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ON THE FOURIER-BOREL TRANSFORMATIONS OF ANALYTIC FUNCTIONALS ON THE COMPLEX SPHERE

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Introduction. Let $\mathcal{O}(\boldsymbol{C}^{d+1})$ and $\mathrm{Exp}(\boldsymbol{C}^{d+1})$ be the spaces of entire functions on $\boldsymbol{C}^{d+1}$ and entire functions of exponential type, respectively. $\mathcal{O}'(\boldsymbol{C}^{d+1})$ and $\mathrm{Exp}'(\boldsymbol{C}^{d+1})$ are the spaces dual to $\mathcal{O}(\boldsymbol{C}^{d+1})$ and $\mathrm{Exp}(\boldsymbol{C}^{d+1})$, respectively. For $T \in \mathrm{Exp}'(\boldsymbol{C}^{d+1})$ the Fourier-Borel transformation P_λ is defined by

$$P_\lambda T(z) := \langle T_\xi, \exp(i\lambda\xi \cdot z)\rangle \quad \text{for} \quad z \in \boldsymbol{C}^{d+1},$$

where $\lambda \in \boldsymbol{C}$, $\lambda \neq 0$, is a fixed constant (Hashizume, Kowata, Minemura and Okamoto [2]). Martineau [4] determined the images of $\mathrm{Exp}'(\boldsymbol{C}^{d+1})$ and some functional spaces on $\boldsymbol{C}^{d+1}$ by the Fourier-Borel transformation P_λ.

Let $S = S^d$ be the unit sphere in $\boldsymbol{R}^{d+1}$ and $\tilde{S}$ denote the complex sphere in $\boldsymbol{C}^{d+1}$. We put $\tilde{S}(r) = \{z \in \tilde{S}; L(z) < r\}$ and $\tilde{S}[r] = \{z \in \tilde{S}; L(z) \leqq r\}$, where $L(z)$ is the Lie norm on $\boldsymbol{C}^{d+1}$. $\mathcal{O}(\tilde{S})$, $\mathcal{O}(\tilde{S}(r))$ and $\mathcal{O}(\tilde{S}[r])$ denote the spaces of holomorphic functions on $\tilde{S}$, $\tilde{S}(r)$, and $\tilde{S}[r]$, respectively. $\mathrm{Exp}(\tilde{S})$ denotes the restriction of $\mathrm{Exp}(\boldsymbol{C}^{d+1})$ to $\tilde{S}$. $\mathrm{Exp}'(\tilde{S})$, $\mathcal{O}'(\tilde{S})$, $\mathcal{O}'(\tilde{S}(r))$ and $\mathcal{O}'(\tilde{S}[r])$ are the spaces dual to $\mathrm{Exp}(\tilde{S})$, $\mathcal{O}(\tilde{S})$, $\mathcal{O}(\tilde{S}(r))$ and $\mathcal{O}(\tilde{S}[r])$, respectively. $\mathrm{Exp}'(\tilde{S})$ can be regarded as a subspace of $\mathrm{Exp}'(\boldsymbol{C}^{d+1})$.

Morimoto [7] determined the images of $\mathrm{Exp}'(\tilde{S})$ and $\mathcal{O}'(\tilde{S})$ by the Fourier-Borel transformation P_λ (Theorem 1.2). In this paper we will determine the images of $\mathcal{O}'(\tilde{S}(r))$ and $\mathcal{O}'(\tilde{S}[r])$ by the Fourier-Borel transformation P_λ. The images are characterized explicitly in terms of the dual Lie norm (Theorem 3.1).

Consider a complex cone $M = \{z \in \boldsymbol{C}^{d+1}; \sum_{j=1}^{d+1} z_j^2 = 0, z \neq 0\}$, which can be identified with the cotangent bundle of S minus its zero section. We define for $f' \in \mathrm{Exp}'(\tilde{S})$

$$Ff'(z) = \langle f'_\xi, \exp(\xi \cdot z)\rangle \quad (z \in M).$$

Ff' is the restriction of $P_{-i}f'$ to M. Ii [3] determined the images of $H_{n,d}$ by F, where $H_{n,d}$ is the space of spherical harmonics of degree n in dimension $d+1$. Moreover if d is even, Ii [3] characterized the image of $L^2(S)$ under this mapping F. In this paper we determine the image of $L^2(S)$ for odd d (Theorem 2.4). We also determine the images of

$\mathrm{Exp}'(\tilde{S})$, $\mathcal{O}'(\tilde{S})$, $\mathcal{O}'(\tilde{S}(r))$, $\mathcal{O}'(\tilde{S}[r])$, $\mathcal{O}(\tilde{S}(r))$, $\mathcal{O}(\tilde{S}[r])$ and $\mathcal{O}(\tilde{S})$ (Theorem 2.1).

To prove our main theorems, we need, among others, Lemmas 1.3 and 1.4. Although Lemma 1.4 was proved in Ii [3], we give here a new proof to it.

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1. Preliminaries. Let d be a positive integer and $d \geqq 2$. $S = S^d = \{x \in \boldsymbol{R}^{d+1};\ \|x\| = 1\}$ denotes the unit sphere in $\boldsymbol{R}^{d+1}$, where $\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_{d+1}^2$. ds denotes the unique $O(d+1)$ invariant measure on S with $\int_S 1 ds = 1$, where $O(k)$ is the orthogonal group of degree k. $\|\ \ \|_2$ is the L^2-norm on S. $H_{n,d}$ is the space of spherical harmonics of degree n in dimension $d+1$. For spherical harmonics, see Müller [8],

The Lie norm $L(z)$ and the dual Lie norm $L^*(z)$ on $\boldsymbol{C}^{d+1}$ are defined as follows:

$$\begin{aligned} L(z) &= L(x+iy) := [\|x\|^2 + \|y\|^2 + 2\{\|x\|^2\|y\|^2 - (x\cdot y)^2\}^{1/2}]^{1/2}\,, \\ L^*(z) &= L^*(x+iy) := \sup\{|\xi\cdot z|;\ L(\xi) \leqq 1\} \\ &= (1/\sqrt{2})[\|x\|^2 + \|y\|^2 + \{(\|x\|^2 - \|y\|^2)^2 + 4(x\cdot y)^2\}^{1/2}]^{1/2}\,, \end{aligned}$$

where z, $\xi \in \boldsymbol{C}^{d+1}$, and $z\cdot\xi = z_1\xi_1 + z_2\xi_2 + \cdots + z_{d+1}\xi_{d+1}$, $x, y \in \boldsymbol{R}^{d+1}$, (see Drużkowski [1]).

We put

$$\tilde{B}(r) := \{z \in \boldsymbol{C}^{d+1};\ L(z) < r\} \qquad \text{for} \quad 0 < r \leqq \infty$$

and

$$\tilde{B}[r] := \{z \in \boldsymbol{C}^{d+1};\ L(z) \leqq r\} \qquad \text{for} \quad 0 \leqq r < \infty\,.$$

Let us denote by $\mathcal{O}(\tilde{B}(r))$ the space of holomorphic functions on $\tilde{B}(r)$. Then $\mathcal{O}(\tilde{B}(r))$ is an FS space. $\mathcal{O}(\tilde{B}(\infty)) = \mathcal{O}(\boldsymbol{C}^{d+1})$ is the space of entire functions on $\boldsymbol{C}^{d+1}$. Let us define

$$\mathcal{O}(\tilde{B}[r]) := \operatorname*{ind\,lim}_{r'>r} \mathcal{O}(\tilde{B}(r'))\,.$$

Then $\mathcal{O}(\tilde{B}[r])$ is a DFS space.

