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DEFORMATIONS OF THREE DIMENSIONAL CUSP SINGULARITIES

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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Introduction. Freitag and Kiehl [1] showed that Hilbert modular cusp singularities of dimensions greater than two are rigid. On the other hand, we saw in [7] that there are many other 3-dimensional cusp singularities. Ogata [2] recently showed that those 3-dimensional cusp singularities are not rigid. The purpose of this paper is to obtain more precise information on deformations of 3-dimensional cusp singularities.

Let (V, p) be a 3-dimensional cusp singularity which is not of the Hilbert modular type. In Section 1, we calculate certain cohomology groups, which are related to deformations of the singularity (V, p). In Section 2, we first construct a family $(\mathscr{U}, \mathscr{X}) \to D$, over a polydisk D, of deformations of a resolution (U, X) of the singularity (V, p). Next, contracting $\mathscr{X}$ simultaneously, we obtain a family $\mathscr{V} \to D$ of deformations of the singularity (V, p). Finally, we see that the family $\mathscr{V} \to D$ is a versal family. Hence the cusp singularity (V, p) is neither taut nor smoothable.

1. Calculations of cohomology groups. We fix a 3-dimensional pair (C, Γ) in $\mathscr{S}$ (see [7]), throughout this paper. Recall that C is an open convex cone in $N_{\boldsymbol{R}}$, that Γ is a subgroup in $\mathrm{Aut}(N)$ preserving C and that $S := (C/\boldsymbol{R}_{>0})/\Gamma$ is a compact topological surface, where $N = \boldsymbol{Z}^3$. Also recall that we obtain from (C, Γ), a 3-dimensional cusp singularity (V, p) with $V \setminus \{p\} \simeq (\boldsymbol{R}^3 + \sqrt{-1}C)/N \cdot \Gamma$, where $N \cdot \Gamma$ is the semi-direct product of N and Γ.

Assume first that $\chi(S) < 0$ and that S is orientable. Let $T = N \otimes \boldsymbol{C}^\times$ and let $CT = N \otimes U(1)$, where $U(1) = \{z \in \boldsymbol{C}^\times \mid |z| = 1\}$. Then we have two Γ-equivariant exact sequences:

$$0 \to N \to N_{\boldsymbol{C}} \to T \to 1 ,$$
$$0 \to N \to N_{\boldsymbol{R}} \to CT \to 1 ,$$

where the third arrows are the maps induced by $\exp(2\pi\sqrt{-1}?)$: $\boldsymbol{C} \to \boldsymbol{C}^\times$. From these short exact sequences, we have the following long exact sequences of the cohomology groups with respect to the Γ-actions:

$$H^0(\Gamma, T) \to H^1(\Gamma, N) \to H^1(\Gamma, N_C) \to H^1(\Gamma, T) \to H^2(\Gamma, N) ,$$
$$H^0(\Gamma, CT) \to H^1(\Gamma, N) \to H^1(\Gamma, N_R) \to H^1(\Gamma, CT) \to H^2(\Gamma, N) .$$

The first purpose of this section is to calculate $H^1(\Gamma, L)$ for $L = N, N_R$ and N_C. Let

$$Z^1(\Gamma, L) = \{\varphi: \Gamma \to L \mid \varphi(\gamma\gamma') = \varphi(\gamma) + \gamma\varphi(\gamma') \quad \text{for} \quad \gamma, \gamma' \in \Gamma\} ,$$
$$B^1(\Gamma, L) = \{\delta l: \Gamma \to L \mid l \in L\} ,$$

where δl is the map sending γ to $\gamma l - l$. Then $Z^1(\Gamma, L)$ and $B^1(\Gamma, L)$ are K-modules and $H^1(\Gamma, L) = Z^1(\Gamma, L)/B^1(\Gamma, L)$, where $K = \boldsymbol{Z}$ (resp. $\boldsymbol{R}$, resp. $\boldsymbol{C}$) if $L = N$ (resp. N_R, resp. N_C).

LEMMA 1.1. $B^1(\Gamma, L) \simeq K^3$.

