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A CHARACTERIZATION OF THE STABLE INVARIANT INTEGRAL

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Contents

1. Introduction. Invariant integrals, stable or not, occupy a central position in harmonic analysis on a reductive Lie group G. For instance, they play a crucial role in Harish-Chandra's derivation of the Plancherel formula. They also figure prominently in the theory centering on the Selberg trace formula. Therefore it is only natural to try to characterize them. One important contribution in this direction is the work of Shelstad [9], who has obtained a "pointwise" description but only within the context of the Schwartz space $\mathscr{C}(G)$. For the applications, it is also necessary to consider other function spaces, e.g., $C_c^\infty(G)$. This, in fact, is one of our objectives. The main result is, however, rather different from Shelstad's in that the characterization is essentially "transform-theoretic" in nature (cf. [10]), the point being that the work of Herb [5-(b)] already gives explicit inversion formulae for the invariant integrals so, in order to study their transforms, a Paley-Wiener type theorem is required. And for this, the recent work of Clozel and Delorme [3-(a)] turns out to be exactly what is needed.

Regarding the organization, §§2–4 set up the preliminaries. In §5, we review the results of Herb and in §6 those of Clozel and Delorme. The characterization itself is the subject of §7. We close in §8 with a series of miscellaneous remarks that point the way to a number of variants on our main theme which can all be treated by the methods introduced here.

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For the sake of simplicity, we shall concentrate throughout on the stable case, thereby eliminating certain technical difficulties. In another paper, I shall give a "function-theoretic" characterization. To some extent, though, it depends on what is to be found in the present note.

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2. Assumptions and Conventions. Let G be a connected reductive Lie group with compact center; let K be a maximal compact subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}_c$ its complexification and G_c a complex analytic group with Lie algebra $\mathfrak{g}_c$ for which the complex analytic subgroup of G_c corresponding to $[\mathfrak{g}_c, \mathfrak{g}_c]$ is simply connected. We shall assume that

(1) $G \subset G_c$;
(2) G is acceptable;
(3) $\mathrm{rank}(G) = \mathrm{rank}(K)$.

Let $A = A_I \cdot A_R$ be a θ-stable Cartan subgroup of G, $P = M \cdot A_R \cdot N$ the associated cuspidal parabolic subgroup of G—then, strictly speaking, the preceding assumptions are not hereditary since, in particular, M need not be connected. However, its identity component M^0 is a connected reductive Lie group with compact center. Moreover, if $\mathfrak{m}$ is the Lie algebra of M, $\mathfrak{m}_c$ its complexification and M_c the complex analytic subgroup of G_c with Lie algebra $\mathfrak{m}_c$, then the complex analytic subgroup of M_c corresponding to $[\mathfrak{m}_c, \mathfrak{m}_c]$ is simply connected (cf. [4-(a), p. 482]) and, of course, $M^0 \subset M_c$. Trivially, the acceptability of G forces the acceptability of M^0. Finally, the identity component A_I^0 of A_I is a compact Cartan subgroup of M^0, thus the last assumption is also met. Needless to say, M itself is a reductive Lie group of Harish-Chandra class.

Let

$$Z(A) = K \cap \exp(\sqrt{-1}\mathfrak{a}_R) ,$$

$\mathfrak{a}_R$ the Lie algebra of A_R—then $Z(A)$ is a finite 2-group, central in M, and, as is well-known,

$$A_I = Z(A) \cdot A_I^0 .$$

Put

$$M^\dagger = Z(A) \cdot M^0 .$$

Then $M^\dagger$ is a normal subgroup of M consisting of the $m \in M$ such that $\mathrm{Int}(m)$ is an inner automorphism of M^0. Viewing $Z(A)$ as a vector space over $\mathbf{Z}_2$, write

$$Z(A) = Z(A) \cap M^0 \times Z_M .$$

Naturally, Z_M need not be unique but now $M^\dagger$ can be displayed as a direct product

$$M^\dagger = Z_M \times M^0 ,$$

as can A_I, namely

$$A_I = Z_M \times A_I^0 .$$

In the sequel, we shall encounter a variety of invariant measures. They are to be normalized according to the conventions of Harish-Chandra [4-(d), pp. 114–115]. We remind the reader only that G and M carry the standard Haar measure, while $\mathscr{F}$ (the real dual of $\mathfrak{a}_R$) carries the Euclidean measure reciprocal to the Haar measure on A_R derived from exponentiation of normalized Lebesgue measure on $\mathfrak{a}_R$ relative to the Euclidean structure associated with the Killing form (so Fourier inversion holds with no constant factors).

3. Stabilization of the Discrete Series. It will be simplest to deal first with G and then with M.

Fix a θ-stable compact Cartan subgroup T of G. Let $W = W(G, T)$ be the quotient of the normalizer of T in K by T itself—then W is a subgroup of $W_{\mathbf{C}} = W_{\mathbf{C}}(G, T)$, the full Weyl group of the pair (G, T). In the case at hand, $W_{\mathbf{C}}$ operates on T. Let $\hat{T}$ be the unitary character group of T—then $\hat{T}$ can be canonically identified with a lattice $\mathscr{L}_T$ in the imaginary dual of the Lie algebra $\mathfrak{t}$ of T. Let $\mathscr{L}_T'$ be the set of regular elements of $\mathscr{L}_T$. If $\mathfrak{Z}$ is, as always, the center of $\mathfrak{G}$, then by a regular integral character of $\mathfrak{Z}$ we understand any $\chi: \mathfrak{Z} \rightarrow \mathbf{C}$ of the form $\chi = \chi_\lambda$ ($\lambda \in \mathscr{L}_T'$), the set of such being parametrized by

$$W_{\mathbf{C}} \backslash \mathscr{L}_T' .$$

Let $\hat{G}_d$ be the discrete series for G—then, according to Harish-Chandra, there is a 1-to-1 correspondence

$$\hat{G}_d \leftrightarrow W \backslash \mathscr{L}_T' .$$

This said, given $\omega \in \hat{G}_d$, let Θ_ω be its character, χ_ω its infinitesimal character. For each regular integral χ, set

$$\hat{G}_d(\chi) = \{\omega : \chi_\omega = \chi\} .$$

Then (cf. [4-(c), p. 94]).

$$\#(\hat{G}_d(\chi)) = [W_{\mathbf{C}}]/[W] .$$

The break up

$$\hat{G}_d = \coprod_{\chi} \hat{G}_d(\chi)$$

can be regarded as the decomposition of $\hat{G}_d$ into "stable" subsets.