Let N be a norm on $\boldsymbol{C}^{d+1}$. For $r > 0$ we put

$$X_{r,N} := \{f \in \mathcal{O}(\boldsymbol{C}^{d+1});\ \sup_{z\in\boldsymbol{C}^{d+1}} |f(z)|\exp(-rN(z)) < \infty\}\,.$$

Then $X_{r,N}$ is a Banach space with respect to the norm

$$\|f\|_{r,N} = \sup_{z\in\boldsymbol{C}^{d+1}} |f(z)|\exp(-rN(z))\,.$$

Define

$$\mathrm{Exp}(\boldsymbol{C}^{d+1}\colon (r\colon N)) := \operatorname*{proj\,lim}_{r'>r} X_{r',N} \qquad \text{for} \quad 0 \leqq r < \infty\ ,$$

$$\mathrm{Exp}(\boldsymbol{C}^{d+1}\colon [r\colon N]) := \operatorname*{ind\,lim}_{r'<r} X_{r',N} \qquad \text{for} \quad 0 < r \leqq \infty\ .$$

$\mathrm{Exp}(\boldsymbol{C}^{d+1}\colon (r\colon N))$ is an FS space and $\mathrm{Exp}(\boldsymbol{C}^{d+1}\colon [r\colon N])$ is a DFS space. $\mathrm{Exp}(\boldsymbol{C}^{d+1}) = \mathrm{Exp}(\boldsymbol{C}^{d+1}\colon [\infty\colon N])$ is independent of the choice of the norm N and is called the space of entire functions of exponential type.

$\mathrm{Exp}'(\boldsymbol{C}^{d+1})$, $\mathscr{O}'(\boldsymbol{C}^{d+1})$, $\mathscr{O}'(\tilde{B}(r))$ and $\mathscr{O}'(\tilde{B}[r])$ denote the spaces dual to $\mathrm{Exp}(\boldsymbol{C}^{d+1})$, $\mathscr{O}(\boldsymbol{C}^{d+1})$, $\mathscr{O}(\tilde{B}(r))$ and $\mathscr{O}(\tilde{B}[r])$, respecrively.

$\tilde{S} = \{z \in \boldsymbol{C}^{d+1};\, z_1^2 + z_2^2 + \cdots + z_{d+1}^2 = 1\}$ is the complex sphere. For $1 < r \leqq \infty$ we put

$$\tilde{S}(r) := \tilde{B}(r) \cap \tilde{S} = \{z = x + iy \in \tilde{S};\, \|y\| < (r - 1/r)/2\}$$

and for $1 \leqq r < \infty$

$$\tilde{S}[r] = \tilde{B}[r] \cap \tilde{S} = \{z = x + iy \in \tilde{S}\colon \|y\| \leqq (r - 1/r)/2\}\ .$$

It is clear that $S = \tilde{S} \cap \boldsymbol{R}^{d+1} = \tilde{S}[1]$ and $\tilde{S} = \tilde{S}(\infty)$.

Let us denote by $\mathscr{O}(\tilde{S}(r))$ the space of holomorphic functions on $\tilde{S}(r)$ equipped with the topology of uniform convergence on every compact subset of $\tilde{S}(r)$. We put

$$\mathscr{O}(\tilde{S}[r]) := \operatorname*{ind\,lim}_{r'>r} \mathscr{O}(\tilde{S}(r'))\ .$$

$\mathscr{O}(\tilde{S}(r))$ is an FS space and $\mathscr{O}(\tilde{S}[r])$ is a DFS space. $\mathscr{O}(\tilde{S}[1])$ is the space of real analytic functions on S. $\mathrm{Exp}(\tilde{S})$ denotes the restriction to $\tilde{S}$ of $\mathrm{Exp}(\boldsymbol{C}^{d+1})$. $\mathscr{O}'(\tilde{S}(r))$, $\mathscr{O}'(\tilde{S}[r])$ and $\mathrm{Exp}'(\tilde{S})$ denote the spaces dual to $\mathscr{O}(\tilde{S}(r))$, $\mathscr{O}(\tilde{S}[r])$ and $\mathrm{Exp}(\tilde{S})$, respectively. We have the following sequence of functional spaces on $\tilde{S}$ (cf. Morimoto [6], [7]):

$$\mathrm{Exp}'(\tilde{S}) \supset \mathscr{O}'(\tilde{S}) \supset \mathscr{O}'(\tilde{S}[r]) \supset \mathscr{O}'(\tilde{S}(r)) \supset \mathscr{O}'(\tilde{S}[1])\ . \tag{1.1}$$

If f is a function or a functional on S, we denote by f_n the n-th spherical harmonic component of f:

$$f_n(s) = N(n, d)\langle f, P_{n,d}(\ \cdot s)\rangle \qquad \text{for} \quad s \in S\ , \tag{1.2}$$

where

$$N(n, d) = \dim H_{n,d} = \frac{(2n + d - 1)(n + d - 2)!}{n!(d - 1)!} \tag{1.3}$$

and $P_{n,d}$ is the Legendre polynomial of degree n and of dimension $d + 1$.

We put $L_n(x) = \|x\|^n\, P_{n,d}(\alpha \cdot x/\|x\|)$ for fixed $\alpha \in S$. Then L_n is the unique homogeneous harmonic polynomial of degree n with the following properties:

$$(1.4) \qquad L_n(Ax) = L_n(x) \quad \text{for all} \quad A \in O(d+1) \quad \text{such that} \quad A\alpha = \alpha .$$

$$(1.5) \qquad L_n(\alpha) = 1 .$$

We see that f_n belongs to $H_{n,d}$ for $n = 0, 1, \cdots$. We can characterize the functional spaces in (1.1) by the behavior of the spherical harmonic development as follows.

LEMMA 1.1 (Morimoto [7, Theorems 5.1 and 6.1]). *If f_n is the n-th spherical harmonic component of f, then*

$$(1.6) \qquad f \in \mathrm{Exp}'(\tilde{S}) \Leftrightarrow \limsup_{n\to\infty} (\|f_n\|_2/n!)^{1/n} = 0 ,$$

$$(1.7) \qquad f \in \mathcal{O}'(\tilde{S}) \Leftrightarrow \limsup_{n\to\infty} \|f_n\|_2^{1/n} < \infty ,$$

$$(1.8) \qquad f \in \mathcal{O}'(\tilde{S}[r]) \Leftrightarrow \limsup_{n\to\infty} \|f_n\|_2^{1/n} \leqq r \quad (1 \leqq r < \infty) ,$$

$$(1.9) \qquad f \in \mathcal{O}'(\tilde{S}(r)) \Leftrightarrow \limsup_{n\to\infty} \|f_n\|_2^{1/n} < r \quad (1 < r \leqq \infty) ,$$

$$(1.10) \qquad f \in L^2(S) \Leftrightarrow \{\|f_n\|_2\}_{n=0,1,2,\ldots} \in l^2 ,$$

$$(1.11) \qquad f \in \mathcal{O}(\tilde{S}(r)) \Leftrightarrow \limsup_{n\to\infty} \|f_n\|_2^{1/n} \leqq 1/r \quad (1 < r \leqq \infty) ,$$

$$(1.12) \qquad f \in \mathcal{O}(\tilde{S}[r]) \Leftrightarrow \limsup_{n\to\infty} \|f_n\|_2^{1/n} < 1/r \quad (1 \leqq r < \infty) ,$$

$$(1.13) \qquad f \in \mathcal{O}(\tilde{S}) \Leftrightarrow \limsup_{n\to\infty} \|f_n\|_2^{1/n} = 0 .$$

