $\tilde{R}'(Z'^N) = -pZ'^N$, we obtain $\mathscr{J}'(Z'^N) = 2B'(U^T, V^T)$. Here $\tilde{A}'$ and $\tilde{R}'$ are defined for ψ' in the same way as in 1.1 of Section 1. q.e.d.

In the above situation we shall prove Theorem 2.1.

PROOF OF THEOREM 2.1. We assume that M is stable. Let Z^N be the component of Z normal to M. Since π is a local isometry, we have $\pi_* \mathscr{J}'(Z'^N) = \mathscr{J}(Z^N)$ and $\langle \mathscr{J}(Z^N), Z^N \rangle \circ \pi_M = \langle \mathscr{J}'(Z'^N), Z'^N \rangle = 2\langle B'(U^T, V^T), V^N \rangle f_u$. We may regard $\langle B'(U^T, V^T), V^N \rangle f_u$ as a function on M. Then we have

$$(2.5) \qquad Q_M(Z) = \int_M \langle \mathscr{J}(Z^N), Z^N \rangle dv = 2\int_M \langle B'(U^T, V^T), V^N \rangle f_u dv .$$

Fix the vector u and regard $Q_M(Z)$ as a quadratic form with respect to $v \in \boldsymbol{E}^{n+1}$. We compute the trace of $Q_M(Z)$ on $v \in \boldsymbol{E}^{n+1}$ with respect to the inner product $\langle \ , \ \rangle$. Let $\{v_1, \cdots, v_{n+1}\}$ be an orthonormal basis of $\boldsymbol{E}^{n+1}$ with respect to $\langle \ , \ \rangle$. Then we have

$$(2.6) \qquad \mathrm{Tr}_v\, Q_M(Z) = 2\int_M \sum_{i=1}^{n+1} \langle B'(U^T, V_i^T), V_i^N \rangle f_u dv ,$$

where $V_i = \mathrm{grad} f_{v_i}$ $(i = 1, \cdots, n+1)$. We fix any $x \in M'$. Since the integrand on the right-hand side of (2.6) is independent of the choice of the orthonormal basis $\{v_1, \cdots, v_{n+1}\}$, we may assume that $v_1, \cdots, v_p$ and $v_{p+1}, \cdots, v_n$ are tangent and normal vectors at $x \in M'$, respectively, and $v_{n+1} = x$. Since $(V_i^T)_x$ and $(V_i^N)_x$ are the $T_x(M')$- and $N_x(M')$-components of v_i in $\boldsymbol{E}^{n+1}$, respectively, we have

$$\sum_{i=1}^{n+1} \langle B'(U^T, V_i^T), V_i^N \rangle f_u = 0 \quad \text{at} \quad x .$$

As x is any point of M', the integrand of (2.6) vanishes identically on M. Therefore we have $\mathrm{Tr}_v\, Q_M(Z) = 0$. Since $Q_M(Z)$ is nonnegative by the stability of M, we have $Q_M(Z) = 0$ for any $u, v \in \boldsymbol{E}^{n+1}$. As $\mathscr{J}$ has no negative eigenvalue by the stability of M, we get $\mathscr{J}(Z^N) = 2\pi_* B'(U^T, V^T) = 0$ for any $u, v \in \boldsymbol{E}^{n+1}$. Hence $B' = 0$ on M'. Thus both M' and M are totally geodesic. Therefore, either M is isometric to $P^p(\boldsymbol{R})$ and ψ is a totally geodesic imbedding, or M is isometric to $S^p(1)$ and ψ covers a projective subspace $P^p(\boldsymbol{R})$ of $P^n(\boldsymbol{R})$. We have only to show that the latter never happens. In the latter case there is a lift $\phi: M \to S^n$ such that $\psi = \pi \circ \phi$. ϕ is a minimal isometric immersion of M into S^n. By Theorem A the second variation of the volume of M is negative for some $V \in C^\infty(\phi^* T(S^n))$. Then the second variation $Q_M(\pi_* V)$ for $\pi_* V \in C^\infty(\psi^* T(P^n(\boldsymbol{R})))$ is negative. This is a contradiction. q.e.d.

Combining (1) of Proposition 1.1 with Theorem 2.1, we obtain Theorem C.