PROOF. It is sufficient to show that the linear map $L \ni l \mapsto \delta l \in B^1(\Gamma, L)$ is injective, because $L = N \otimes K$ and $N \simeq \boldsymbol{Z}^3$. Suppose not. Then there exists a nonzero element n in L such that $\gamma n = n$ for all γ in Γ. Hence for any point x_0 in C^*, the orbit $\Gamma x_0 := \{\gamma x_0 \mid \gamma \in \Gamma\}$ under Γ must be contained in the plane $\{x \in N_R^* \mid \langle x, n\rangle = \langle x_0, n\rangle\}$, where $C^* := \{x \in N_R^* \mid \langle x, y\rangle > 0$ for all $y \in \bar{C} \setminus \{0\}\}$ is the dual cone of C. However, (C^*, Γ) is in $\mathscr{S}$ by [7, Lemma 1.6], a contradiction (see the proof of [7, Lemma 1.1]). q.e.d.

Let χ be the Euler number of the compact orientable surface $S = (C/\boldsymbol{R}_{>0})/\Gamma$ and let $\gamma_1, \gamma_2, \cdots, \gamma_s$ be generators of Γ with the relation $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} \cdots \gamma_{s-1}^{-1}\gamma_s^{-1} = 1$, where $s = -\chi + 2$.

LEMMA 1.2. $Z^1(\Gamma, L) \simeq K^{3s-3}$.

PROOF. Let φ be an element in $Z^1(\Gamma, L)$. Then by the cocycle condition, we have

$$\begin{aligned} 0 &= \varphi(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} \cdots \gamma_{s-1}^{-1}\gamma_s^{-1}) \\ &= g_1\varphi(\gamma_1) + g_2\varphi(\gamma_2) + \cdots + g_s\varphi(\gamma_s) , \end{aligned}$$

where

$$g_{2k+1} := h_{2k+1}\alpha_k \qquad \text{with} \quad h_{2k+1} := (1 - \alpha_{k+1}\gamma_{2k+2}\alpha_k^{-1}) ,$$
$$g_{2k+2} := h_{2k+2}\alpha_k\gamma_{2k+1} \quad \text{with} \quad h_{2k+2} := (1 - \alpha_{k+1}\gamma_{2k+1}^{-1}\alpha_k^{-1})$$

for $k = 0$ through $s/2 - 1$ and $\alpha_k := \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} \cdots \gamma_{2k-1}\gamma_{2k}\gamma_{2k-1}^{-1}\gamma_{2k}^{-1} \in \Gamma$ for $k > 0$ and $\alpha_0 = 1$. Hence we have the exact sequence:

$$0 \to Z^1(\Gamma, L) \to L^s \xrightarrow{G} L ,$$

where the second arrow sends φ to $(\varphi(\gamma_1), \varphi(\gamma_2), \cdots, \varphi(\gamma_s))$ and the third arrow G sends $(l_1, l_2, \cdots, l_s)$ to $g_1l_1 + g_2l_2 + \cdots + g_sl_s$. Therefore, it is sufficient to show that the rank of the image of G is equal to 3. Suppose not. Then $h_1L + h_2L + \cdots + h_sL$ must be contained in a submodule M of rank 2. On the other hand $\beta_{2k+1} := \alpha_{k+1}\gamma_{2k+2}\alpha_k^{-1} = (1 - h_{2k+1})$ and $\beta_{2k+2} := \alpha_{k+1}\gamma_{2k+1}^{-1}\alpha_k^{-1} = (1 - h_{2k+2})$ with k running from 0 through $s/2 - 1$ are generators of Γ, because $\beta_{2k+1}^{-1}\beta_{2k+2}^{-1}\beta_{2k+1} = \alpha_k\gamma_{2k+1}\alpha_k^{-1}$ and $\beta_{2k+1}^{-1}\beta_{2k+2}\beta_{2k+1}\beta_{2k+2}^{-1}\beta_{2k+1} = \alpha_k\gamma_{2k+2}\alpha_k^{-1}$. Hence the orbit Γy under Γ of any point y in $C \subset N_{\boldsymbol{R}}$ must be contained in the plane $y + M'$, where $M' = M \otimes \boldsymbol{R}$, M or $M \cap N_{\boldsymbol{R}} (\subsetneqq N_{\boldsymbol{R}})$ according as $L = \boldsymbol{Z}, \boldsymbol{R}$ or $\boldsymbol{C}$. Thus we have the same contradiction as in the proof of Lemma 1.1. q.e.d.