The Fourier-Borel transformation P_λ for a functional $T \in \mathrm{Exp}'(\boldsymbol{C}^{d+1})$ is defined by

$$P_\lambda T(z) := \langle T_\xi, \exp(i\lambda\xi \cdot z)\rangle \qquad \text{for} \quad z \in \boldsymbol{C}^{d+1} ,$$

where $\lambda \in \boldsymbol{C}$, $\lambda \neq 0$, is a fixed constant. We define the transformation P_λ for a functional $f' \in \mathrm{Exp}'(\tilde{S})$ by

$$P_\lambda f'(z) := \langle f'_\xi, \exp i\lambda(\xi \cdot z)\rangle .$$

The following is known:

THEOREM 1.2 (Morimoto [7, Theorem 7.1]). *The transformation P_λ establishes the linear topological isomorphisms*

$$(1.14) \qquad P_\lambda\colon \mathrm{Exp}'(\tilde{S}) \xrightarrow{\sim} \mathcal{O}_\lambda(\boldsymbol{C}^{d+1}) ,$$

$$(1.15) \qquad P_\lambda\colon \mathcal{O}'(\tilde{S}) \xrightarrow{\sim} \mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}) ,$$

where we put

$$\mathcal{O}_\lambda(\boldsymbol{C}^{d+1}) := \{F \in \mathcal{O}(\boldsymbol{C}^{d+1}); (\Delta_z + \lambda^2)F(z) = 0\} ,$$

$$\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}) := \mathrm{Exp}(\boldsymbol{C}^{d+1}) \cap \mathcal{O}_\lambda(\boldsymbol{C}^{d+1}) ,$$

and $\Delta_z = (\partial/\partial z_1)^2 + (\partial/\partial z_2)^2 + \cdots + (\partial/\partial z_{d+1})^2$.

We define a complex cone M by

$$M = \{z \in \boldsymbol{C}^{d+1}; z_1^2 + z_2^2 + \cdots + z_{d+1}^2 = 0, z \neq 0\} .$$

M is identified with the cotangent bundle of S minus its zero section (cf. Ii [3], Rawnsley [9], [10]). $P_n(\boldsymbol{C}^{d+1})$ denotes the space of homogeneous polynomials of degree n on $\boldsymbol{C}^{d+1}$. $\mathrm{Holo}(M)$ and $P_n(M)$ denote the restriction to M of $\mathcal{O}(\boldsymbol{C}^{d+1})$ and $P_n(\boldsymbol{C}^{d+1})$, respectively. We define the subset N of M by

$$N = \{z = x + iy \in M; \|x\| = \|y\| = 1\} ,$$

where $x, y \in \boldsymbol{R}^{d+1}$. The unit cotangent bundle to S is identified with the subset N and we have $N \simeq O(d+1)/O(d-1)$. dN denotes the unique $O(d+1)$ invariant measure on N with $\int_N 1 dN = 1$. We define the inner product

$$\langle \varphi, \psi \rangle_N := \int_N \varphi(z)\overline{\psi(z)} dN$$

and the norm

$$\|\varphi\|_N = \langle \varphi, \varphi \rangle_N^{1/2} .$$

LEMMA 1.3. *If α and β belong to S, the following formula is valid.*

$$\int_N (z \cdot \alpha)^n \overline{(z \cdot \beta)^m} dN = \frac{n!\,\Gamma((d+1)/2)}{\Gamma(n + (d+1)/2)} \delta_{nm} P_{n,d}(\alpha \cdot \beta) . \tag{1.16}$$

PROOF. Denote by $F(\alpha, \beta)$ the left hand side of (1.16). Then for any orthogonal matrix A

$$\begin{aligned} F(A\alpha, A\beta) &= \int_N (z \cdot A\alpha)^n \overline{(z \cdot A\beta)^m} dN \\ &= \int_{z=x+iy \in N} (x \cdot A\alpha + iy \cdot A\alpha)^n \overline{(x \cdot A\beta + iy \cdot A\beta)^m} dN \\ &= \int_N (A^{-1}z \cdot \alpha)^n \overline{(A^{-1}z \cdot \beta)^m} dN . \end{aligned}$$

Since dN is $O(d+1)$-invariant we get

$$F(A\alpha, A\beta) = F(\alpha, \beta) \tag{1.17}$$

for any $A \in O(d+1)$. As a function of α, $F(\alpha, \beta)$ belongs to $H_{n,d}$, since

$(z\cdot\alpha)^n \in H_{n,d}$ if $z \in M$. Similarly, as a function of β, $F(\alpha, \beta)$ belongs to $H_{m,d}$.

Suppose $n \neq m$. There exists an $A \in O(d+1)$ such that $A\alpha = \beta$ and $A\beta = \alpha$. Then (1.17) gives

$$F(\alpha, \beta) = F(\beta, \alpha) . \tag{1.18}$$

If we fix α, (1.18) implies that $F(\alpha, \beta) \in H_{n,d} \cap H_{m,d}$. Since $H_{n,d} \cap H_{m,d} = \{0\}$, we have

$$F(\alpha, \beta) \equiv 0 \quad \text{if} \quad n \neq m . \tag{1.19}$$

Next we assume $n = m$. For all $A \in O(d+1)$ such that $A\alpha = \alpha$ we have from (1.17) $F(\alpha, A\beta) = F(A\alpha, A\beta) = F(\alpha, \beta)$. Therefore $F(\alpha, \beta)$, as a function of β, is a homogeneous harmonic polynomial of degree n and satisfies (1.4). So we obtain

$$F(\alpha, \beta) = CP_{n,d}(\alpha\cdot\beta) , \tag{1.20}$$

where

$$C = \int_N |z\cdot\alpha|^{2n} dN = \frac{n!\,\Gamma((d+1)/2)}{\Gamma(n+(d+1)/2)}$$

(c.f. Rawnsley [10, Appendix]). (1.16) follows from (1.19) and (1.20).

q.e.d.

We put for $f' \in \mathrm{Exp}'(\tilde{S})$ and $z \in M$.

$$Ff'(z) := \langle f'_\xi, e^{\xi\cdot z}\rangle .$$

Ff' is the restriction of $P_{-i}f'$ to M.