3. Stable minimal submanifolds in $P^n(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$. The purpose of this section is to show the following theorems:

THEOREM 3.1. *If M is a p-dimensional compact stable minimal submanifold immersed in the quaternionic projective space $P^n(\boldsymbol{H})$, then $p = 4l$ for some integer l and M is an l-dimensional quaternionic projective subspace $P^l(\boldsymbol{H})$ of $P^n(\boldsymbol{H})$.*

THEOREM 3.2. *If M is a p-dimensional compact stable minimal submanifold immersed in the Cayley projective plane $P^2(\boldsymbol{Cay})$, then $p = 8$ and M is a Cayley projective line $P^1(\boldsymbol{Cay})$ of $P^2(\boldsymbol{Cay})$.*

Put $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$ and $\boldsymbol{Cay}$, and let $P^n(\boldsymbol{F})$ be the n-dimensional projective space over $\boldsymbol{F}$ with the standard metric of the maximum c of the sectional curvatures. Here $n = 2$ when $\boldsymbol{F} = \boldsymbol{Cay}$.

We shall give the unified proof of the following theorem.

THEOREM 3.3. *Let $\bar{M}$ be a compact rank one symmetric space, that is, $\bar{M} = S^n$ or $P^n(\boldsymbol{F})$ and $\mathscr{S} \in \mathscr{R}_p(\bar{M})$ a stable p-current.*

(1) *If $\bar{M} = S^n$, then $p = 0$ or $p = n$ (Lawson and Simons).*

(2) *If $\bar{M} = P^n(\boldsymbol{C})$, then $p = 2l$ for some integer l and $\mathscr{S}$ is a complex current (Lawson and Simons).*

(3) *If $\bar{M} = P^n(\boldsymbol{H})$, then $p = 4l$ for some integer l and $\mathscr{S}$ is a quaternionic current.*

(4) *If $\bar{M} = P^2(\boldsymbol{Cay})$, then $p = 0$, 16 or $p = 8$ and $\mathscr{S}$ is a Cayley current.*

In particular, we obtain the following.

COROLLARY 3.4. *Let $\bar{M}$ be a p-dimensional compact stable minimal submanifold immersed in $P^n(\boldsymbol{F})$.*

(1) *If $\boldsymbol{F} = \boldsymbol{C}$, then $p = 2l$ for some integer l and M is a complex submanifold (Lawson and Simons).*

(2) *If $\boldsymbol{F} = \boldsymbol{H}$, the $p = 4l$ for some integer l and M is a quaternionic submanifold.*

(3) *If $\boldsymbol{F} = \boldsymbol{Cay}$, then $p = 8$ and M is a Cayley submanifold.*

First we derive the following trace formula for a submanifold in a Euclidean space.

PROPOSITION 3.5. *Let $\bar{M}$ be an n-dimensional Riemannian manifold isometrically immersed in the Euclidean space $\boldsymbol{E}^m$ with the canonical inner product $\langle\ ,\ \rangle$ and denote by Φ the immersion. Assume that the image of $\bar{M}$ does not lie in any hyperplane of $\boldsymbol{E}^m$. We define*

$$\mathscr{V} = \{\operatorname{grad} f_v \in \mathfrak{X}(\bar{M});\ v \in \boldsymbol{E}^m\}\ ,$$

where $f_v(x) = \langle \Phi(x), v \rangle$ $(x \in \bar{M})$. *Making use of the natural isomorphism*

$$\mathscr{V} \cong \boldsymbol{E}^m , \tag{3.1}$$

we introduce an inner product on $\mathscr{V}$. *To any unit simple p-vector* $\xi \in \wedge^p T_x(\bar{M})$, $x \in \bar{M}$, *we associate a quadratic form* Q_ξ *on* $\mathscr{V}$ *in the same way as in* 1.4. *of Section* 1. *Then we have*

$$\operatorname{Tr} Q_\xi = \sum_{j=1}^{p} \sum_{k=1}^{p} (2\| \bar{B}(e_j, n_k) \|^2 - \langle \bar{B}(e_j, e_j) \bar{B}(n_k, n_k) \rangle) , \tag{3.2}$$

where $\bar{B}$ *is the second fundamental form of* Φ *and* $\{e_1, \cdots, e_p, n_1, \cdots, n_q\}$ *is an orthonormal basis of* $T_x(\bar{M})$ *with* $\xi = e_1 \wedge \cdots \wedge e_p$.

