Tôhoku Math. Journ.
38 (1986), 365-370.

OUTRADII OF THE TEICHMÜLLER SPACES OF FUCHSIAN GROUPS OF THE SECOND KIND

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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(Received July 3, 1985)

1. Introduction. Let $o(\Gamma)$ be the outradius of the Teichmüller space $T(\Gamma)$ of a Fuchsian group Γ. Then $o(\Gamma)$ is strictly greater than 2 (Earle [5]) and not greater than 6 (Nehari [7]). A Fuchsian group is said to be of the first kind (resp. second kind) if its region of discontinuity is not connected (resp. connected). If Γ is a finitely generated Fuchsian group of the first kind, then $o(\Gamma)$ is strictly less than 6 ([9]). Recently the authors proved, by using a basic result on the stability of finitely generated Fuchsian groups (Bers [3]), that $o(\Gamma)$ is equal to 6 for a finitely generated Fuchsian group Γ of the second kind ([10]). In this paper we give an alternative proof of it, which works also for an *infinitely generated* Fuchsian group of the second kind.

THEOREM. *If Γ is a Fuchsian group of the second kind, then $o(\Gamma)$ is equal to* 6.

The authors would like to thank the referee for pointing out some errors in the original manuscript.

2. Definitions. Let Δ be the open unit disc and Δ^* be the exterior of Δ in the Riemann sphere $\hat{C}$. For each function f which is conformal in Δ^* let $\{f, z\}$ be the Schwarzian derivative of f, that is, $\{f, z\} = (f''/f')' - (1/2)(f''/f')^2$. Let Γ be a Fuchsian group keeping Δ invariant. A quasiconformal automorphism w of $\hat{C}$ is said to be compatible with Γ if $w\circ\gamma\circ w^{-1}$ is a Möbius transformation for each $\gamma\in\Gamma$. Let w be a quasiconformal automorphism of $\hat{C}$ which is compatible with Γ and which is conformal in Δ^*. The Teichmüller space $T(\Gamma)$ of Γ is the set of the Schwarzian derivatives $\{w\,|\,\Delta^*, z\}$ of such w's restricted to Δ^*. Let $\lambda(z) = (|z|^2 - 1)^{-1}$ be a Poincaré density of Δ^*. For a function ϕ defined in Δ^* let $\|\phi\| = \sup_{z\in\Delta^*}\lambda(z)^{-2}|\phi(z)|$. The outradius $o(\Gamma)$ of $T(\Gamma)$ is defined to be $\sup\|\phi\|$, where the supremum is taken over all ϕ in $T(\Gamma)$.

3. Lemmas. In this section we state two lemmas without proof.

Lemma 1 is due to Chu [4]. Lemma 2 is proved in §§5–6. Let $k(z) = z + z^{-1}$. Then k maps Δ^* conformally onto $\hat{C}$ with the closed real segment $[-2, 2]$ removed. Let S_r be the circle of radius r (>1) around the origin. Then the image of S_r under k is the ellipse

$$E_r\colon \xi^2/(r + r^{-1})^2 + \eta^2/(r - r^{-1})^2 = 1 ,$$

where $\zeta = k(z)$ and $\zeta = \xi + \eta\sqrt{-1}$.

For two Jordan loops J_1 and J_2 in the finite complex plane C we define the Fréchet distance $\delta(J_1, J_2)$ as $\inf \max_{0\leqq t\leqq 1}|z_1(t) - z_2(t)|$, where the infimum is taken over all possible parametrizations $z_i(t)$ of J_i ($i = 1, 2$).

LEMMA 1 (Chu [4]). *For each positive* ε *there exist constants* $r_1 > 1$ *and* $d_1 > 0$ *so that if* $E_{r_1} = k(S_{r_1})$ *and if* J *is a Jordan loop in* C *with* $\delta(J, E_{r_1}) \leqq d_1$, *then a conformal mapping* f *of* Δ^* *onto the exterior of* J *satisfies* $\|\{f, z\}\| > 6 - \varepsilon$.

Denote by $\mu[w]$ the complex dilatation of a quasiconformal mapping w.