Then we have:

LEMMA 1.4 (cf. Ii [3]). *The transformation $F\colon f' \to Ff'$ is a one-to-one linear mapping of $H_{n,d}$ onto $P_n(M)$ and we have*

$$\langle f, g\rangle_S = C_n \langle Ff, Fg\rangle_N \quad \textit{for} \quad f, g \in H_{n,d} , \tag{1.21}$$

where

$$\langle f, g\rangle_S = \int_S f(s)\overline{g(s)}ds$$

and

$$C_n = \frac{n!\,\Gamma(n+(d+1)/2)N(n, d)}{\Gamma((d+1)/2)} . \tag{1.22}$$

PROOF. It is known that there exists a system of $N(n, d)$ points $\alpha_1, \alpha_2, \cdots, \alpha_{N(n,d)} \in S$ such that $P_{n,d}(\alpha_k\cdot\ \ )$, $k = 1, 2, \cdots, N(n, d)$, is a basis of $H_{n,d}$. Therefore for every $f \in H_{n,d}$, there exist $a_1, a_2, \cdots, a_{N(n,d)} \in \boldsymbol{C}$ such that

$$f(s) = \sum_{k=1}^{N(n,d)} a_k P_{n,d}(\alpha_k \cdot s) \quad s \in S \tag{1.23}$$

(see, for example, Müller [8, Theorem 3]). If z belongs to M, then

$$\begin{aligned} Ff(z) &= \sum_{k=1}^{N(n,d)} a_k \int_S P_{n,d}(\alpha_k \cdot s) e^{s \cdot z} ds \\ &= \sum_{k=1}^{N(n,d)} a_k \sum_{m=0}^{\infty} (m!)^{-1} \int_S P_{n,d}(\alpha \cdot s)(s \cdot z)^m ds \\ &= \sum_{k=1}^{N(n,d)} \frac{a_k}{n!} \int_S P_{n,d}(\alpha_k \cdot s)(s \cdot z)^n ds , \end{aligned}$$

since $(s \cdot z)^m \in H_{m,d}$ and $H_{n,d} \perp H_{m,d}$ if $m \neq n$. This shows that

$$Ff(z) = \sum_{k=1}^{N(n,d)} \frac{a_k}{n!\, N(n, d)} (\alpha_k \cdot z)^n . \tag{1.24}$$

Thus Ff belongs to $P_n(M)$. For $f(s) = \sum_{k=1}^{N(n,d)} a_k P_{n,d}(\alpha_k \cdot s)$ and $g(s) = \sum_{k=1}^{N(n,d)} b_k P_{n,d}(\alpha_k \cdot s) \in H_{n,d}$ we have

$$\begin{aligned} \langle f, g \rangle_S &= \sum_{1 \leq k, l \leq N(n,d)} a_k \bar{b}_l \int_S P_{n,d}(\alpha_k \cdot s) P_{n,d}(\alpha_l \cdot s) ds \\ &= \sum_{1 \leq k, l \leq N(n,d)} \frac{a_k \bar{b}_l}{N(n, d)} P_{n,d}(\alpha_k \cdot \alpha_l) . \end{aligned} \tag{1.25}$$

On the other hand we have from (1.24) and (1.16)

$$\begin{aligned} \langle Ff, Fg \rangle_N &= \sum_{1 \leq k, l \leq N(n,d)} \frac{a_k \bar{b}_l}{(n!\, N(n, d))^2} \frac{n!\, \Gamma((d+1)/2)}{\Gamma(n + (d+1)/2)} P_{n,d}(\alpha_k \cdot \alpha_l) \\ &= \frac{\Gamma((d+1)/2)}{n!\, N(n, d) \Gamma(n + (d+1)/2)} \sum_{k,l=1}^{N(n,d)} \frac{a_k \bar{b}_l}{N(n, d)} P_{n,d}(\alpha_k \cdot \alpha_l) . \end{aligned} \tag{1.26}$$

(1.25) and (1.26) give (1.21) and (1.22). (1.21) shows that F is injective. Since $\dim P_n(M) = N(n, d)$, we can prove the surjectivity of F. q.e.d.

2. Integral transformation F. Now we define the following subspaces of Holo(M):

$$\operatorname{Exp}(M, r) := \bigcap_{r' > r} \{\psi \in \operatorname{Holo}(M); \sup_{z \in M} |\psi(z)| \exp(-r' \|z\|) < \infty\} , \tag{2.1}$$

$$\operatorname{Exp}[M, r] := \bigcup_{r' < r} \{\psi \in \operatorname{Holo}(M); \sup_{z \in M} |\psi(z)| \exp(-r' \|z\|) < \infty\} , \tag{2.2}$$

$$\operatorname{Exp}(M) = \operatorname{Exp}[M, \infty] , \tag{2.3}$$

where $\|z\| = \|x + iy\| = (\|x\|^2 + \|y\|^2)^{1/2}$ for $x, y \in \boldsymbol{R}^{d+1}$.

Our first main theorem in this paper is the following:

THEOREM 2.1.

(2.4) *F is a one-to-one linear mapping of* $\mathrm{Exp}'(\tilde{S})$ *onto* $\mathrm{Holo}(M)$.

(2.5) *F is a one-to-one linear mapping of* $\mathcal{O}'(\tilde{S})$ *onto* $\mathrm{Exp}(M)$.

(2.6) *F is a one-to-one linear mapping of* $\mathcal{O}'(\tilde{S}[r])$ *onto* $\mathrm{Exp}(M, r/\sqrt{2})$ *for* $1 \leqq r < \infty$.

(2.7) *F is a one-to-one linear mapping of* $\mathcal{O}'(\tilde{S}(r))$ *onto* $\mathrm{Exp}[M, r/\sqrt{2}]$ *for* $1 < r \leqq \infty$.

(2.8) *F is a one-to-one linear mapping of* $\mathcal{O}(\tilde{S}(r))$ *onto* $\mathrm{Exp}(M, 1/(\sqrt{2}\, r))$ *for* $1 \leqq r < \infty$.

(2.9) *F is a one-to-one linear mapping of* $\mathcal{O}(\tilde{S}[r])$ *onto* $\mathrm{Exp}[M, 1/(\sqrt{2}\, r)]$ *for* $1 < r \leqq \infty$.

(2.10) *F is a one-to-one linear mapping of* $\mathcal{O}(\tilde{S})$ *onto* $\mathrm{Exp}(M, 0)$.

PROOF. By (1.14) F is a linear mapping of $\mathrm{Exp}'(\tilde{S})$ into $\mathrm{Holo}(M)$. Conversely, if ψ belongs to $\mathrm{Holo}(M)$ there exist $\tilde{\psi} \in \mathcal{O}(C^{d+1})$ and $\tilde{\psi}_n \in P_n(C^{d+1})$ $(n = 0, 1, \cdots,)$ such that

$$\tilde{\psi}|_M = \psi \quad \text{and} \quad \tilde{\psi}(z) = \sum_{n=0}^{\infty} \tilde{\psi}_n(z)$$

for any $z \in C^{d+1}$. It is known that

$$\tilde{\psi}_n(z) = \frac{1}{2i\pi} \oint_{|t|=\rho} \frac{\tilde{\psi}(tz)}{t^{n+1}} dt \tag{2.11}$$

for any $\rho > 0$. We put $\|\tilde{\psi}\|_{\infty,\sqrt{2}\rho} = \sup_{\|z\|=\sqrt{2}\rho} |\tilde{\psi}(z)|$ and $\psi_n = \tilde{\psi}_n|_M$. If z belongs to N then $\|z\| = \sqrt{2}$. Hence we get from (2.11)

$$\sup_{z \in N} |\psi_n(z)| = \sup_{z \in N} \left| \frac{1}{2i\pi} \oint_{|t|=\rho} \frac{\tilde{\psi}(tz)}{t^{n+1}} dt \right| \leqq \rho^{-n} \|\tilde{\psi}\|_{\infty,\sqrt{2}\rho} . \tag{2.12}$$