PROOF. Assume that $V \in \mathscr{V}$ corresponds to $v \in \boldsymbol{E}^m$ under the isomorphism (3.1). Then at any $x \in \bar{M}$ we have $V_x = v^T$, where $(\ )^T$ denotes the orthogonal projection $T_x(\boldsymbol{E}^m) \to T_x(\bar{M})$. By a simple computation it follows that

$$(\bar{\nabla}^2 f_v)(X, Y) = \langle \bar{B}(X, Y), v \rangle , \tag{3.3}$$

$$(\bar{\nabla}^3 f_v)(X, Y, Z) = -\langle \bar{B}(X, Y), \bar{B}(Z, V) \rangle + \langle (\bar{\nabla}^* \bar{B})(X, Y, Z), v \rangle \tag{3.4}$$

for $X, Y, Z \in T_x(\bar{M})$. Here $\bar{\nabla}$ is the Riemannian connection of $\bar{M}$ and $\bar{\nabla}^* \bar{B}$ is the covariant derivative of $\bar{B}$ defined in the same way as in Section 2. We define $\mathscr{A}^V$ and $\bar{\nabla}_{V,\cdot} V$ in the same way as in 1.4. of Section 1. Since $(\bar{\nabla}^2 f_v)(X, Y) = \langle \bar{\nabla}_Y V, X \rangle$ and $(\bar{\nabla}^3 f_v)(X, Y, Z) = \langle \bar{\nabla}_{Z,Y} V, X \rangle$, it follows from (3.3) and (3.4) that

$$\langle \mathscr{A}^V(X), Y \rangle = \langle \bar{B}(X, Y), v \rangle , \tag{3.5}$$

$$\langle \bar{\nabla}_{V,X} V, Y \rangle = -\langle \bar{B}(Y, X), \bar{B}(V, V) \rangle + \langle (\bar{\nabla}^* \bar{B})(Y, X, V), v \rangle \tag{3.6}$$

for $X, Y, Z \in T_x(\bar{M})$. Thus from (1.5), (1.6) and (1.7) we have

$$\begin{aligned} Q_\xi(V) = &\left(\sum_{j=1}^{p} \langle \bar{B}(e_j, e_j), v \rangle \right)^2 + 2 \sum_{j=1}^{p} \sum_{k=1}^{q} \langle \bar{B}(e_j, n_k), v \rangle^2 \\ &- \sum_{j=1}^{p} \langle \bar{B}(e_j, e_j), \bar{B}(V, V) \rangle + \sum_{j=1}^{p} \langle (\bar{\nabla}^* \bar{B})(e_j, e_j, V), v \rangle . \end{aligned} \tag{3.7}$$

We now choose an orthonormal basis $\{e_1, \cdots, e_p, n_1, \cdots, n_q, \zeta_1, \cdots, \zeta_{m-n}\}$ for $\boldsymbol{E}^m$ and let $\{V_1, V_2, \cdots, V_m\}$ be an orthonormal basis of $\mathscr{V}$ corresponding to $\{e_1, \cdots, e_p, n_1, \cdots, n_q, \zeta_1, \cdots, \zeta_{m-n}\}$ via (3.1). Hence from (3.7) we obtain

$$\begin{aligned} \operatorname{Tr} Q_\xi &= \sum_{l=1}^{m} Q_\xi(V_l) \\ &= 2 \sum_{j=1}^{p} \sum_{k=1}^{q} \langle \bar{B}(e_j, n_k), \bar{B}(e_j, n_k) \rangle - \sum_{j=1}^{p} \sum_{k=1}^{q} \langle \bar{B}(e_j, e_j), \bar{B}(n_k, n_k) \rangle . \quad \text{q.e.d.} \end{aligned}$$

Next we review quickly the definition of the standard minimal immersions of compact irreducible symmetric spaces (cf. [14], [11]). Let $\bar{M} = G/K$ be an n-dimensional compact irreducible symmetric space represented by a symmetric pair (G, K) and g_0 a G-invariant Riemmannian metric on $\bar{M}$ induced by the Killing form of the Lie algebra of G. We should note that the scalar curvature of $(\bar{M}, g_0)$ is equal to $n/2$ (cf. [16]). Let Δ be the Laplacian of $(\bar{M}, g_0)$ acting on functions. For the k-th eigenvalue λ_k of Δ, we choose an orthonormal basis $\{f_0, \cdots, f_{m(k)}\}$ of the k-th eigenspace V_k with respect to the L^2-inner product defined by g_0. We define a mapping Φ_k of M into $\boldsymbol{E}^{m(k)+1}$ by

$$\Phi_k\colon \bar{M} \ni x \mapsto C\cdot(f_0(x), \cdots, f_{m(k)}) \in \boldsymbol{E}^{m(k)+1} ,$$

where $C = (\mathrm{Vol}(\bar{M}, g_0)/(m(k)+1))^{1/2}$. Then $\Phi_k = \iota\circ\phi_k$ is the composite of a G-equivariant minimal isometric immersion ϕ_k of $(\bar{M}, (\lambda_k/n)g_0)$ into the unit sphere $S^{m(k)}(1)$ and the inclusion map ι of $S^{m(k)}(1)$ into $\boldsymbol{E}^{m(k)+1}$. ϕ_k is called the *k-th standard minimal immersion* of $\bar{M}$. $\Phi_k(\bar{M})$ is not contained in any hyperplane of $\boldsymbol{E}^{m(k)+1}$. We have $V_k = \{f_v;\, v \in \boldsymbol{E}^{m(k)+1}\}$, where $f_v(x) = \langle \Phi_k(x), v\rangle$ $(x \in \bar{M})$.