Thus, let

$$\Omega = \{\omega : \omega \in \hat{G}_d(\chi)\}$$

and then put

$$\Theta_\Omega = \sum_{\omega \in \Omega} \Theta_\omega \ .$$

Obviously, Θ_Ω is a central eigendistribution on G.

In general, a central $\mathfrak{Z}$-finite distribution Θ on G is said to be stable if $\Theta | G'$, qua an analytic function, is invariant in the following sense (cf. [9, p. 38]): For every $x \in G'$,

$$\Theta(wx) = \Theta(x) \quad (w \in W_I(G, A_x)) \ .$$

Here, A_x is the Cartan subgroup of G containing x, $W_I(G, A_x)$ the subgroup of the full Weyl group of the pair (G, A_x) generated by the reflections in the imaginary roots. By way of notation, in what follows we shall write

$$(f, \Theta) = \int_G f(x)\overline{\Theta(x)} d_G(x) \quad (f \in C_c^\infty(G)) \ .$$

LEMMA 3.1. *Θ_Ω is stable.*

PROOF. Fix an $\omega \in \Omega$ and attach to it a parameter $\lambda \in \mathscr{L}_T'$ so that

$$\Theta_\omega = (-1)^d \varepsilon(\lambda) \Theta_\lambda \quad (d = 2^{-1} \dim(G/K)) \ .$$

Then

$$\Theta_\Omega = (-1)^d \varepsilon(\lambda) \cdot \sum_{W \backslash W_{\mathbf{C}}} \varepsilon(w_i) \Theta_{w_i \lambda}$$

and the stability of

$$\Theta_\lambda^* = \sum_{W \backslash W_{\mathbf{C}}} \varepsilon(w_i) \Theta_{w_i \lambda}$$

is a lemma of Harish-Chandra [4-(b), p. 307].

The stabilized discrete series for G will be denoted by

$$\mathrm{ST} - \hat{G}_d \ ,$$

its elements being the Ω, its characters the Θ_Ω.

Put

$$\Phi_\omega = \Delta_T \cdot \Theta_\omega \ .$$

Then Φ_ω is a C^∞ function on T. We recall that

(1) Φ_ω is W-skew, i.e.,

$$\Phi_\omega^w = \varepsilon(w)\Phi_\omega \quad (w \in W) ;$$

(2) $(\Phi_\omega, \Phi_\omega) = [W]$;

(3) $\omega' \neq \omega'' \Rightarrow (\Phi_{\omega'}, \Phi_{\omega''}) = 0$.

To stabilize these considerations, take an Ω and choose any $\omega \in \Omega$. We then let

$$\Phi_\Omega = \sum_{W \backslash W_{\mathbf{C}}} \varepsilon(w_i)\Phi_\omega^{w_i} .$$

It is clear that Φ_Ω is well-defined in the sense that it is independent of the choice of ω. Again, Φ_Ω is a C^∞ function on T and

(1) Φ_Ω is $W_{\mathbf{C}}$-skew, i.e.,

$$\Phi_\Omega^w = \varepsilon(w)\Phi_\Omega \quad (w \in W_{\mathbf{C}}) ;$$

(2) $(\Phi_\Omega, \Phi_\Omega) = [W_{\mathbf{C}}]$;

(3) $\Omega' \neq \Omega'' \Rightarrow (\Phi_{\Omega'}, \Phi_{\Omega''}) = 0$.

LEMMA 3.2. *Suppose that $\phi \in C^\infty(T)$ is $W_{\mathbf{C}}$-skew—then ϕ admits an absolutely convergent expansion*

$$\phi(t) = \sum_\Omega \Phi_\Omega(t) \cdot \hat{\phi}(\Omega) ,$$

where, by definition,

$$\hat{\phi}(\Omega) = \frac{1}{[W_{\mathbf{C}}]} \cdot \int_T \phi(t)\overline{\Phi_\Omega(t)}d_T(t) .$$

[We omit the elementary verification.]

The preceding remarks also apply to M provided that certain modifications are made but it is easiest to work in stages: M^0 to $M^\dagger$ to M.

First of all, everything that has been said above applies verbatim to M^0 (since M^0 and G satisfy the same general assumptions). Passing on to $M^\dagger$, from the fact that

$$M^\dagger = Z_M \times M^0 ,$$

it follows that the elements $\sigma^\dagger \in \hat{M}_d^\dagger$ are tensor products

$$\sigma^\dagger = \zeta \otimes \sigma^0 \quad \begin{cases} \zeta \in \hat{Z}_M \\ \sigma^0 \in \hat{M}_d^0 , \end{cases}$$

where the characters are connected by the relation

$$\Theta_{\sigma^\dagger}(zm^0) = \zeta(z)\Theta_{\sigma^0}(m^0) .$$

As for M itself, the elements $\sigma \in \hat{M}_d$ are of the form

$$\sigma = \mathrm{Ind}_{M^\dagger}^M(\sigma^\dagger) \quad (\sigma^\dagger \in \hat{M}_d^\dagger) .$$

The character Θ_σ of σ is supported in $M^\dagger$, being given there, after Frobenius, by the prescription

$$\Theta_\sigma(zm^0) = \sum_\mu \Theta_{\sigma^\dagger}(m_\mu z m^0 m_\mu^{-1})$$

or still

$$\Theta_\sigma(zm^0) = \zeta(z) \cdot \sum_\mu \Theta_{\sigma^0}(m_\mu m^0 m_\mu^{-1})$$

if

$$M = \coprod_\mu m_\mu M^\dagger .$$

In addition (cf. [7, p. 55]), the restriction $\sigma | M^\dagger$ is a direct sum $\sum_\mu \sigma_\mu^\dagger$ of mutually inequivalent $\sigma_\mu^\dagger$ and for every μ,

$$\sigma = \mathrm{Ind}_{M^\dagger}^{M}(\sigma_\mu^\dagger) .$$

Therefore the natural map $\hat{M}_d^\dagger \to \hat{M}_d$ is a surjection of order $[M: M^\dagger]$.