Put $K_n := \sup_{z \in N} |\psi_n(z)|$. (2.12) implies that $\limsup_{n\to\infty} K_n^{1/n} \leqq 1/\rho$ for any $\rho > 0$. Hence we see

$$\limsup_{n\to\infty} K_n^{1/n} = 0 . \tag{2.13}$$

From Lemma 1.4 there exist $f_n \in H_{n,d}$ $(n = 0, 1, \cdots)$ such that

$$F f_n = \psi_n \tag{2.14}$$

and

$$\|f_n\|_2 = \sqrt{C_n}\,\|\psi_n\|_N\ . \tag{2.15}$$

Since $\sqrt{C_n} = \{(n!\,\Gamma(n+(d+1)/2)N(n,d))/\Gamma((d+1)/2)\}^{1/2} < a\Gamma(n+d)$, where a is a constant independent of n, (2.13) and (2.15) give

$$\|f_n\|_2 \leqq a\Gamma(n+d)K_n \tag{2.16}$$

and

$$\limsup_{n\to\infty}\left(\frac{1}{n!}\|f_n\|_2\right)^{1/n} = 0\ . \tag{2.17}$$

$f' := \sum_{n=0}^{\infty} f_n$ belongs to $\mathrm{Exp}'(\tilde{S})$ by (1.6) and (2.17). Moreover, (2.14) implies that

$$Ff'(z) = \langle f'_\xi, e^{\xi\cdot z}\rangle = \sum_{n=0}^{\infty}\int_S f_n(s)e^{s\cdot z}ds = \sum_{n=0}^{\infty} Ff_n(z) = \psi(z)\ .$$

Therefore, we get $F(\mathrm{Exp}'(\tilde{S})) = \mathrm{Holo}(M)$.

Let $f' = \sum_{n=0}^{\infty} f'_n \in \mathrm{Exp}'(\tilde{S})$ and $Ff' = 0$. From the proof of Lemma 1.4, $\{(z\cdot\alpha)^n;\ \alpha\in S\}$ spans $P_n(M)$. From this fact and (1.16) we see that $P_n(M)\perp P_m(M)$ with respect to $\langle\ ,\ \rangle_N$ if $m\neq n$. Hence $Ff'_n = 0$ on N, because Ff'_n is in $P_n(M)$. Thus $Ff'_n = 0$ on M, since Ff'_n is a homogeneous polynomial. Therefore, we obtain $f'_n = 0$ and $f' = 0$ by Lemma 1.4. Hence we have (2.4).

F is a one-to-one linear mapping of $\mathcal{O}'(\tilde{S})$ into $\mathrm{Exp}(M)$ from (1.15) and (2.4). Conversely, if ψ belongs to $\mathrm{Exp}(M)$, there exists $\tilde{\psi}\in\mathcal{O}(\boldsymbol{C}^{d+1})$ such that $\tilde{\psi}|_M = \psi$ and that for some positive constants C and A

$$|\tilde{\psi}(z)| \leqq Ce^{A\|z\|} \qquad \text{for any} \quad z\in M\ . \tag{2.18}$$

We put $\tilde{\psi} = \sum_{n=0}^{\infty}\tilde{\psi}_n$ and $\tilde{\psi}_n|_M = \psi_n$, where $\tilde{\psi}_n$ is given by (2.11). (2.11) and (2.18) imply

$$\begin{aligned} K_n = \sup_{z\in N}|\psi_n(z)| &\leqq \sup_{z\in N,\,|t|=\rho}\rho^{-n}|\psi(tz)| \\ &\leqq \sup_{z\in N,\,|t|=\rho}\rho^{-n}Ce^{A\|tz\|} \leqq \sup_{\|z\|=\sqrt{2}}\rho^{-n}Ce^{A\rho\|z\|}\ , \end{aligned}$$

since $tN\subset M$ for any $t\in\boldsymbol{C}\setminus\{0\}$. Hence we have

$$K_n \leqq \rho^{-n}Ce^{\sqrt{2}A\rho} \qquad \text{for any} \quad \rho>0\ . \tag{2.19}$$

Since $\inf\{\rho^{-n}e^{\sqrt{2}A\rho};\ \rho>0\} = (\sqrt{2}Ae/n)^n$ we get

$$K_n \leqq C(\sqrt{2}Ae/n)^n\ . \tag{2.20}$$

There exist $f_n\in H_{n,d}$ $(n = 0, 1, 2, \cdots)$ which satisfy (2.14) and (2.15). By (2.16) and (2.20) we have

$$\|f_n\|_2 \leqq aC\Gamma(n+d)(\sqrt{2}Ae/n)^n\ .$$

Since $\limsup_{n\to\infty}(n^ne^{-n}\sqrt{2\pi n}/n!)^{1/n} = 1$ by Stirling's formula, we have

$$\limsup_{n\to\infty} \|f_n\|_2^{1/n} \leqq \limsup_{n\to\infty} \{aC\Gamma(n+d)(\sqrt{2}Ae/n)^n n^n e^{-n}\sqrt{2\pi n}/n!\}^{1/n} = \sqrt{2}A < \infty . \tag{2.21}$$

(2.21) and (1.7) show that $f' = \sum_{n=0}^{\infty} f_n \in \mathcal{O}'(\tilde{S})$ and we have (2.5).

Let $f' = \sum_{n=0}^{\infty} f'_n$ be in $\mathcal{O}'(\tilde{S}[r])$ $(1 \leqq r < \infty)$ and put $\psi = \sum_{n=0}^{\infty} \psi_n = Ff'$. Then we have for $z \in M$

$$\begin{aligned}\psi(z) &= \langle f'_\xi, \exp(\xi \cdot z)\rangle = \sum_{n=0}^{\infty} \int_S f'_n(s) e^{s \cdot z} ds \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} \int_S f'_n(s)(s \cdot z)^m ds \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_S f'_n(s)(s \cdot z)^n ds ,\end{aligned} \tag{2.22}$$

since $(s \cdot z)^m \in H_{n,d}$ and $H_{m,d} \perp H_{n,d}$ if $n \neq m$. (2.22) implies that

$$\psi_n(z) = \frac{1}{n!} \int_S f'_n(s)(s \cdot z)^n ds . \tag{2.23}$$

For $z = x + iy \in M$ we get

$$\sup_{s \in S} |s \cdot z|^2 = \sup_{s \in S} \|x\|^2 |s \cdot (x/\|x\|) + is \cdot (y/\|y\|)|^2 \leqq \|x\|^2 \leqq \|z\|^2/2 . \tag{2.24}$$

From (2.23) and (2.24) we see that

$$|\psi_n(z)| \leqq \frac{1}{n!} \|f'_n\|_2 (\|z\|/\sqrt{2})^n . \tag{2.25}$$

If we put $\rho := \limsup_{n\to\infty} \|f'_n\|_2^{1/n}$, then $\rho \leqq r$ by (1.8) and for any $\varepsilon > 0$ there exists $k_\varepsilon > 0$ such that

$$\sup_{k \geqq k_\varepsilon} \|f'_k\|_2^{1/k} < \rho + \varepsilon \leqq r + \varepsilon . \tag{2.26}$$