We consider the case of $\bar{M} = S^n$ or $P^n(\boldsymbol{F})$. Let ϕ_1 be the first standard minimal immersion of $\bar{M}$. If $\bar{M} = S^n$, then ϕ_1 is the identity map of S^n. If $\bar{M} = P^n(\boldsymbol{F})$, then ϕ_1 is the generalized Veronese imbedding (cf. Sakamoto [12]). Then $\Phi_1 = \iota\circ\phi_1$ has the following properties.

PROPOSITION 3.6.

(i) *Φ_1 is an isotropic immersion, that is, there is a positive constant λ such that*

$$\|\bar{B}(X, X)\|^2 = \lambda^2 \quad \textit{for any unit vector } X \textit{ on } \bar{M}\,, \tag{3.8}$$

where $\bar{B}$ denotes the second fundamental form of Φ_1.

(ii) *Let c be the maximum of the sectional curvatures of the Riemannian metric on $\bar{M}$ induced by Φ_1. Then the values of c and λ^2 in each case are given as in Table 2.*

TABLE 2.

$\bar{M}$	$\dim \bar{M}$	c	λ^2
S^n	n	1	1
$P^n(\boldsymbol{R})$	n	$n/2(n+1)$	$2n/(n+1)$
$P^n(\boldsymbol{C})$	$2n$	$2n/(n+1)$	$2n/(n+1)$
$P^n(\boldsymbol{H})$	$4n$	$2n/(n+1)$	$2n/(n+1)$
$P^2(\boldsymbol{Cay})$	16	$4/3$	$4/3$

(3.8) is equivalent to

$$\begin{aligned}(3.9)\quad &\langle \bar{B}(X, Y), \bar{B}(Z, W)\rangle + \langle \bar{B}(Y, Z), \bar{B}(X, W)\rangle + \langle \bar{B}(Z, X), \bar{B}(Y, W)\rangle \\ &\qquad = \lambda^2(\langle X, Y\rangle\langle Z, W\rangle + \langle Y, Z\rangle\langle X, W\rangle + \langle Z, X\rangle\langle Y, W\rangle)\end{aligned}$$

for $X, Y, Z, W \in T_x(\bar{M})$. Applying the Gauss equation for Φ_1 to the second and third terms on the left-hand side of (3.9), we obtain

$$\begin{aligned}(3.10)\quad &3\langle \bar{B}(X, Y), \bar{B}(Z, W)\rangle \\ &\qquad = \langle \bar{R}(X, Z)W, Y\rangle + \bar{R}\langle (X, W)Z, Y\rangle + \lambda^2\langle X, Y\rangle\langle Z, W\rangle \\ &\qquad\quad + \lambda^2\langle X, W\rangle\langle Y, Z\rangle + \lambda^2\langle X, Z\rangle\langle W, Y\rangle\end{aligned}$$

for $X, Y, Z, W \in T_x(\bar{M})$, where $\bar{R}$ is the curvature tensor of $\bar{M}$. We denote by $\bar{K}$ the sectional curvature of $\bar{M}$: $\bar{K}(X\wedge Y) = \langle \bar{R}(X, Y)Y, X\rangle$ for orthonormal vectors X, Y.

PROOF OF THEOREM 3.3. We apply Proposition 3.5 to the imbedding Φ_1 of $\bar{M}$. For any unit simple p-vector $\xi \in \wedge^p T_x(\bar{M})$, we choose an orthonormal basis $\{e_1, \cdots, e_p, n_1, \cdots, n_q\}$ of $T_x(\bar{M})$ with $\xi = e_1\wedge\cdots\wedge e_p$. By (3.10) we have

$$(3.11)\qquad 3\sum_{j=1}^{p}\sum_{k=1}^{q}\|\bar{B}(e_j, n_k)\|^2 = -\sum_{j=1}^{p}\sum_{k=1}^{q}\bar{K}(e_j\wedge n_k) + pq\lambda^2\ ,$$

$$(3.12)\qquad 3\sum_{j=1}^{p}\sum_{k=1}^{q}\langle \bar{B}(e_j, e_j), \bar{B}(n_k, n_k)\rangle = 2\sum_{j=1}^{p}\sum_{k=1}^{q}\bar{K}(e_j\wedge n_k) + pq\lambda^2\ .$$