Let

$$\begin{cases} W^0 = W(M^0, A_I^0) \\ W = W(M, A_I) \end{cases}$$

be the quotient of the normalizer of

$$\begin{cases} A_I^0 \text{ in } K \cap M^0 \text{ by } A_I^0 \text{ itself} \\ A_I \text{ in } K \cap M \text{ by } A_I \text{ itself} . \end{cases}$$

Then W^0 and W are subgroups of $W_{\mathbf{C}} = W_{\mathbf{C}}(M, A_I)$, the full Weyl group of the pair (M, A_I). It is not difficult to see that W^0 is a normal subgroup of W and actually (cf. [13, p. 46]),

$$W/W^0 \sim M/M^\dagger ,$$

implying that the m_μ can always be chosen to represent distinct elements w_μ in W.

[Note: There is nothing to be gained by introducing $W^\dagger$: For $W^\dagger \sim W^0$....]

The infinitesimal character χ of σ^0, $\sigma^\dagger$, and σ is one and the same. The cardinality of $\hat{M}_d^0(\chi)$ is $[W_{\mathbf{C}}]/[W^0]$. However, a given σ^0 determines $\#(\hat{Z}_M)$ possible $\sigma^\dagger$, hence the cardinality of $\hat{M}_d^\dagger(\chi)$ is $\#(\hat{Z}_M) \cdot [W_{\mathbf{C}}]/[W^0]$. Thus the cardinality of $\hat{M}_d(\chi)$ is

$$\frac{\#(\hat{Z}_M) \cdot [W_{\mathbf{C}}]/[W^0]}{[W]/[W^0]} = \#(\hat{Z}_M) \cdot [W_{\mathbf{C}}]/[W] .$$

So, while

$$\hat{M}_d^0 = \coprod_{\chi} \hat{M}_d^0(\chi)$$

is already the decomposition of $\hat{M}_d^0$ into stable subsets Σ^0, to effect the splitting for $\hat{M}_d^\dagger$ and $\hat{M}_d$, it is necessary to reflect the presence of $\hat{Z}_M$, say by writing

$$\hat{M}_d^\dagger(\chi) = \coprod_{\zeta} \hat{M}_d^\dagger(\zeta, \chi) , \qquad \hat{M}_d(\chi) = \coprod_{\zeta} \hat{M}_d(\zeta, \chi) ,$$

providing decompositions

$$\hat{M}_d^\dagger = \coprod_{\zeta,\chi} \hat{M}_d^\dagger(\zeta, \chi) , \qquad \hat{M}_d = \coprod_{\zeta,\chi} \hat{M}_d(\zeta, \chi) ,$$

into stable subsets $\Sigma^\dagger$, Σ, respectively. The symbols

$$\begin{cases} \mathrm{ST} - \hat{M}_I^0 \\ \Theta_{\Sigma^0} \end{cases} , \qquad \begin{cases} \mathrm{ST} - \hat{M}_d^\dagger \\ \Theta_{\Sigma^\dagger} \end{cases} , \qquad \begin{cases} \mathrm{ST} - \hat{M}_d \\ \Theta_{\Sigma} \end{cases} ,$$

are then to be given the evident meanings.

Let

$$\pi_P(\sigma, \nu) = \mathrm{Ind}_P^G(\sigma, \nu) \quad (\nu \in \mathscr{F})$$

be a unitary principal series representation, $\Theta_{\sigma,\nu}$ its character. Put

$$\Theta_{\Sigma,\nu} = \sum_{\sigma \in \Sigma} \Theta_{\sigma,\nu} .$$

LEMMA 3.3. *$\Theta_{\Sigma,\nu}$ is stable.*

PROOF. Owing to the formulae of Hirai [6-(a), p. 358] and Wolf [13, p. 73] for $\Theta_{\Sigma,\nu}$ on the Cartan subgroups of $L = M \cdot A_R$, it is enough to check the stability of Θ_Σ (vis-à-vis M), itself readily reducible to the stability of Θ_{Σ^0} (vis-à-vis M^0), placing us back in a situation already covered by Lemma 3.1.

We agree that both A_I and A_I^0 come equipped with the Haar measure assigning to each total volume one. To ensure compatibility, each point in Z_M is to have mass $1/\#(Z_M)$.

Attached to each σ^0 is the W^0-skew C^∞ function Φ_{σ^0} on A_I^0 with

$$(\Phi_{\sigma^0}, \Phi_{\sigma^0}) = [W^0] .$$

If $\sigma^\dagger = \zeta \otimes \sigma^0$, then seemingly it is natural to let

$$\Phi_{\sigma^\dagger}(za_I^0) = \zeta(z)\Phi_{\sigma^0}(a_I^0) .$$

Doing this though will lead to problems later on with the invariant integral, thus we might just as well remedy the situation now. Implicit in the definition of acceptability is the quasicharacter ξ_ρ of A. The restriction of ξ_ρ to Z_M is an element of $\hat{Z}_M$. That being the case, we twist the data and put instead

$$\Phi_{\sigma^\dagger}(za_I^0) = \xi_\rho(z)\zeta(z)\Phi_{\sigma^0}(a_I^0) .$$

If

$$\sigma = \mathrm{Ind}_{M^\dagger}^M(\sigma^\dagger) ,$$

then

$$\Phi_\sigma(za_I^0) = \sum_\mu \varepsilon(w_\mu)\Phi_{\sigma^\dagger}^{w_\mu}(za_I^0)$$

or still

$$\Phi_\sigma(za_I^0) = \xi_\rho(z)\zeta(z) \cdot \sum_\mu \varepsilon(w_\mu)\Phi_{\sigma^0}^{w_\mu}(a_I^0) .$$

Φ_σ is a W-skew C^∞ function on A_I such that

$$\begin{cases}(\Phi_\sigma, \Phi_\sigma) = [W] \\ \sigma' \neq \sigma'' \Rightarrow (\Phi_{\sigma'}, \Phi_{\sigma''}) = 0 .\end{cases}$$

Turning to the stabilization, we need only work with M: Simply take a Σ, choose any $\sigma \in \Sigma$, and let

$$\Phi_\Sigma = \sum_{W \backslash W_{\mathbf{C}}} \varepsilon(w_i)\Phi_\sigma^{w_i} .$$

Then Φ_Σ has the necessary expected properties.

In the present context, a generalization of Lemma 3.2 is valid. To formulate it, note that $W_{\mathbf{C}}$ operates on all of A, the action on A_R being, of course, trivial.