By Lemmas 1.1 and 1.2, we have:

PROPOSITION 1.3. *$H^1(\Gamma, N) \simeq \boldsymbol{Z}^{-3\chi} \oplus$ torsion, $H^1(\Gamma, N_{\boldsymbol{R}}) \simeq \boldsymbol{R}^{-3\chi}$ and $H^1(\Gamma, N_{\boldsymbol{C}}) \simeq \boldsymbol{C}^{-3\chi}$.*

PROPOSITION 1.4. *The connected components of the unit elements in $H^1(\Gamma, T)$ and $H^1(\Gamma, CT)$ are an algebraic torus $(\boldsymbol{C}^\times)^{-3\chi}$ and a compact real torus $U(1)^{-3\chi}$, respectively, of dimensions -3χ.*

PROOF. The map $H^1(\Gamma, N) \to H^1(\Gamma, L)$ is induced by the injective map $Z^1(\Gamma, N) \to Z^1(\Gamma, L)$ and $Z^1(\Gamma, N) \otimes K = Z^1(\Gamma, L)$, where $K = \boldsymbol{R}$ or $\boldsymbol{C}$ and $L = N \otimes K$. Hence $\operatorname{coker}(H^1(\Gamma, N) \to H^1(\Gamma, L)) \simeq (K/\boldsymbol{Z})^{-3\chi}$ q.e.d.

Now we consider the case where $S = (C/\boldsymbol{R}_{>0})/\Gamma$ is not orientable with the Euler number χ. Also in this case, Lemma 1.1 continues to hold, $\dim_{\boldsymbol{Z}} Z^1(\Gamma, N) = \dim_{\boldsymbol{R}} Z^1(\Gamma, N_{\boldsymbol{R}}) = \dim_{\boldsymbol{C}} Z^1(\Gamma, N_{\boldsymbol{C}})$, by the proof of Lemma 1.2 and hence $\dim_{\boldsymbol{Z}} H^1(\Gamma, N) = \dim_{\boldsymbol{R}} H^1(\Gamma, N_{\boldsymbol{R}}) = \dim_{\boldsymbol{C}} H^1(\Gamma, N_{\boldsymbol{C}})$. Therefore, we see as in the proof of the above proposition that the connected components of the unit elements in $H^1(\Gamma, T)$ and $H^1(\Gamma, CT)$ are an algebraic torus and a compact real torus, respectively. Moreover, the dimensions of the tori are not smaller than -3χ, by [2, Theorems 1 and 3]. Thus we conclude that 3-dimensional cusp singularities are not taut, by [8, Proposition 3.2], if they are not Hilbert modular cusp singularities, because then $\chi < 0$, by [7, Theorem 3.1 and Corollary 3.2].

2. Versal families of deformations of 3-dimensional cusp singularities. We keep the notations in the previous section. Recall that we have a resolution $(U, X) \to (V, p)$ of the cusp singularity (V, p) such that the exceptional set X is a toric divisor (see [7] and [8]). Here U and X are the quotient spaces under Γ of an open set $\tilde{U}$ of a non-singular torus embedding $T\operatorname{emb}(\Sigma)$ of T and the union of its 2-dimensional orbits $\tilde{X}$, respectively, such that $\tilde{U} \setminus \tilde{X} = \operatorname{ord}^{-1}(C)$ is the inverse image of the cone C under the map ord: $T \to N_{\boldsymbol{R}}$ induced by $-\log|\ |: \boldsymbol{C}^\times \to \boldsymbol{R}$.