Hence it follows from (3.11), (3.12) and (3.2) that

$$(3.13)\qquad \mathrm{Tr}\, Q_\xi = pq\lambda^2/3 - (4/3)\sum_{j=1}^{p}\sum_{k=1}^{q}\bar{K}(e_j\wedge n_k)\ .$$

If $\bar{M} = S^n$, then from (ii) of Proposition 3.6 we have

$$\mathrm{Tr}\, Q_\xi = pq/3 - (4/3)pq = -pq\ .$$

Hence for any p-current $\mathscr{S} \in \mathscr{R}_p(S^n)$ we have

$$\mathrm{Tr}\, Q_{\mathscr{S}} = -pq\boldsymbol{M}(\mathscr{S})$$

to obtain (1). Suppose $\bar{M} = P^n(\boldsymbol{F})$ with $\boldsymbol{F} = \boldsymbol{C}$, $\boldsymbol{H}$ or $\boldsymbol{Cay}$. From (ii) of Proposition 3.6 we have $c = \lambda^2$. By this together with the fact that the sectional curvature $\bar{K}$ of $P^n(\boldsymbol{F})$ is 1/4-pinched, we obtain

$$(3.14)\qquad \mathrm{Tr}\, Q_\xi = pqc/3 - (4/3)\sum_{j=1}^{p}\sum_{k=1}^{q}K(e_j\wedge n_k) \leqq 0\ .$$

Hence for any p-current $\mathscr{S} \in \mathscr{R}_p(P^n(\boldsymbol{F}))$ we have

$$(3.15)\qquad \mathrm{Tr}\, Q_{\mathscr{S}} = \int \mathrm{Tr}\, Q_{\vec{\mathscr{S}}_x} d\|\mathscr{S}\|$$

$$= \int \Big(pqc/3 - (4/3) \sum_{j=1}^{p} \sum_{k=1}^{q} \bar{K}(e_j \wedge n_k) \Big) d\|\mathscr{S}\| ,$$

where $\{e_1, \cdots, e_p, n_1, \cdots, n_q\}$ is an orthonormal basis of $T_x(P^n(\boldsymbol{F}))$ with $\vec{\mathscr{S}}_x = e_1 \wedge \cdots \wedge e_p$ at $\|\mathscr{S}\|$-almost all $x \in P^n(\boldsymbol{F})$. Suppose now that $\mathscr{S}$ is stable. Then by virtue of (3.14) we have $\operatorname{Tr} Q_{\mathscr{S}} = 0$. Again by (3.14) we have $\operatorname{Tr} Q_{\vec{\mathscr{S}}_x} = 0$ for $\|\mathscr{S}\|$-almost all $x \in P^n(\boldsymbol{F})$. Since $\bar{K}$ is 1/4-pinched, we get $K(e_j \wedge n_k) = c/4$ for $1 \leqq j \leqq p$ and $1 \leqq k \leqq q$. Hence any stable p-current $\mathscr{S} \in \mathscr{R}_p(P^n(\boldsymbol{F}))$ satisfies

$$K(X \wedge \zeta) = c/4 \tag{3.16}$$

for any two unit vectors $X \in \vec{\mathscr{S}}_x$, $\zeta \in \vec{\mathscr{S}}_x^{\perp}$ at $\|\mathscr{S}\|$-almost all $x \in P^n(\boldsymbol{F})$. Here $\vec{\mathscr{S}}_x^{\perp}$ is the orthogonal complement of $\vec{\mathscr{S}}_x$ in $T_x(P^n(\boldsymbol{F}))$.

Case 1. Suppose $\boldsymbol{F} = \boldsymbol{C}$ or $\boldsymbol{H}$. Substituting (1.1) or (1.2) into the left-hand side of (3.16), we obtain

$$\langle JX, \zeta \rangle = 0 \qquad \text{or} \qquad \sum_{i=1}^{3} \langle J_i X, \zeta \rangle^2 = 0$$

for any $X \in \vec{\mathscr{S}}_x$ and $\zeta \in \vec{\mathscr{S}}_x^{\perp}$, at $\|\mathscr{S}\|$-almost all $x \in P^n(\boldsymbol{F})$. Hence, for $\|\mathscr{S}\|$-almost all $x \in P^n(\boldsymbol{F})$, the tangent space $\vec{\mathscr{S}}_x$ of $\mathscr{S}$ is invariant under J or J_i $(i = 1, 2, 3)$. Therefore if $\boldsymbol{F} = \boldsymbol{C}$, then p is even and $\mathscr{S}$ is a complex current. If $\boldsymbol{F} = \boldsymbol{H}$, then p is a multiple of 4 and $\mathscr{S}$ is a quaternionic current.