LEMMA 3.4. *Suppose that $\phi \in C_c^\infty(A)$ is $W_{\mathbf{C}}$-skew—then ϕ admits an absolutely convergent expansion*

$$\phi(a_I a_R) = \sum_\Sigma \Phi_\Sigma(a_I) \cdot \int_{\mathscr{F}} \hat{\phi}(\Sigma, \nu) a_R^{\sqrt{-1}\nu} d\nu$$

where, by definition,

$$\hat{\phi}(\Sigma, \nu) = \frac{1}{[W_{\mathbf{C}}]} \cdot \int_{A_I}\int_{A_R} \phi(a_I a_R)\overline{\Phi_\Sigma(a_I) a_R{}^{\sqrt{-1}\nu}} d_{A_I}(a_I) d_{A_R}(a_R) .$$

[We omit the elementary verification.]

4. **The Invariant Integral and its Stabilization.** Fix a θ-stable Cartan subgroup $A = A_I \cdot A_R$ of G—then, for any $f \in C_c^\infty(G)$, the invariant integral F_f^A of f relative to A is defined by

$$F_f^A(a) = \varepsilon_R(a)\Delta_A(a) \cdot \int_{G/A} f(xax^{-1}) d_{G/A}(x) \quad (a \in A') ,$$

A' the set of regular elements in A. Here, we employ the usual notation (see, e.g., [12-(a)] and [12-(b)]), which can therefore remain unexplained.

It will be recalled that a priori, $F_f^A \in C^\infty(A')$, its support being compact in A but not necessarily in A'. In reality, it is possible to extend the domain of F_f^A. Thus let A_I' be the set of points in A_I that are regular in M—then $A_I' \cdot A_R$ is the set of points in A lying outside the kernel of every ξ_α (α imaginary). This said, one knows that F_f^A extends to an element of $C^\infty(A_I' \cdot A_R)$. But more is true (cf. [11, p. 373] or [12-(b), p. 248]). We may, in the obvious way, view F_f^A as an element of $C^\infty(A_I'; C_c^\infty(A_R))$. Suppose now that $\mathscr{B}^\infty(A_I'; C_c^\infty(A_R))$ is the vector space of all

$$u \in C^\infty(A_I'; C_c^\infty(A_R))$$

such that for every invariant differential operator D on A_I,

$$|u|_{\Lambda; D} = \sup_{A_I'} \Lambda(Du) < +\infty \ ,$$

Λ a seminorm on $C_c^\infty(A_R)$. When topologized by the $|?|_{\Lambda; D}$, $\mathscr{B}^\infty(A_I'; C_c^\infty(A_R))$ becomes a complete LCTVS. With this in mind, it can then be shown that

$$F_f^A \in \mathscr{B}^\infty(A_I'; C_c^\infty(A_R)) \ ,$$

the assignment

$$\begin{cases} f \rightarrow F_f^A \\ C_c^\infty(G) \rightarrow \mathscr{B}^\infty(A_I'; C_c^\infty(A_R)) \end{cases}$$

being continuous.

[Note: When $A = T$, the result is to be regarded as saying that $F_f^T \in \mathscr{B}^\infty(T')$ in the sense that for every invariant differential operator D on T,

$$\sup_{T'} |DF_f^T| < +\infty \ .$$

At the other extreme, when A is of Iwasawa type, there are no singularities and it is actually the case that $F_f^A \in C_c^\infty(A)$ (cf. [11, p. 400]).]

It is clear that A_I' is invariant under the action of $W_{\mathbf{C}}$. Therefore the same is true of $A_I' \cdot A_R$

LEMMA 4.1. *F_f^A is W-skew.*

In fact, since Δ_A is W-skew and

$$\int_{G/A} f(xax^{-1}) d_{G/A}(x)$$

remains unchanged with respect to the operations of W, everything reduces to showing that ε_R is W-invariant, which is easy to establish using definitions only (cf. [6-(c), p. 37]).

[Note: The proof shows that ε_R is even $W_{\mathbf{C}}$-invariant, a point that will be used below.]

Thanks to the lemma, it then makes sense to form

$$\mathscr{S}_f^A = \sum_{W\backslash W_{\mathbf{C}}} \varepsilon(w_i)(F_f^A)^{w_i} ,$$

a function in $\mathscr{B}^\infty(A'_I; C_c^\infty(A_R))$. We shall refer to $\mathscr{S}_f^A$ as the stabilized invariant integral of f relative to A. As such, $\mathscr{S}_f^A$ is obviously $W_{\mathbf{C}}$-skew.

The interpretation of the procedure is simply this. Given $a \in A'$, put

$$\mathscr{O}_a(f) = \int_{G/A} f(xax^{-1})d_{G/A}(x) .$$

Then $\mathscr{O}_a(f)$ is essentially the orbital integral of f relative to a ("essentially" because the centralizer G_a of a in G can very well be larger than A, although the index $[G_a: A]$ must be finite (cf. [12-(b), p. 228])). Accordingly,

$$\mathscr{S}_f^A(a) = \varepsilon_R(a)\Delta_A(a)\cdot(\sum_{W\backslash W_{\mathbf{C}}} \mathscr{O}_{w,a}(f)) .$$

5. On the Results of R. Herb. Let $\mathrm{Car}_\theta(G)$ be the set of θ-stable Cartan subgroups of G. If A and B are two elements of $\mathrm{Car}_\theta(G)$, then we write $A \succ B$ (or $B \prec A$) if $\mathfrak{w}(\mathfrak{a}_R|\mathfrak{b}_R) \neq \varnothing$ (cf. [4-(f), p. 151]). Let

$$\mathfrak{C} = \{A_1 = T, A_2, \cdots, A_r\}$$

be a complete set of representatives for

$$K\backslash\mathrm{Car}_\theta(G) .$$

[Note: Conventionally,

$$\mathfrak{w} = \mathfrak{w}(\mathfrak{a}_R|\mathfrak{a}_R) ,$$

the "little" Weyl group of (G, A). The order $\succ$ is plainly transitive but need not be linear. If $A \succ B$ and $B \succ A$, then A and B are K-conjugate (cf. [12-(a), p. 91]). Within $\mathfrak{C}$, it is therefore permissible to use the symbol $\geqq$, $\succ$ being understood in the strict sense, i.e.,

$$A_i \succ A_j \Rightarrow i \neq j .]$$

THEOREM 5.1. *Fix $A \in \mathfrak{C}$—then there exist slowly increasing, C^∞ functions*

$$\mathbf{H}_{A|A_i} \quad on \quad A'_I\cdot A_R\times \mathrm{ST} - \hat{M}_{id}\times\mathscr{F}_i$$

such that for all $f \in C_c^\infty(G)$,

$$\mathscr{S}_f^A(a) = \sum_{A_i \in \mathfrak{C}} \sum_{\Sigma_i} \int_{\mathscr{F}_i} (f, \Theta_{\Sigma_i, \nu_i}) \mathbf{H}_{A|A_i}(a: (\Sigma_i, \nu_i)) d\nu_i \, .$$

In substance, this theorem is due to Herb [5-(b)]. However, she does not explicitly formulate it in this manner so it will be necessary to make some comments on how one goes about making the transition.