By (2.25) and (2.26) we have

$$\begin{aligned}|\psi(z)| &\leqq \sum_{n=0}^{\infty} |\psi_n(z)| \leqq \sum_{n=0}^{k_\varepsilon - 1} (1/n!) \|f'_n\|_2 (\|z\|/\sqrt{2})^m \\ &\quad + \sum_{n=k_\varepsilon}^{\infty} (1/n!)(r+\varepsilon)^n (\|z\|/\sqrt{2})^n \leqq C_\varepsilon \exp((r+\varepsilon)\|z\|/\sqrt{2})\end{aligned} \tag{2.27}$$

for all $z \in M$, where C_ε is a constant. From (2.27) we see that $\psi \in \mathrm{Exp}(M, r/\sqrt{2})$. Therefore, F is a one-to-one linear mapping of $\mathcal{O}'(\tilde{S}[r])$ into $\mathrm{Exp}(M, r/\sqrt{2})$. Conversely, if $\psi = \sum_{n=0}^{\infty} \psi_n$ belongs to $\mathrm{Exp}(M, r/\sqrt{2})$, then there exists $\tilde{\psi} = \sum_{n=0}^{\infty} \tilde{\psi}_n \in \mathcal{O}(C^{d+1})$ such that $\tilde{\psi}|_M = \psi$, $\tilde{\psi}_n|_M = \psi_n$ and that

$$\sup_{z \in N} |\tilde{\psi}(z) \exp(-r' \|z\|/\sqrt{2})| < \infty \qquad \text{for any} \quad r' > r . \tag{2.28}$$

(2.18), (2.20) and (2.28) imply

$$K_n \leqq C_{r'}(r'e/n)^n \tag{2.29}$$

for any $r' > r$ and a constant $C_{r'}$. If $Ff_n = \psi_n$ for $f_n \in H_{n,d}$ ($n = 0, 1, 2, \cdots$), from (2.16) and (2.29) we have

$$\limsup_{n\to\infty} \|f_n\|_2^{1/n} \leqq r'$$

for any $r' > r$. Hence we get

$$\limsup_{n\to\infty} \|f_n\|_2^{1/n} \leqq r \tag{2.30}$$

and (1.8) and (2.30) imply $f' = \sum_{n=0}^\infty f_n \in \mathcal{O}'(\tilde{S}[r])$. Thus we have (2.6).

Similarly, we get from (1.9)

$$F(\mathcal{O}'(\tilde{S}(r))) \subset \mathrm{Exp}[M, r/\sqrt{2}]\,.$$

On the other hand, if $\psi = \sum_{n=0}^\infty \psi_n$ belongs to $\mathrm{Exp}[M, r/\sqrt{2}]$, there exists $\tilde{\psi} = \sum_{n=0}^\infty \tilde{\psi}_n \in \mathcal{O}(C^{d+1})$ such that $\tilde{\psi}|_M = \psi$, $\psi_n|_M = \psi_n$ and that

$$\sup_{z\in M} |\tilde{\psi}(z)\exp(-r'\|z\|/\sqrt{2})| < \infty \tag{2.31}$$

for some $r' < r$. (2.31) implies

$$K_n \leqq C(r'e/n)^n\,, \tag{2.32}$$

where C is a constant. For $f_n \in H_{n,d}$ ($n = 0, 1, \cdots$) such that $Ff_n = \psi_n$, (2.16) and (2.32) give

$$\limsup_{n\to\infty} \|f_n\|_2^{1/n} \leqq r' < r\,. \tag{2.33}$$

(1.9) and (2.33) show $f' = \sum_{n=0}^\infty f_n \in \mathcal{O}'(\tilde{S}(r))$ and we obtain (2.7).

Using (1.11), (1.12) and (1.13) we can prove (2.8)–(2.10) similarly.

q.e.d.

Next we consider the image of $L^2(S)$ by F.

LEMMA 2.2 (c.f. Ii [3, Lemma 2.1]). *We denote the modified Bessel function K_ν by*

$$K_\nu(r) = \int_0^\infty \exp(-r\cosh t)\cosh \nu t dt \quad (\mathrm{Re}\,\nu > -(1/2),\ 0 < r < \infty)\,,$$

$$K_{-\nu}(r) = K_\nu(r)$$

and define the function $\rho_d(r)$ as follows:

$$\rho_d(r) := \begin{cases} \sum_{l=0}^{k} a_l r^{l+1} K_l(2r) & (\text{if } d \text{ is odd}) \\ \sum_{l=0}^{k} a_l r^{l+(1/2)} K_{l-(1/2)}(2r) & (\text{if } d \text{ is even})\,. \end{cases} \tag{2.34}$$

Then we can uniquely determine k and a_l $(l = 0, 1, \cdots, k)$ which satisfy

$$\int_0^\infty r^{2n+d-1}\rho_d(r)dr = C_n \qquad \text{for all} \quad n = 0, 1, 2, \cdots . \tag{2.35}$$

PROOF. It is known that

$$\int_0^\infty r^{\mu-1}K_\nu(ar)dr = 2^{\mu-2}a^{-\mu}\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right), \tag{2.36}$$

where $a > 0$ and $\operatorname{Re}\mu > |\operatorname{Re}\nu|$.

First we assume that d is odd. From (2.34) and (2.36) we get

$$\int_0^\infty r^{2n+d-1}\rho_d(r)dr = (1/4)\sum_{l=0}^{k} a_l\Gamma\left(n + \frac{d+1}{2}\right)\Gamma\left(n + l + \frac{d+1}{2}\right), \tag{2.37}$$

If (2.35) is valid, from (1.3), (1.22) and (2.37) we have

$$\begin{aligned}\sum_{l=0}^{k}\frac{1}{4}a_l\Gamma\left(n+\frac{d+1}{2}\right)\Gamma\left(n+l+\frac{d+1}{2}\right)\\ = C\Gamma\left(n+\frac{d+1}{2}\right)\Gamma(n+d-1)(2n+d-1)\end{aligned} \tag{2.38}$$

for any $n = 0, 1, 2, \cdots$, where C is a positive constant. Thus we have

$$\begin{aligned}\sum_{l=0}^{k} a_l\Gamma\left(n+l+\frac{d+1}{2}\right)\Big/\Gamma\left(n+\frac{d+1}{2}\right)\\ = 4C(2n+d-1)\Gamma(n+d-1)/\Gamma\left(n+\frac{d+1}{2}\right).\end{aligned} \tag{2.39}$$

Since $d \geqq 3$, we have $d - 1 \geqq (d+1)/2$. Hence the right hand side of (2.39) is a polynomial of n of degree $(d-1)/2$. Thus we obtain

$$k = (d-1)/2\,, \tag{2.40}$$

and

$$a_k = 8C > 0\,, \tag{2.41}$$

and we can determine $a_0, a_1, \cdots, a_{k-1}$ uniquely.

Next we assume that d is even. (2.34) and (2.36) imply

$$\int_0^\infty r^{2n+d-1}\rho_d(r)dr = \frac{1}{4}\sum_{l=0}^{k} a_l\Gamma\left(n+\frac{d+1}{2}\right)\Gamma\left(n+l+\frac{d}{2}\right) \tag{2.42}$$

and we get similarly

$$\begin{aligned}\sum_{l=0}^{k} a_l\Gamma\left(n+l+\frac{d}{2}\right)\Big/\Gamma\left(n+\frac{d}{2}\right)\\ = 4C(2n+d-1)\Gamma(n+d-1)/\Gamma\left(n+\frac{d}{2}\right)\end{aligned} \tag{2.43}$$

for $n = 0, 1, 2, \cdots$. Therefore we get

$$k = d/2 \tag{2.44}$$

and

$$a_k = 8C > 0 , \tag{2.45}$$

and $a_0, a_1, \cdots, a_{k-1}$ are determined uniquely. q.e.d.