Case 2. Suppose $\boldsymbol{F} = \boldsymbol{Cay}$. We fix any point $x \in P^2(\boldsymbol{Cay})$ such that $\vec{\mathscr{S}}_x$ satisfies (3.16). We have to show that $\vec{\mathscr{S}}_x$ is of Cayley type. We can identify $T_x(P^2(\boldsymbol{Cay}))$ with $\boldsymbol{Cay} \oplus \boldsymbol{Cay}$ as in 1.2 of Section 1. Let $\{e_1, \cdots, e_p, n_1, \cdots, n_q\}$ be an orthonormal basis of $T_x(P^2(\boldsymbol{Cay}))$ with $\vec{\mathscr{S}}_x = e_1 \wedge \cdots \wedge e_p$. Transforming it by an isometry of $P_2(\boldsymbol{Cay})$, if necessary, we may assume that $e_1 = (1, 0)$. Put $n_k = (c_k, d_k)$ $(1 \leqq k \leqq q)$, where $c_k, d_k \in \boldsymbol{Cay}$. Note that $\langle e_1, n_k \rangle = \langle 1, c_k \rangle = 0$ and $\|c_k\|^2 + \|d_k\|^2 = 1$. Substituting (1.3) into the left-hand side of (3.16), we have

$$\bar{K}(e_1 \wedge n_k) = c(\|c_k\|^2 + \|d_k\|^2/4) = (c/4)(3\|c_k\|^2 + 1) = c/4 .$$

Hence we have $c_k = 0$. Thus $n_k = (0, d_k)$ $(1 \leqq k \leqq q)$, that is, $\{n_1, \cdots, n_q\} \subset \{0\} \oplus \boldsymbol{Cay}$. Put $e_i = (a_i, b_i)$ for $2 \leqq i \leqq p$, where $a_i, b_i \in \boldsymbol{Cay}$. Note that $\langle e_i, e_i \rangle = \|a_i\|^2 + \|b_i\|^2 = 1$, $\langle n_k, n_k \rangle = \|d_k\|^2 = 1$ and $\langle e_i, n_k \rangle = \langle b_i, d_k \rangle = 0$, Again using (1.3), we have

$$\begin{aligned} \bar{K}(e_i \wedge n_k) &= c(\|b_i\|^2\|d_k\|^2 + \|a_i\|^2/4) = c(\|b_i\|^2 + \|a_i\|^2/4) \\ &= (c/4)(3\|b_i\|^2 + 1) = c/4 \end{aligned}$$

for $2 \leqq i \leqq p$. Hence we have $b_i = 0$. Thus $e_i = (a_i, 0)$ for $2 \leqq i \leqq p$, that is, $\{e_1, \cdots, e_p\} \subset \boldsymbol{Cay} \oplus \{0\}$. Hence we obtain $p=8$ and $\vec{\mathscr{S}}_x = \boldsymbol{Cay} \oplus \{0\}$. Therefore $\vec{\mathscr{S}}_x$ is of Cayley type. Thus $\mathscr{S}$ is a Cayley current. q.e.d.

Applying the same method to a submanifold, with $d\|\mathscr{S}\|$ replaced by the Riemannian measure, we obtain Corollary 3.4.

Combining Corollary 3.4 with the following two propositions, we get Theorems 3.1 and 3.2.

PROPOSITION 3.7 (Alekseevskii [1], Gray [5]). *Any quaternionic submanifold of a quaternionic Kaehler manifold is totally geodesic.*

PROPOSITION 3.8. *Any Cayley submanifold of the Cayley projective plane $P^2(\boldsymbol{Cay})$ is totally geodesic.*

PROOF. Let M be a Cayley submanifold of $P^2(\boldsymbol{Cay})$. We denote by $\bar{\nabla}$ and ∇ the Riemannian connections of $P^2(\boldsymbol{Cay})$ and M, respectively. Since M is curvature invariant, that is, $\bar{R}(X, Y)Z \in T_x(M)$ for any $X, Y, Z \in T_x(M)$ and any $x \in M$, and since $P^2(\boldsymbol{Cay})$ is locally symmetric, the curvature tensor $\bar{R}$ of $P^2(\boldsymbol{Cay})$ and the second fundamental form B of M satisfy the equation

$$B(W, \bar{R}(X, Y)Z) = \bar{R}(B(W, X), Y)Z + \bar{R}(X, B(W, Y))Z + \bar{R}(X, Y)B(W, Z) \tag{3.17}$$