It is in the nature of things that

$$\mathbf{H}_{A|A_i} = 0$$

unless $A \leqq A_i$. Moreover, the procedure itself immediately furnishes $\mathbf{H}_{A|A}$:

$$\mathbf{H}_{A|A}(a: (\Sigma, \nu)) = \Phi_\Sigma(a_I) a_R^{\sqrt{-1}\nu} \, .$$

Therefore, we can write

$$\mathscr{S}_f^A(a) = \sum_{\Sigma} \Phi_\Sigma(a_I) \cdot \int_{\mathscr{F}} (f, \Theta_{\Sigma,\nu}) a_R^{\sqrt{-1}\nu} d\nu$$
$$+ \sum_{A_i; A_i \succ A} \sum_{\Sigma_i} \int_{\mathscr{F}_i} (f, \Theta_{\Sigma_i, \nu_i}) \mathbf{H}_{A|A_i}(a: (\Sigma_i, \nu_i)) d\nu_i \, .$$

Herb also explicitly computes the $\mathbf{H}_{A|A_i}$ when $A_i \succ A$. For our purposes, it will not be necessary to recall these formulae. Suffice it to say that they are ultimately expressible in terms of "elementary" functions, $W_{\mathbf{C}}$-skew in a. This is because Herb's theory of two-structures reduces the computation of the $\mathbf{H}_{A|A_i}$ to root systems of type A_1 or B_2, where one ends up with sums of products of quotients of hyperbolic sine functions (cf. [5-(b), pp. 13–15] and [5-(c), pp. 245–246]). Consequently, the slow growth of the $\mathbf{H}_{A|A_i}$ and its derivatives can be established by direct calculation.

The most important case is that of $A = T$, so we shall first look at it. The key is to employ at all stages of the argument the stabilized invariant integral rather than just the invariant integral (even in rank 1 (cf. [8]) or rank 2 (cf. [2])). As always, one begins by expanding F_f^T on T' into a Fourier series:

$$F_f^T(t) = \sum_{\lambda \in \mathscr{L}_T} \xi_\lambda(t) \cdot (f, \Theta_\lambda) - \sum_{\lambda \in \mathscr{L}_T} \xi_\lambda(t) \cdot \sum_{A_i; A_i \succ T} \int_{G_{A_i}} f(x) \overline{\Theta_\lambda(x)} d_G(x) \, ,$$

where, for any $A \in \mathfrak{C}$,

$$G_A = \bigcup_{x \in G} x A' x^{-1} \, .$$

Form now

$$\mathscr{S}_f^T(t) = \sum_{W \setminus W_{\mathbf{C}}} \varepsilon(w_i) F_f^T(w_i t) = \sum_{\lambda \in \mathscr{L}_T} \xi_\lambda(t) \cdot (f, \Theta_\lambda^*)$$
$$- \sum_{\lambda \in \mathscr{L}_T} \xi_\lambda(t) \cdot \sum_{A_i; A_i \succ T} \int_{G_{A_i}} f(x) \overline{\Theta_\lambda^*(x)} d_G(x) \, ,$$

Θ_λ^* as in the proof of Lemma 3.1. Since $\Theta_\lambda^* = 0$ if $\lambda \notin \mathscr{L}'_T$, we may replace $\sum_{\lambda \in \mathscr{L}_T}$ by $\sum_{\lambda \in \mathscr{L}'_T}$. Then, on the basis of the definitions

$$\sum_{\lambda \in \mathscr{L}'_T} \xi_\lambda(t) \cdot (f, \Theta_\lambda) = \sum_{\Omega} \Phi_\Omega(t) \cdot (f, \Theta_\Omega) .$$

As for what is left, fix an $A \neq T$ in $\mathfrak{C}$ and consider

$$\int_{G_A} f(x)\overline{\Theta_\lambda^*(x)} d_G(x)$$

or still, by Weyl's integration formula,

$$[W(G, A)]^{-1} \cdot \int_A F_f^A(a)\overline{\varepsilon_R(a)\Delta_A(a)\Theta_\lambda^*(a)} d_A(a) .$$

It has been noted earlier that Θ_λ^* is stable. Accordingly, taking into account what has been said in §4, we can average the last expression over $W \backslash W_{\mathrm{C}}$ (per M) to get

$$\frac{1}{[W(G, A)] \cdot [W \backslash W_{\mathrm{C}}]} \cdot \int_A \mathscr{S}_f^A(a)\overline{\varepsilon_R(a)\Delta_A(a)\Theta_\lambda^*(a)} d_A(a) ,$$

thereby incorporating the stabilized invariant integral into the "remainder". From this point on, one can, modulo a few minor changes, simply copy Herb's argument to come up with the sought for conclusion.