First, we construct a finite open covering of X. We note that (N, Σ) is a Γ-invariant non-singular r.p.p. decomposition of N_R with $|\Sigma|$ $(:= \bigcup_{\sigma\in\Sigma}\sigma) = C\cup\{0\}$. For each 3-dimensional cone $\sigma = \boldsymbol{R}_{\geqq 0}l^1 + \boldsymbol{R}_{\geqq 0}l^2 + \boldsymbol{R}_{\geqq 0}l^3$ in Σ, let

$$\sigma(\eta, \delta) = \{x^1l^1 + x^2l^2 + x^3l^3 \,|\, x^1 + x^2 + x^3 > \eta,\ x^1, x^2, x^3 > -\delta\}$$

and let $\tilde{U}_\sigma(\eta, \delta)$ be the interior of the closure of $\mathrm{ord}^{-1}(\sigma(\eta, \delta))$ in $T\mathrm{emb}(\Sigma)$. Let $\sigma_1, \sigma_2, \cdots, \sigma_I$ be representatives of 3-dimensional cones in Σ modulo Γ, i.e., $\bigcup_{1\leqq j\leqq I, \gamma\in\Gamma}\gamma\sigma_j = C\cup\{0\}$ and $\sigma_i \neq \gamma\sigma_j$ for any γ in Γ, if $i \neq j$. Let

$$U_j = q(\tilde{U}_{\sigma_j}(\eta, \delta)) , \qquad U_j' = q(\tilde{U}_{\sigma_j}(\eta', \delta')) ,$$

for large enough $\eta > \eta' > 0$ and for small enough $\delta' > \delta > 0$, where $q\colon \tilde{U} \to U$ is the quotient map under Γ. Then $\bar{U}_j \subset U_j'$ and $\{U_j\}$ is an open covering of X. Moreover, we may impose the following assumption, replacing Σ by a non-singular subdivision of it, if necessary.

ASSUMPTION 1. For each pair (i, j), the set $\{\gamma\in\Gamma \,|\, \sigma_i\cap\gamma\sigma_j \neq \{0\}\}$ is not empty if and only if $U_i'\cap U_j' \neq \varnothing$ and then it consists of only one element, which we denote by γ_{ij}. Then clearly $\gamma_{ii} = 1$ and $\gamma_{ji} = \gamma_{ij}^{-1}$. Moreover, $\gamma_{ki} = \gamma_{kj}\gamma_{ji}$, if $U_k'\cap U_j'\cap U_i' \neq \varnothing$, because then $\sigma_k\cap\gamma_{kj}\sigma_j\cap\gamma_{ki}\sigma_i \neq \{0\}$ and $\sigma_j\cap\gamma_{ji}\sigma_i \neq \{0\}$.

By this assumption, the restriction $q_i\colon \tilde{U}_{\sigma_i}(\eta', \delta') \to U_i'$ to $\tilde{U}_{\sigma_i}(\eta', \delta')$ of the quotient map $q\colon \tilde{U} \to U$ is a biholomorphic map and $U_i'\cap U_j'$ is connected or empty.

Next, we define a local coordinate on each U_i'. Fix a basis (n^1, n^2, n^3) of N. Let $\sigma_i = \boldsymbol{R}_{\geqq 0}l_i^1 + \boldsymbol{R}_{\geqq 0}l_i^2 + \boldsymbol{R}_{\geqq 0}l_i^3$, let $(l_i^1, l_i^2, l_i^3) = (n^1, n^2, n^3)A_i$ $(A_i \in GL(N))$ and let (m_1, m_2, m_3) be the basis of $\mathrm{Hom}(N, \boldsymbol{Z})$ dual to (l_i^1, l_i^2, l_i^3). Then we have the holomorphic immersion:

$$\psi_i\colon U_i' \hookrightarrow T\mathrm{emb}(\{\text{faces of } \sigma_i\}) \simeq \boldsymbol{C}^3$$