REMARK 2.3. (1) Since it is known that

$$K_{n+1/2}(r) = (\pi/2r)^{1/2} e^{-r} \sum_{j=0}^{n} \frac{(n+j)!}{j!\,(n-j)!\,(2r)^j}$$

for $n = 0, 1, 2, \cdots$, there exists a polynomial $P_{d/2}(r)$ of degree $d/2$ such that $\rho_d(r) = e^{-2r} P_{d/2}(r)$, if d is even. This fact coincides with a result of Ii ([3, Lemma 2.1]). Though $K_\nu(r)$ is not defined at $r = 0$, $\rho_d(0)$ is well defined for even d by this fact.

(2) If d is odd, we have for $r > 0$

$$\begin{aligned} |\rho_d(r)| &\leqq \sum_{l=0}^{(d-1)/2} |a_l|\, r^{l+1} K_l(2r) \\ &\leqq \sum_{l=0}^{(d-1)/2} |a_l|\, r^{l+1} K_{l+1/2}(2r) = e^{-2r} r^{1/2} P_{(d-1)/2}(r) , \end{aligned} \tag{2.46}$$

where $P_{(d-1)/2}$ is a polynomial of degree $(d-1)/2$, since $0 \leqq K_l(r) \leqq K_{l+1/2}(r)$. Hence $\rho_d(r)$ is well defined at $r = 0$.

(3) If d is odd, by (2.41) $a_{(d-1)/2} > 0$. Hence we have for $r > 0$

$$\begin{aligned} \rho_d(r) &\geqq a_k r^{k+1} K_k(2r) - \sum_{l=0}^{k-1} |a_l|\, r^{l+1} K_l(2r) \\ &\geqq K_k(2r)\Big(a_k r^{k+1} - \sum_{l=0}^{k-1} |a_l|\, r^{l+1}\Big) , \end{aligned}$$

where we put $k := (d-1)/2$. Therefore $\rho_d(r) > 0$ for r sufficiently large.

For even d it is trivial by (1) that $\rho_d(r) > 0$ for r sufficiently large.

Now we define a measure μ_d on M by

$$\int_M f(z) d\mu_d(z) = \int_0^\infty r^{d-1} \Big(\int_N f(rz') dN(z')\Big) \rho_d(r) dr . \tag{2.47}$$

We define a subspace $P(M)$ of $\mathrm{Holo}(M)$ by

$$P(M) := \{\psi \in \mathrm{Holo}(M);\ \langle \psi, \psi \rangle_M < \infty\} , \tag{2.48}$$

where

$$\langle \psi, \varphi \rangle_M = \int_M \psi(z) \overline{\varphi(z)} d\mu_d(z) . \tag{2.49}$$

By Remark 2.3, (3) we can prove the following in the same way as in the proof of Ii [3, Theorem 2.5].

THEOREM 2.4 (cf. Ii [3, Theorem 2.5]). *F is a unitary isomorphism of $L^2(S)$ onto $P(M)$ with respect to $\langle\ ,\ \rangle_S$ and $\langle\ ,\ \rangle_M$.*

REMARK 2.5. Similarly, we can prove for odd d the results in Ii [3, Corollary 2.6-Theorem 2.11] given for even d.

3. The Fourier-Borel transformations of $\mathcal{O}'(\tilde{S}(r))$ and $\mathcal{O}'(\tilde{S}[r])$. In this section we consider the images of $\mathcal{O}'(\tilde{S}(r))$ and $\mathcal{O}'(\tilde{S}[r])$ by the Fourier-Borel transformation P_λ. Our second main theorem in this paper is the following:

THEOREM 3.1. *The transformation P_λ establishes linear topological isomorphisms*

$$(3.1)\qquad P_\lambda\colon \mathcal{O}'(\tilde{S}(r)) \xrightarrow{\sim} \mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}\colon [|\lambda| r\colon L^*]) \quad (1 < r \leqq \infty)\ ,$$

$$(3.2)\qquad P_\lambda\colon \mathcal{O}'(\tilde{S}[r]) \xrightarrow{\sim} \mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}\colon (|\lambda| r\colon L^*)) \quad (1 \leqq r < \infty)\ ,$$

where

$$\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}\colon [|\lambda| r\colon L^*]) := \mathcal{O}_\lambda(\boldsymbol{C}^{d+1}) \cap \mathrm{Exp}(\boldsymbol{C}^{d+1}\colon [|\lambda| r\colon L^*])$$

and

$$\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}\colon (|\lambda| r\colon L^*)) := \mathcal{O}_\lambda(\boldsymbol{C}^{d+1}) \cap \mathrm{Exp}(\boldsymbol{C}^{d+1}\colon (|\lambda| r\colon L^*))\ .$$

We need the following theorem in order to prove the theorem.

THEOREM 3.2 (Martineau [4]). *Suppose $\lambda \in \boldsymbol{C}$, $\lambda \neq 0$. The Fourier-Borel transformation P_λ establishes the linear topological isomorphisms*

$$(3.3)\qquad P_\lambda\colon \mathcal{O}'(\tilde{B}[r]) \xrightarrow{\sim} \mathrm{Exp}(\boldsymbol{C}^{d+1}\colon (|\lambda| r\colon L^*))\ ,$$

$$(3.4)\qquad P_\lambda\colon \mathcal{O}'(\tilde{B}(r)) \xrightarrow{\sim} \mathrm{Exp}(\boldsymbol{C}^{d+1}\colon [|\lambda| r\colon L^*])\ .$$

PROOF OF THEOREM 3.1. Since $\mathcal{O}'(\tilde{S}(r)) \subset \mathrm{Exp}'(\tilde{S}) \cap \mathcal{O}'(\tilde{B}(r))$ we have

$$P_\lambda(\mathcal{O}'(\tilde{S}(r))) \subset \mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}\colon [|\lambda| r\colon L^*])$$

by (1.14) and (3.4). Hence P_λ is a one-to-one linear mapping of $\mathcal{O}'(\tilde{S}(r))$ into $\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}\colon [|\lambda| r\colon L^*])$.