If $A \neq T$, then the position is slightly more complicated due to the disconnectedness of M. Following Harish-Chandra [4-(d), pp. 145 and 152], let

$$f^P(m: a_R) = d_P(a_R) \cdot \int_N f_K(m a_R n) d_N(n) ,$$

f_K the K-centralization of f:

$$f_K(x) = \int_K f(kxk^{-1}) d_K(k) .$$

As is well-known (cf. [4-(d), p. 146]),

$$\begin{aligned} F_f^A(z a_I^0 a_R) &= \xi_\rho(z)\Delta_{A_I^0}(a_I^0) \cdot \int_{M/A_I} f^P(z m a_I^0 m^{-1}: a_R) d_{M/A_I}(m) \\ &= \xi_\rho(z) \cdot \sum_\mu \Delta_{A_I^0}(a_I^0) \cdot \int_{M^0/A_I^0} f^P(z m^0 (w_\mu a_I^0) m^{-0}: a_R) d_{M^0/A_I^0}(m^0) . \end{aligned}$$

If

$$f_z^P(m^0: a_R) = f^P(z m^0: a_R) ,$$

then

$$\Delta_{A_I^0}(w_\mu a_I^0) \cdot \int_{M^0/A_I^0} f^P(z m^0 (w_\mu a_I^0) m^{-0}: a_R) d_{M^0/A_I^0}(m^0)$$

is the invariant integral of $f_z^P(?: a_R)$ at $w_\mu a_I^0$ calculated relative to the compact Cartan subgroup A_I^0 of M^0. In view of the normality of W^0 in W,

$$\sum_{W^0\backslash W_{\mathbf{C}}} = \sum_{W\backslash W_{\mathbf{C}}} \sum_{\mu} .$$

So, taking into account the relation

$$\Delta_{A_I^0}(w_\mu a_I^0) = \varepsilon(w_\mu)\Delta_{A_I^0}(a_I^0) ,$$

it follows that

$$\mathscr{S}_f^A(za_I^0 a_R)$$

is $\xi_\rho(z)$ times the stabilized invariant integral of $f_z^P(?: a_R)$ at a_I^0 calculated relative to the compact Cartan subgroup A_I^0 of M^0, the "leading term" of which is, by the foregoing discussion,

$$\sum_{\Sigma^0} \Phi_{\Sigma^0}(a_I^0) \cdot (f_z^P(?: a_R), \Theta_{\Sigma^0}) .$$

Our claim is that this, multiplied by $\xi_\rho(z)$, is the same as

$$\sum_{\Sigma} \Phi_\Sigma(za_I^0) \cdot \int_{\mathscr{F}} (f, \Theta_{\Sigma,\nu}) a_R^{\sqrt{-1}\nu} d\nu .$$

Thus, in the notation of Harish-Chandra [4-(e), p. 162], we have

$$\begin{aligned}(f_z^P(?: a_R), \Theta_{\Sigma^0}) &= \int_{M^0} f_z^P(m^0: a_R)\overline{\Theta_{\Sigma^0}(m^0)}d_{M^0}(m^0) = \int_{M^0} f^P(zm^0 a_R)\overline{\Theta_{\Sigma^0}(m^0)}d_M(m^0) \\ &= \int_{\mathscr{F}} \left[\int_{M^0} f_\nu^P(zm^0)\overline{\Theta_{\Sigma^0}(m^0)}d_{M^0}(m^0)\right] a_R^{\sqrt{-1}\nu} d\nu .\end{aligned}$$

Suppose that $\Sigma \rightarrow \Sigma^\dagger \leftrightarrow (\zeta, \Sigma^0)$—then

$$\begin{aligned}(f, \Theta_{\Sigma,\nu}) = (f_\nu^P, \Theta_\Sigma) &= \int_M f_\nu^P(m)\overline{\Theta_\Sigma(m)}d_M(m) = \int_{M^\dagger} f_\nu^P(m^\dagger)\overline{\Theta_\Sigma(m^\dagger)}d_{M^\dagger}(m^\dagger) \\ &= \int_{Z_M} \left[\sum_\mu \int_{M^0} f_\nu^P(zm^0)\overline{\Theta_{\Sigma^0}(m_\mu m^0 m_\mu^{-1})}d_{M^0}(m^0)\right]\overline{\zeta(z)}d_{Z_M}(z) .\end{aligned}$$

On the other hand (§3),

$$\Phi_\Sigma(za_I^0) = \xi_\rho(z)\zeta(z)\Phi_{\Sigma^0}(a_I^0) .$$

So, using the orthogonality relations on $\hat{Z}_M$, we find that

$$\sum_{\Sigma} \Phi_\Sigma(za_I^0) \cdot \int_{\mathscr{F}} (f, \Theta_{\Sigma,\nu}) a_R^{\sqrt{-1}\nu} d\nu .$$

equals

$$\frac{\xi_\rho(z)}{[M: M^\dagger]} \cdot \sum_{\Sigma^0} \Phi_{\Sigma^0}(a_I^0) \int_{\mathscr{F}} \left[\sum_\mu \int_{M^0} f_\nu^P(zm^0)\overline{\Theta_{\Sigma^0}(m_\mu m^0 m_\mu^{-1})}d_{M^0}(m^0)\right] a_R^{\sqrt{-1}\nu} d\nu .$$

But obviously

$$\frac{1}{[M:M^{\dagger}]}\cdot\int_{\mathscr{F}}\left[\sum_{\mu}\int_{M^0} f_\nu^P(zm^0)\overline{\Theta_{\Sigma^0}(m_\mu m^0 m_\mu^{-1})}d_{M^0}(m^0)\right]a_R^{\sqrt{-1}\nu}d\nu$$
$$=\int_{\mathscr{F}}\left[\int_{M^0} f_\nu^P(zm^0)\overline{\Theta_{\Sigma^0}(m^0)}d_{M^0}(m^0)\right]a_R^{\sqrt{-1}\nu}d\nu$$

if we assume, as is permissible, that the $m_\mu \in K\cap M$. Hence the claim. This sets the stage once again for the methods of Herb.

Herb's formulae readily lead to a result of independent interest (cf. [1, p. 388], [3-(a), p. 452], [9, p. 40]).

PROPOSITION 5.2. *Fix* $f\in C_c^\infty(G)$*—then the following are equivalent*:
(i) *For every* $A\in\mathfrak{C}$ *and for every* $a\in A'$,

$$\mathscr{S}_f^A(a)=0\ ;$$

(ii) *For every stable* Θ,

$$(f,\Theta)=0\ .$$

PROOF. (i) $\Rightarrow$ (ii). Owing to the stability of Θ, we have

$$(f,\Theta)=\int_G f(x)\overline{\Theta(x)}d_G(x)=\sum_{A\in\mathfrak{C}}C_A\int_A \mathscr{S}_f^A(a)\overline{\varepsilon_R(a)\Delta_A(a)\Theta(a)}d_A(a)\ ,$$

C_A a positive constant. Since $\mathscr{S}_f^A$ vanishes identically, (f,Θ) must be zero.

(ii) $\Rightarrow$ (i). Because the Θ_Ω and $\Theta_{\Sigma,\nu}$ are stable (Lemmas 3.1 and 3.3), this follows from Theorem 5.1.