sending z to $(e(m_1)(q_i^{-1}(z)), e(m_2)(q_i^{-1}(z)), e(m_3)(q_i^{-1}(z)))$, where $e(m)\colon T\mathrm{emb}(\{\text{faces of } \sigma_i\}) \to \boldsymbol{C}$ is the natural extension of the character $m\otimes\boldsymbol{C}^\times\colon T\to\boldsymbol{C}^\times$ of $m\in\mathrm{Hom}(N, \boldsymbol{Z})$. For each pair (i, j) with $U_i'\cap U_j' \neq \varnothing$, let $f_{ji}\colon \psi_i(U_i'\cap U_j') \to \psi_j(U_i'\cap U_j')$ be the composite of the restriction of ψ_i^{-1} to $\psi_i(U_i'\cap U_j')$ and ψ_j. Then f_{ji} is written in terms of monomials, i.e.,

$$f_{ji}(w^1, w^2, w^3) = \left(\prod_{\alpha=1}^{3}(w^\alpha)^{a_{\alpha 1}} ,\quad \prod_{\alpha=1}^{3}(w^\alpha)^{a_{\alpha 2}} ,\quad \prod_{\alpha=1}^{3}(w^\alpha)^{a_{\alpha 3}}\right) ,$$

where $(a_{\alpha\beta}) = {}^t(A_j^{-1}\gamma_{ji}A_i)$. Hence we have the maximal set W_{ji} among open sets in $\boldsymbol{C}^3$ on which the analytic continuations of f_{ji} are holomorphic. Clearly W_{ji} is defined by $w^\alpha \neq 0$ or $w^\alpha w^\beta \neq 0$ according as $\sigma_i\cap\gamma_{ij}\sigma_j$ is a

2-dimensional cone or a 1-dimensional cone and $W_{ii} = C^3$. We denote by $\bar{f}_{ji}$, the analytic continuation of f_{ji} to W_{ji}. Then we easily see that $\bar{f}_{ji}(W_{ji}) = W_{ij}$ and that $\{w \in \psi_i(U_i) \cap W_{ji} \mid \bar{f}_{ji}(w) \in \psi_j(U_j)\} = \psi_i(U_i \cap U_j)$.

Let H be a complementary subspace of $B^1(\Gamma, N_C)$ in $Z^1(\Gamma, N_C)$ and let D be a polydisc in H. In the following, we construct a family over D of deformations of the pair (U, X) by patching up $\{\psi_i(U_i)\times D\}_{1\leq i\leq I}$. For each pair (i, j) with $U'_i \cap U'_j \neq \emptyset$, let

$$F_{ji}(w, \varphi) = (\bar{\varphi}\bar{f}_{ji}(w), \varphi) \qquad (w, \varphi) \in W_{ji}\times D ,$$

where $\bar{\varphi} = \exp(2\pi\sqrt{-1}\,{}^t\{A_j^{-1}\varphi(\gamma_{ji})\})$ and $(\varphi^1, \varphi^2, \varphi^3)(z^1, z^2, z^3) = (\varphi^1 z^1, \varphi^2 z^2, \varphi^3 z^3)$. Then F_{ji} is a biholomorphic map from $W_{ji}\times D$ to $W_{ij}\times D$. If $U'_i \cap U'_j \cap U'_k \neq \emptyset$, then $F_{ki} = F_{kj}\circ F_{ji}$ on $(W_{ki}\cap W_{ji})\times D \neq \emptyset$, because $\gamma_{ki} = \gamma_{kj}\gamma_{ji}$. If D is small enough, then we may assume the following:

ASSUMPTION 2. The closures of $\{(w, \varphi) \in (\psi_i(U_i)\cap W_{ji})\times D \mid F_{ji}(w, \varphi) \in \psi_j(U_j)\times D\}$ and $F_{ji}((\psi_i(U_i)\cap W_{ji})\times D) \cap \psi_j(U_j)\times D$ are contained in $\psi_i(U'_i \cap U'_j)\times D$ and $\psi_j(U'_i \cap U'_j)\times D$, respectively, for each pair (i, j) with $U'_i \cap U'_j \neq \emptyset$.

DEFINITION 2.1. $p \sim q$ for two points p and q in $\psi_i(U_i)\times D$ and $\psi_j(U_j)\times D$, respectively, if $U'_i \cap U'_j \neq \emptyset$, if $p \in W_{ji}\times D$ and if $F_{ji}(p) = q$.