Conversely, let $\tilde{\psi}$ be in $\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}\colon [|\lambda| r\colon L^*])$. If we put $\tilde{\psi}|_M = \psi$, there exist $r' < r$ and $C > 0$ such that

$$|\psi(z)| \leqq C \exp(|\lambda| r' L^*(z)) = C \exp(|\lambda| r' \|z\|/\sqrt{2})$$

for any $z \in M$. So we get

$$(3.5)\qquad |\psi(-iz/\lambda)| \leqq C \exp(r' \|z\|/\sqrt{2}) \qquad \text{for } \forall z \in M\ .$$

Now we put $\tilde{\psi}_{-i/\lambda}(z):=\tilde{\psi}(-iz/\lambda)$. Then $\tilde{\psi}_{-i/\lambda}$ belongs to $\mathrm{Exp}[M, r/\sqrt{2}]$ from (3.5). By (2.7) there exists $f'\in\mathcal{O}'(\tilde{S}(r))$ such that

$$Ff'=\tilde{\psi}_{-i/\lambda}\,. \tag{3.6}$$

Since $\tilde{\psi}\in\mathcal{O}_\lambda(C^{d+1})$, we can find $h'\in\mathrm{Exp}'(\tilde{S})$ such that $\tilde{\psi}=P_\lambda h'$ by (1.14). Since $\tilde{\psi}(-iz/\lambda)=P_\lambda h'(-iz/\lambda)=Fh'(z)$ for all $z\in M$, we have from (3.6)

$$Fh'=Ff'\,. \tag{3.7}$$

By Theorem 2.1 and (3.7) we get $h'=f'$ and $\tilde{\psi}\in P_\lambda(\mathcal{O}'(\tilde{S}(r)))$. P_λ and P_λ^{-1} are continuous by (3.4) and the closed graph theorem. Therefore, we obtain (3.1). Using (3.3) and (2.6), we can prove (3.2) similarly. q.e.d.

Now we define the topology of $\mathrm{Holo}(M)$ to be the quotient topology $\mathcal{O}(C^{d+1})/\mathcal{J}(M)$ since $\mathrm{Holo}(M)=\mathcal{O}(C^{d+1})|_M$, where we put $\mathcal{J}(M):=\{f\in\mathcal{O}(C^{d+1});\ f=0 \text{ on } M\}$. We also define the topologies of $\mathrm{Exp}(M)$, $\mathrm{Exp}(M, r/\sqrt{2})$ $(1\leqq r<\infty)$ and $\mathrm{Exp}[M, r/\sqrt{2}]$ $(1<r\leqq\infty)$ similarly since we have $\mathrm{Exp}(M)=\mathrm{Exp}(C^{d+1})|_M$, $\mathrm{Exp}(M, r/\sqrt{2})=\mathrm{Exp}(C^{d+1}\colon (r\colon L^*))|_M$ $(1\leqq r<\infty)$ and $\mathrm{Exp}[M, r/\sqrt{2}]=\mathrm{Exp}(C^{d+1}\colon [r\colon L^*])|_M$ $(1<r\leqq\infty)$ by Theorem 2.1.

Then by Theorems 1.2, 2.1 and 3.1 and the closed graph theorem, we have:

COROLLARY 3.3. *The transformation F establishes the following linear topological isomorphisms*

$$F\colon \mathrm{Exp}'(\tilde{S})\xrightarrow{\sim}\mathrm{Holo}(M)\,. \tag{3.8}$$

$$F\colon \mathcal{O}'(\tilde{S})\xrightarrow{\sim}\mathrm{Exp}(M)\,. \tag{3.9}$$

$$F\colon \mathcal{O}'(\tilde{S}[r])\xrightarrow{\sim}\mathrm{Exp}(M, r/\sqrt{2})\quad \textit{for}\quad 1\leqq r<\infty\,. \tag{3.10}$$

$$F\colon \mathcal{O}'(\tilde{S}(r))\xrightarrow{\sim}\mathrm{Exp}[M, r/\sqrt{2}]\quad \textit{for}\quad 1<r\leqq\infty\,. \tag{3.11}$$

COROLLARY 3.4. (i) *For any $f\in\mathcal{O}(C^{d+1})$ there exists a unique $g\in\mathcal{O}_\lambda(C^{d+1})$ such that $f=g$ on M.*

(ii) *For any $f\in\mathcal{O}(C^{d+1})$ such that $\sup_{z\in M}|f(z)|\exp(-A\|z\|)<\infty$ for an $A>0$, there exists a unique $g\in\mathrm{Exp}_\lambda(C^{d+1})$ such that $f=g$ on M.*

(iii) *Assume that $1\leqq r<\infty$. For any $f\in\mathcal{O}(C^{d+1})$ such that $\sup_{z\in M}|f(z)|\exp(-|\lambda|r'\|z\|/\sqrt{2})<\infty$ for $\forall r'>r$, there exists a unique $g\in\mathrm{Exp}_\lambda(C^{d+1}\colon (|\lambda|r\colon L^*))$ such that $f=g$ on M.*

(iv) *Assume that $1<r\leqq\infty$. For any $f\in\mathcal{O}(C^{d+1})$ such that $\sup_{z\in M}|f(z)|\exp(-|\lambda|r'\|z\|/\sqrt{2})<\infty$ for some $r'<r$, there exists a unique $g\in\mathrm{Exp}_\lambda(C^{d+1}\colon [|\lambda|r\colon L^*])$ such that $f=g$ on M.*

PROOF. (i) If f belongs to $\mathcal{O}(C^{d+1})$ $f_{-i/\lambda}$ also belongs to $\mathcal{O}(C^{d+1})$. Then by Corollary 3.3 there exists $f' \in \mathrm{Exp}'(\tilde{S})$ such that $Ff' = f_{-i/\lambda}$ on M. If we put $g = P_\lambda f'$, g belongs to $\mathcal{O}_\lambda(C^{d+1})$ and $f = g$ on M by (1.14). The uniqueness follows from the injectivity of F.

By Theorem 1.2, Theorem 3.1 and Corollary 3.3 we can prove (ii), (iii), (iv) similarly. q.e.d.

REMARK. When $d = 1$ (the case of the unit circle), Corollary 3.4 is known (see Morimoto [5]).

REFERENCES

[1] L. DRUŻKOWSKI, Effective formula for the crossnorm in the complexified unitary spaces, Zeszyty Nauk. Uniw. Jagielloń., Prace Mat. 16 (1974), 47–53.

[2] M. HASHIZUME, A. KOWATA, K. MINEMURA AND K. OKAMOTO, An integral representation of an eigenfunction of the Laplacian on the Euclidean space, Hiroshima Math. J. 2 (1972), 535–545.

[3] K. II, On a Bargmann-type transform and a Hilbert space of holomorphic functions, Tôhoku Math. J. 38 (1986), 57–69.

[4] A. MARTINEAU, Équations differéntielles d'ordre infini, Bull. Soc. Math. France 95 (1967), 109–154.

[5] M. MORIMOTO, A generalization of the Fourier-Borel transformation for the analytic functionals with non convex carrier, Tokyo J. Math. 2 (1979), 301–322.

[6] M. MORIMOTO, Hyperfunctions on the Sphere, Sophia Kokyuroku in Mathematics, 12, Sophia University, Department of Math. Tokyo, 1982 (in Japanese).

[7] M. MORIMOTO, Analytic functionals on the sphere and their Fourier-Borel transformations, Complex Analysis, Banach Center Publications, 11, PWN-Polish Scientific Publishers, Warsaw, 1983, 223–250.

[8] C. MÜLLER, Spherical Harmonics, Lecture Notes in Math. 17, Springer-Verlag, Berlin, Heidelberg, New York, 1966.

[9] J. H. RAWNSLEY, Coherent states and Kähler manifolds, Quart. J. Math. Oxford Ser. (2) 28 (1977), no. 112, 403–415.

[10] J. H. RAWNSLEY, A nonunitary pairing of polarizations for the Kepler problem, Trans. Amer. Math. Soc. 250 (1979), 167–180.

[11] R. WADA, The Fourier-Borel transformations af analytic functionals on the complex sphere, Proc. Japan Acad., 61A, (1985), 298–301.

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