The proof is therefore complete.

6. The Theorem of Clozel and Delorme. The theorem in question provides a Paley-Wiener type characterization for the "invariant Fourier transform" of the K-finite functions $C_c^\infty(G,K)$ in $C_c^\infty(G)$.

With $P=M\cdot A_R\cdot N$ determined by $A=A_I\cdot A_R$ as in §2, let $\mathscr{F}_c$ be the complexification of $\mathscr{F}$ and call $\Theta_{\sigma,\nu}$ ($\nu\in\mathscr{F}_c$) the character of the non-unitary principal series representation

$$\pi_P(\sigma,\nu)=\mathrm{Ind}_P^G(\sigma,\nu)\ .$$

Given $f\in C_c^\infty(G,K)$, put

$$\hat{f}_P(\sigma,\nu)=(f,\Theta_{\sigma,\nu})\ .$$

Then, by definition,

$$\hat{f}_P\colon \hat{M}_d\times\mathscr{F}_c\to\mathbf{C}$$

is the invariant Fourier transform of f at P. It has three basic properties.

(1) As a function of σ, $\hat{f}_P$ has finite support.
(2) As a function of ν, $\hat{f}_P$ is in P-W $(\mathscr{F}_c)$.
(3) For every $w \in \mathfrak{w}$,

$$\hat{f}_P(w\sigma, w\nu) = \hat{f}_P(\sigma, \nu) .$$

[Note: Here, P-W $(\mathscr{F}_c)$ is the Paley-Wiener space on $\mathscr{F}_c$, i.e., the image under Fourier transformation of $C_c^\infty(\mathfrak{a}_R)$. These properties obtain, of course, when $P = G$ since then $A = T$.]

It is not hard to see that $\hat{f}_P$ does indeed satisfy (1)-(3). There is, however, a converse that is substantially more difficult to establish (cf. [3-(a)]), namely:

THEOREM 6.1. *Suppose given functions*

$$\hat{f}_i\colon \hat{M}_{id} \times \mathscr{F}_{ic} \to \mathbf{C}$$

such that for all i:
(1) *As a function of* σ_i, $\hat{f}_i$ *has finite support.*
(2) *As a function of* ν_i, $\hat{f}_i$ *is in* P-W $(\mathscr{F}_{ic})$.
(3) *For every* $w \in \mathfrak{w}_i$,

$$\hat{f}_i(w\sigma_i, w\nu_i) = \hat{f}_i(\sigma_i, \nu_i) .$$

Then there exists an $f \in C_c^\infty(G, K)$ *with the property that for all* i,

$$\hat{f}_{P_i} = \hat{f}_i$$

[Note: Needless to say, f is far from unique.]

It is easy to stabilize this result (cf. [3-(a), p. 449]). Thus, one can define the stable invariant Fourier transform of f at P by putting

$$\hat{F}_P(\Sigma, \nu) = (f, \Theta_{\Sigma,\nu}) .$$

Then

$$\hat{F}_P\colon \mathrm{ST}\text{—}\hat{M}_d \times \mathscr{F}_c \to \mathbf{C}$$

satisfies:
(1) As a function of Σ, $\hat{F}_P$ has finite support.
(2) As a function of ν, $\hat{F}_P$ is in P-W $(\mathscr{F}_c)$.
(3) For every $w \in \mathfrak{w}$,

$$\hat{F}_P(w\Sigma, w\nu) = \hat{F}_P(\Sigma, \nu) .$$

Conversely, suppose given functions

$$\hat{F}_i\colon \mathrm{ST}\text{—}\hat{M}_{id} \times \mathscr{F}_{ic} \to \mathbf{C}$$

such that for all i:
(1) As a function of Σ_i, $\hat{F}_i$ has finite support.

(2) As a function of ν_i, $\hat{F}_i$ is in P-W $(\mathscr{F}_{ic})$.
(3) For every $w \in \mathfrak{w}_i$,

$$\hat{F}_i(w\Sigma_i, w\nu_i) = \hat{F}_i(\Sigma_i, \nu_i) .$$

Then there exists an $f \in C_c^\infty(G, K)$ with the property that for all i,

$$\hat{F}_{P_i} = \hat{F}_i .$$

[In fact, the cardinality of each Σ_i is the same, viz. $[W_{ic}]/[W_i]$, so one has only to set

$$\hat{f}_i(\sigma_i, \nu_i) = \frac{1}{\#(\Sigma_i)} \cdot \hat{F}_i(\Sigma_i, \nu_i) \quad (\sigma_i \in \Sigma_i)$$

and then apply the theorem to the $\hat{f}_i$ to produce an $f \in C_c^\infty(G, K)$ fulfilling all the requirements.]

The support condition in σ or Σ can be relaxed but this is not the place to go into detail. Instead, we shall address that problem in a future publication.

7. Formulation and Proof of the Characterization. Fix $A \in \mathfrak{C}$—then, as we have seen in §5, for any $f \in C_c^\infty(G)$,

$$\mathscr{S}_f^A(a) = \sum_\Sigma \Phi_\Sigma(a_I) \cdot \int_{\mathscr{F}} (f, \Theta_{\Sigma,\nu}) a_R^{\sqrt{-1}\nu} d\nu$$
$$+ \sum_{A_i; A_i > A} \sum_{\Sigma_i} \int_{\mathscr{F}_i} (f, \Theta_{\Sigma_i,\nu_i}) \mathbf{H}_{A|A_i}(a\colon (\Sigma_i, \nu_i)) d\nu_i .$$

Let

$${}^0\mathscr{S}_f^A(a) = \sum_\Sigma \Phi_\Sigma(a_I) \cdot \int_{\mathscr{F}} (f, \Theta_{\Sigma,\nu}) a_R^{\sqrt{-1}\nu} d\nu .$$

Then it is clear that ${}^0\mathscr{S}_f^A \in C_c^\infty(A)$ is W_{C}-skew and, in the notation of Lemma 3.2 (if $A = T$) and Lemma 3.4 (if $A \neq T$),

$${}^0\hat{\mathscr{S}}_f^A(\Sigma, \nu) = (f, \Theta_{\Sigma,\nu}) .$$

Consequently,

$$\mathscr{S}_f^A(a) = \sum_\Sigma \Phi_\Sigma(a_I) \cdot \int_{\mathscr{F}} {}^0\hat{\mathscr{S}}_f^A(\Sigma, \nu) a_R^{\sqrt{-1}\nu} d\nu$$
$$+ \sum_{A_i; A_i > A} \sum_{\Sigma_i} \int_{\mathscr{F}_i} {}^0\hat{\mathscr{S}}_f^{A_i}(\Sigma_i, \nu_i) \mathbf{H}_{A|A_i}(a\colon (\Sigma_i, \nu_i)) d\nu_i .$$

Thus the upshot is that to each $f \in C_c^\infty(G)$ we can attach functions

$$\mathscr{S}_f^A \in \mathscr{B}^\infty(A_I'; C_c^\infty(A_R)) ,$$

one for each $A \in \mathfrak{C}$, all linked by Herb's formulae via the transforms ${}^0\hat{\mathscr{S}}_f^{A_i}$. But the latter can be characterized by the theorem of Clozel and Delorme, at least in the K-finite case.