LEMMA 2.2. *The relation in Definition* 2.1 *is an equivalence relation in the disjoint union of* $\{\psi_i(U_i)\times D\}_{1\leq i\leq I}$.

PROOF. Since the reflexive law and the symmetric law are trivial, we only prove the transitive law. Let p, q and r be points in $\psi_i(U_i)\times D$, $\psi_j(U_j)\times D$ and $\psi_k(U_k)\times D$, respectively, and assume that $p \sim q$ and that $q \sim r$. Then by Assumption 2, q is contained in both $\psi_j(U'_i \cap U'_j)\times D$ and $\psi_j(U'_j \cap U'_k)\times D$. Hence $U'_i \cap U'_j \cap U'_k \neq \emptyset$ and $F_{ki}(p) = F_{kj}(F_{ji}(p)) = F_{kj}(q) = r$. Thus we have $p \sim r$. q.e.d.

Let $\mathscr{U} = (\coprod_{i=1}^{I} \psi_i(U_i)\times D)/\sim$ be the quotient space of $\coprod_{i=1}^{I} \psi_i(U_i)\times D$ by the above equivalence relation.

LEMMA 2.3. *$\mathscr{U}$ is a Hausdorff space.*

PROOF. Let p and q be points in $\psi_i(U_i)\times D$ and $\psi_j(U_j)\times D$, respectively, and suppose that $U_p \cap U_q \neq \emptyset$ for any neighborhoods U_p and U_q of p and q, respectively. Then there exist sequences $\{p_a\}$ and $\{q_a\}$ of points in $(\psi_i(U_i)\cap W_{ji})\times D$ and $(\psi_j(U_j)\cap W_{ij})\times D$ converging to p and q, respectively, with $F_{ji}(p_a) = q_a$. By Assumption 2, $p \in \psi_i(U'_i \cap U'_j)\times D \subset W_{ji}\times D$ and $F_{ji}(p) = q$. Hence $p \sim q$. q.e.d.

By this lemma, $\mathscr{U}$ is a complex manifold. Let $\pi\colon \mathscr{U} \to D$ be the natural projection and let $\mathscr{X} = \bigcup_{i=1}^{I}\{(w^1, w^2, w^3, \varphi) \in \psi_i(U_i)\times D \mid w^1w^2w^3 = 0\}$. Then π is a smooth holomorphic map and $X_\varphi := \mathscr{X} \cap \pi^{-1}(\varphi)$ are compact divisors in $U_\varphi := \pi^{-1}(\varphi)$ for all φ in D, if D is small enough. Clearly, there is an immersion $U_0 \hookrightarrow U$ mapping X_0 onto X. Hence X_0 is contractible to a point.

PROPOSITION 2.4. *$X_\varphi \simeq X_{[\theta]}$ $(:= \tilde{X}/\{\theta(\gamma)\gamma \mid \gamma \in \Gamma\})$ for any φ in D, where θ is the image of φ under the map $Z^1(\Gamma, N_C) \to Z^1(\Gamma, T)$ induced by* $\exp(2\pi\sqrt{-1}?)$: *$N_C \to T$. Hence X_φ is a toric divisor.* (*See* [8, §3].)

PROOF. Let $X_i := \{\psi_i(U_i)\times D\} \cap X_\varphi$ and let r_i be the restriction to X_i of the composite $q_i^{-1}\circ\psi_i^{-1}\circ p_i$ of the maps $p_i\colon \psi_i(U_i)\times D \to \psi_i(U_i)$, $\psi_i^{-1}\colon \psi_i(U_i) \xrightarrow{\sim} U_i$ and $q_i^{-1}\colon U_i' \xrightarrow{\sim} \tilde{U}_{\sigma_i}(\eta', \delta') \subset \tilde{U}$, where p_i is the natural projection. Then $\bigcup_{i=1}^{I} X_i = X_\varphi$ and the image $r_i(X_i)$ under r_i is contained in $\tilde{X}$. Let s_i be the composite of r_i and the quotient map $\tilde{X} \to X_{[\theta]}$ under $\{\theta(\gamma)\gamma \mid \gamma \in \Gamma\}$. Then $s_i\colon X_i \hookrightarrow X_{[\theta]}$ is a holomorphic immersion. Moreover, we see by an easy calculation that $s_i(p_i) = s_j(p_j)$ for any points p_i and p_j in X_i and X_j, respectively, if and only if $F_{ji}(p_i) = p_j$. Hence we have a holomorphic immersion $s\colon X_\varphi \to X_{[\theta]}$. Since X_φ is compact, s is an isomorphism. q.e.d.