for $X, Y, Z, W \in T_x(M)$. Indeed, for any local vector fields X, Y, Z, W on M around x with $(\nabla X)_x = (\nabla Y)_x = (\nabla Z)_x = 0$, we have

$$\begin{aligned} &B(W, \bar{R}(X, Y)Z) \\ &\quad = \{\bar{\nabla}_W(\bar{R}(X, Y)Z)\}^N \\ &\quad = \{(\bar{\nabla}_W\bar{R})(X, Y)Z + \bar{R}(\bar{\nabla}_W X, Y)Z + \bar{R}(X, \bar{\nabla}_W Y)Z + \bar{R}(X, Y)\bar{\nabla}_W Z\}^N \\ &\quad = \{\bar{R}(B(W, X), Y)Z + \bar{R}(X, B(W, Y))Z + \bar{R}(X, Y)B(W, Z)\}^N . \end{aligned}$$

By the curvature invariance of M we get (3.17).

Fix any point $x \in M$. We identify $T_x(\boldsymbol{Cay})$ with $\boldsymbol{Cay} \oplus \boldsymbol{Cay}$ as before. We may assume that $T_x(M) = \boldsymbol{Cay} \oplus \{0\}$. Then by (1.3) we have $\bar{R}(X, Y) = c(X \wedge Y)$ for $X, Y \in T_x(M)$, where c is the maximum of the sectional curvatures of $P^2(\boldsymbol{Cay})$. Now we shall show that $B(X, X) = -B(Y, Y)$ for any two orthonormal vectors $X, Y \in T_x(M)$. If we put $Z = Y$, $W = X$ in (3.17), then we have

$$cB(X, X) = \bar{R}(B(X, X), Y)Y + \bar{R}(X, B(X, Y))Y + \bar{R}(X, Y)B(X, Y) . \tag{3.18}$$

Interchanging X and Y in (3.18), we have

$$cB(Y, Y) = \bar{R}(B(Y, Y), X)X + \bar{R}(Y, B(Y, X))X + \bar{R}(Y, X)B(Y, X) . \tag{3.19}$$

Adding (3.18) to (3.19), we have

$$c(B(X, X) + B(Y, Y)) = \bar{R}(B(X, X), Y)Y + \bar{R}(B(Y, Y), X)X + \bar{R}(X, B(X, Y))Y + \bar{R}(Y, B(X, Y))X . \tag{3.20}$$

We put $X = (x, 0)$ and $Y = (y, 0)$ for some $x, y \in \boldsymbol{Cay}$. Then $B(X, Y) = (0, \beta(x, y))$, where β is a symmetric $\boldsymbol{R}$-bilinear mapping of $\boldsymbol{Cay} \times \boldsymbol{Cay}$ into $\boldsymbol{Cay}$. Then substituting (1.3) into (3.20) we get

$$\beta(x, x) + \beta(y, y) = (\beta(x, x) + \beta(y, y))/4 - \{y^*(x\beta(x, y) + x^*(y\beta(x, y))\}/4 .$$

Hence, if we define the associator of the Cayley algebra $\boldsymbol{Cay}$ by $(a, b, c) = a(bc) - (ab)c$ $(a, b, c \in \boldsymbol{Cay})$, we have

$$3(\beta(x, x) + \beta(y, y)) = -(x^* + y^*, x + y, \beta(x, y)) - 2\langle x, y\rangle\beta(x, y) . \tag{3.21}$$

Since $(a^*, a, b) = 0$ $(a, b \in \boldsymbol{Cay})$ and $\langle X, Y\rangle = \langle x, y\rangle = 0$, the right-hand side of (3.21) vanishes. Thus we obtain $B(X, X) = -B(Y, Y)$. Since $\dim T_x(M) \geqq 3$, this implies that $B = 0$. Hence M is totally geodesic. q.e.d.

Combining Theorems 3.1 and 3.2 with (3) and (4) of Proposition 1.1, we get Theorems D and E.

4. Remarks on the nonexistence of stable currents. In this section we show two theorems on the nonexistence of stable currents. Now we assume that $\bar{M}$ is an n-dimensional compact Riemannian manifold isometrically immersed in the $(n+1)$-dimensional Euclidean space $\boldsymbol{E}^{n+1}$ with the inner product $\langle\ ,\ \rangle$. Let $\bar{R}$ and $\bar{A}$ be the curvature tensor and the shape operator of $\bar{M}$, respectively.