Suppose, therefore, that we are given functions

$$\phi^{A_1} = \phi^T, \phi^{A_2}, \cdots, \phi^{A_r} .$$

We then seek to impose conditions on the elements of this collection so as to guarantee the existence of an $f \in C_c^\infty(G, K)$ such that for every i,

$$\mathscr{S}_f^{A_i} = \phi^{A_i} .$$

To begin with, it is necessary to assume that for each $A \in \mathfrak{C}$,

$$\phi^A \in \mathscr{B}^\infty(A_I'; C_c^\infty(A_R))$$

and is W_C-skew. The other conditions involve ${}^0\phi^A$ and its transform ${}^0\hat{\phi}^A$, which will be defined recursively, starting at the "top" and working down. When A is of Iwasawa type, set

$${}^0\phi^A = \phi^A .$$

To define ${}^0\phi^A$ in general, put

$$n(A) = l - \dim(A_R) ,$$

$l = \operatorname{rank}(G/K)$. Obviously,

$$\sum_{A_i; A_i \succ A} = \sum_{n; n < n(A)} \ \sum_{A_i; A_i \succ A \ \& \ n(A) = n} .$$

Accordingly, if the ${}^0\phi^{A_i}$ have been defined for all $A_i \succ A$ with $n(A_i) < n(A)$, then we may let

$${}^0\phi^A = \phi^A - \sum_{A_i; A_i \succ A} \sum_{\Sigma_i} \int_{\mathscr{F}_i} {}^0\hat{\phi}^{A_i}(\Sigma_i, \nu_i)\mathbf{H}_{A|A_i}(?: (\Sigma_i, \nu_i))d\nu_i ,$$

the recursive demand being throughout that ${}^0\phi^A \in C_c^\infty(A)$. As such, ${}^0\phi^A$ is certainly W_C-skew. It will also be necessary to ask that

$${}^0\hat{\phi}^A(\Sigma, \nu)$$

be finitely supported in Σ and satisfy the invariance condition

$${}^0\hat{\phi}^A(w\Sigma, w\nu) = {}^0\hat{\phi}^A(\Sigma, \nu) \quad (w \in \mathfrak{w}) .$$

Because the Paley-Wiener requirement is already built in, it follows from the stable version of Theorem 6.1 that there exists an $f \in C_c^\infty(G, K)$ such that for each $A \in \mathfrak{C}$,

$${}^0\hat{\phi}^A(\Sigma, \nu) = (f, \Theta_{\Sigma,\nu}) .$$

And our main contention is:

THEOREM 7.1. *For every* $A \in \mathfrak{C}$,

$$\mathscr{S}_f^A = \phi^A .$$

PROOF. We have

$$
\begin{aligned}
{}^0\mathcal{S}_f^A(a) &= \sum_{\Sigma} \Phi_\Sigma(a_I)\cdot\int_{\mathcal{F}} (f, \Theta_{\Sigma,\nu}) a_R^{\sqrt{-1}\nu} d\nu \\
&= \sum_{\Sigma} \Phi_\Sigma(a_I)\cdot\int_{\mathcal{F}} {}^0\hat{\phi}^A(\Sigma, \nu) a_R^{\sqrt{-1}\nu} d\nu = {}^0\phi^A(a)\,.
\end{aligned}
$$

But then

$$
\begin{aligned}
\mathcal{S}_f^A(a) &= {}^0\mathcal{S}_f^A(a) + \sum_{A_i : A_i \succ A} \sum_{\Sigma_i} \int_{\mathcal{F}_i} (f, \Theta_{\Sigma_i,\nu_i}) \mathbf{H}_{A|A_i}(a\colon (\Sigma_i, \nu_i)) d\nu_i \\
&= {}^0\phi^A(a) + \sum_{A_i : A_i \succ A} \sum_{\Sigma_i} \int_{\mathcal{F}_i} {}^0\hat{\phi}^{A_i}(\Sigma_i, \nu_i) \mathbf{H}_{A|A_i}(a\colon (\Sigma_i, \nu_i)) d\nu_i \\
&= \phi^A(a)\,,
\end{aligned}
$$

as desired.

8. **Concluding Remarks.** The present methods can easily be adapted to give other characterizations of a similar sort. To do so, one just needs to have a suitable version of Theorem 6.1 (modulo Herb's formulae). Here, we shall briefly touch upon some of the possibilities along these lines, albeit informally.

It has already been mentioned at the end of §6 that the support requirement in the discrete parameter can be weakened, the new requirement being one of "rapid decrease", while retaining, of course, the Paley-Wiener condition.

Another variant arises by working with $\mathcal{C}(G)$, the Schwartz space on G. In this setting, Theorem 6.1 would be replaced by a theorem of Arthur [1], where, in the "real" domain, one demands rapid decrease in both the discrete and continuous parameters (K-finiteness in irrelevant). Since Herb's formulae are still valid, everything goes through as before.

Finally, one could also characterize F_f^A rather than its stable counterpart $\mathcal{S}_f^A$. The requisite inversion formulae are contained in [5-(a)] and [5-(c)] (see also [4-(f)]). But now the analysis is complicated by the fact that it is also necessary to take into account limits of discrete series. However, this problem can be handled by using the main result from [3-(b)].

It would be interesting but apparently rather difficult to find $\mathcal{C}^p(G)$-analogues of the above. The only work in this direction is that of Trombi [10], who has a characterization when $\mathrm{rank}(G/K) = 1$.

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