LEMMA 2.5. *For each positive integer i, $\dim H^i(U_\varphi, \mathcal{O}_{U_\varphi})$ are constant for φ small enough.*

PROOF. Consiser the exact sequences:

$$0 \to \mathcal{O}_{U_\varphi}(-X_\varphi) \to \mathcal{O}_{U_\varphi} \to \mathcal{O}_{X_\varphi} \to 0\ .$$

Let $f\colon (U_0, X_0) \to (V_0, p_0)$ be the contraction map. If we choose an open set in $\mathscr{U}$ so that $f(U_0) = V_0$ is a Stein space, then $H^i(U_0, \mathcal{O}_{U_0}(-X_0)) = R^if_*\mathcal{O}_{U_0}(-X_0) = 0$ for $i > 0$, by [7, Theorem 2.3]. Then by [5, Satz 1] and [4, Theorem 1.6], we have $H^i(U_\varphi, \mathcal{O}_{U_\varphi}(-X_\varphi)) = 0$ for $i > 0$ and for φ small enough. Hence we have $H^i(U_\varphi, \mathcal{O}_{U_\varphi}) \simeq H^i(X_\varphi, \mathcal{O}_{X_\varphi})$ for $i > 0$. On the other hand, $\dim H^i(X_\varphi, \mathcal{O}_{X_\varphi}) = \dim H^i(X_0, \mathcal{O}_{X_0})$ $(=\dim H^i(S, \boldsymbol{C}))$ for $i > 0$, because X_φ are toric divisors whose dual graphs are equal to that of X (see the proof of [7, Proposition 2.7]). Hence $\dim H^i(U_\varphi, \mathcal{O}_{U_\varphi}) = \dim H^i(U_0, \mathcal{O}_{U_0})$. q.e.d.

By this lemma and [3], for D small enough, $\mathscr{X}$ can be simultaneously blown-down in $\mathscr{U}$. Hence we obtain a family $\mathscr{V} \to D$ over D of deformations of the isolated 3-dimensional singularity (V_0, p_0), which is isomorphic to some open set of (V, p).

THEOREM 2.6. *The family $\mathscr{V} \to D$ is versal, i.e., the infinitesimal deformation map (the Kodaira-Spencer map) $\rho\colon T_0(D) \to T^1_{V_0}$ is bijective.*

PROOF. Since $\mathscr{U}\setminus\mathscr{X}\to D$ is a family of deformations of the complex manifold $U_0\setminus X_0$, we have the infinitesimal deformation map ρ': $T_0(D)\to H^1(U_0\setminus X_0, \Theta)$, where Θ is the sheaf of germs of vector fields on U_0. Since D is a polydisc in $H^1(\Gamma, N_C)$ and since there is a canonical isomorphism $H^1(\Gamma, N_C)\simeq H^1(U_0\setminus X_0, \Theta)$ ([2, Theorem 1]), the map ρ' is bijective, by the construction of $\mathscr{U}$. Hence the map ρ must be bijective, because a canonical injection $T^1_{V_0}\to H^1(U_0\setminus X_0, \Theta)$ is bijective, by [6] and [2, Theorem 1]. q.e.d.

COROLLARY 2.7. *The cusp singularity (V, p) is not smoothable.*

REMARK. Also for any higher dimensional pair (C, Γ) in $\mathscr{S}$, we can construct a versal family, over a small polydisc in $H^1(\Gamma, N_C)$, of deformations of the cusp singularity $(V, p) = \mathrm{Cusp}(C, \Gamma)$, in the same way.

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