Let δ be a constant with $0<\delta \leqq 1$, and suppose that at each x of $\bar{M}$ every principal curvature κ_i of $\bar{M}$ with respect to a suitable unit normal vector field ζ satisfies $\sqrt{\delta} \leqq \kappa_i \leqq 1$, $i = 1, \cdots, n$. The assumption implies that $\bar{M}$ has the sectional curvature satisfying $\delta \leqq K_\sigma \leqq 1$ for any tangent 2-plane σ. The above assumption also implies that $\bar{M}$ is orientable. Therefore we can choose a global field ζ of unit normals on $\bar{M}$ which satisfies the above condition, and then we can write $\bar{A}_\zeta = \bar{A}$. We use the same notation as in Proposition 3.5. We show the following.

LEMMA 4.1. *For any simple unit p-vector $\xi \in \wedge^p T_x(\bar{M})$, $x \in \bar{M}$, we have*

$$\operatorname{Tr} Q_\xi = -pq(2\delta - 1) , \quad \textit{where} \quad p + q = n .$$

PROOF. By virtue of Proposition 3.5 we have

$$\operatorname{Tr} Q_\xi = 2 \sum_{j=1}^{p} \sum_{k=1}^{q} \langle \bar{A}(e_j), n_k \rangle^2 - \sum_{j=1}^{p} \sum_{k=1}^{q} \langle \bar{A}(e_j), e_j \rangle \langle \bar{A}(n_k), n_k \rangle .$$

Applying the Gauss equation to the first term of the above equation, we have

$$\operatorname{Tr} Q_\xi = -2 \sum_{j=1}^{p} \sum_{k=1}^{q} \langle \bar{R}(e_j, n_k) n_k, e_j \rangle + \sum_{j=1}^{p} \sum_{k=1}^{q} \langle \bar{A}(e_j), e_j \rangle \langle \bar{A}(n_k), n_k \rangle .$$

By the assumption $\sqrt{\delta} \leqq \kappa_i \leqq 1$ $(i = 1, \cdots, n)$, we get $\sqrt{\delta} \leqq \langle \bar{A}(e_j), e_j \rangle$, $\langle \bar{A}(n_k), n_k \rangle \leqq 1$ and $\delta \leqq \langle \bar{R}(e_j, n_k) n_k, e_j \rangle \leqq 1$, for $j = 1, \cdots, p$ and $k = 1, \cdots, q$. Thus we have

$$\operatorname{Tr} Q_\xi \leqq -2pq\delta + pq = -pq(2\delta - 1) . \qquad \text{q.e.d.}$$

Combining Proposition 1.2 and Lemma 4.1 we get the following theorem, from which Theorem F follows immediately,

THEOREM 4.2. *Let $\bar{M}$ be an n-dimensional compact Riemannian manifold satisfying the conditions above. Then for any $\mathscr{S} \in \mathscr{R}_p(\bar{M})$*

$$\operatorname{Tr} Q_{\mathscr{S}} \leqq -pq(2\delta - 1) \boldsymbol{M}(\mathscr{S}) ,$$

where $q = n - p$.

Our next interesting problem is to classify stable minimal submanifolds and stable currents in compact symmetric spaces of rank greater than one. We here show a theorem on the nonexistence of stable currents on some compact rank two symmetric spaces.

THEOREM 4.3. *Let $\bar{M}$ be an n-dimensional simply connected compact rank two symmetric space of type A_2, that is, one of the following symmetric spaces: $SU(3)/SO(3)$ $(n = 5)$, $SU(3)$ $(n = 8)$, $SU(6)/Sp(3)$ $(n = 14)$ and E_6/F_4 $(n = 26)$. Let $n = p + q$ where p and q are positive integers. If $p < n/3$ or $q < n/3$, then there exist no rectifiable stable p-currents on $\bar{M}$.*

PROOF. Let $\phi_1 \colon \bar{M} \to S^{m(1)}$ be the first standard minimal isometric immersion of $\bar{M}$ into a unit sphere $S^{m(1)}$ where $m(1) + 1$ denotes the multiplicity of the first eigenvalue for the Laplacian of $\bar{M}$. We denote by ι the inclusion of $S^{m(1)}$ into $\boldsymbol{E}^{m(1)+1}$. Then it is not difficult to verify that $\iota \circ \phi_1$ is an isotropic immersion. The square λ^2 of its isotropic constant is equal to 3/2 and the maximum c of the sectional curvatures of $\bar{M}$ is equal to 3/2. Applying Proposition 3.5 to $\iota \circ \phi_1$, straightforward computations show that, for any unit simple p-vector $\xi \in \wedge^p T_x(\bar{M})$,

$$\operatorname{Tr} Q_\xi \leqq \operatorname{Min}\{-p(n - 3p)/2, \ -q(n - 3q)/2\} . \qquad \text{q.e.d.}$$

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