LEMMA 2. *Let* Γ *be a Fuchsian group of the second kind keeping* Δ *invariant. Then for each* $r > 1$ *and* $d > 0$ *there exist a sequence* $\{\sigma_n\}_{n=1}^\infty$ *of Möbius transformations and a sequence* $\{F_n\}_{n=1}^\infty$ *of quasiconformal automorphisms of* $\hat{C}$ *which satisfy the following.*

$$F_n \circ \gamma = \gamma \circ F_n \quad \text{for all} \quad \gamma \in \Gamma . \tag{3.1}$$

$$F_n \circ \sigma_n(\infty) \in \Delta^* . \tag{3.2}$$

$$\lim_{n\to\infty} \|\mu[F_n^{-1} | \Delta^*]\|_\infty = 0 . \tag{3.3}$$

$$\delta(\sigma_n^{-1} \circ F_n^{-1}(\partial\Delta), E_r) \leqq d . \tag{3.4}$$

4. Proof of Theorem. For each $\varepsilon > 0$ let r_1 and d_1 be the constants in Lemma 1. Let $\{\sigma_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ be sequences of Möbius transformations and quasiconformal automorphisms, respectively, obtained from Lemma 2 for $r = r_1$ and $d = d_1/2$.

Set $\nu_n(z) = \mu[F_n^{-1} | \Delta](z)$ for $z \in \Delta$ and $=0$ for $z \in \Delta^*$. Let w_n be the ν_n-conformal automorphism of $\hat{C}$ which sends $F_n \circ \sigma_n(0)$, $F_n \circ \sigma_n(1)$ and $F_n \circ \sigma_n(\infty)$ to 0, 1 and ∞, respectively (Ahlfors [1, p. 98]). Then w_n is compatible with Γ by (3.1) and the quasiconformal automorphism $W_n = w_n \circ F_n \circ \sigma_n$ of $\hat{C}$ keeps 0, 1, and ∞ fixed. Since $W_n(\infty) = \infty$, (3.2) implies $w_n^{-1}(\infty) = F_n \circ \sigma_n \circ W_n^{-1}(\infty) = F_n \circ \sigma_n(\infty) \in \Delta^*$. Hence w_n maps Δ^* conformally onto the exterior of $w_n(\partial\Delta)$. Since both $\mu[w_n | \Delta]$ and $\mu[\sigma_n^{-1} \circ F_n^{-1} | \Delta]$ are equal to $\nu_n | \Delta$, $\mu[W_n | \sigma_n^{-1} \circ F_n^{-1}(\Delta)]$ vanishes ([1, p. 9]). Hence

$$\|\mu[W_n]\|_\infty = \|\mu[W_n | \sigma_n^{-1} \circ F_n^{-1}(\varDelta^*)]\|_\infty = \|\mu[F_n | F_n^{-1}(\varDelta^*)]\|_\infty = \|\mu[F_n^{-1} | \varDelta^*]\|_\infty .$$

Therefore $\lim_{n\to\infty} \|\mu[W_n]\|_\infty = 0$ by (3.3). By a result on quasiconformal mappings (Ahlfors-Bers [2, Lemma 17]), we see the existence of a positive integer n_1 so that

$$|W_{n_1}(z) - z| \leqq d_1/2$$

for all z with $\mathrm{dist}(z, E_{r_1}) \leqq d_1/2$. This shows

$$\delta(w_{n_1}(\partial\varDelta), \sigma_{n_1}^{-1} \circ F_{n_1}^{-1}(\partial\varDelta)) \leqq d_1/2 .$$

Hence this together with (3.4) implies that $\delta(w_{n_1}(\partial\varDelta), E_{r_1}) \leqq d_1$. Now Lemma 1 shows $\|\{w_{n_1} | \varDelta^*, z\}\| > 6 - \varepsilon$. Recall that $\{w_{n_1} | \varDelta^*, z\}$ is in $T(\Gamma)$. Then we see $o(\Gamma) > 6 - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $o(\Gamma) \geqq 6$. On the other hand $o(\Gamma) \leqq 6$ (Nehari [7]). Therefore $o(\Gamma) = 6$. This completes the proof of Theorem.

5. A sequence of quasiconformal mappings. Let $\{\delta_n\}_{n=1}^\infty (\subset (0, 1))$ be a decreasing sequence with $\lim_{n\to\infty} \delta_n = 0$. Let $V_n = \{z \in \boldsymbol{C}; |z| < \delta_n\}$. Let j_n be a smooth closed Jordan arc in $\mathrm{Cl}\, V_n$ which joins $-\delta_n$ to δ_n. Set $l_n = [-1, -\delta_n) \cup j_n \cup (\delta_n, 1]$. Let U and L be the upper and lower half-planes, respectively. Let $B = \{z \in \boldsymbol{C}; |\mathrm{Re}\, z| < 1,\ 0 < \mathrm{Im}\, z < 1\}$. Then both $\alpha_n = l_n \cup (L \cap \partial\varDelta)$ and $\beta_n = l_n \cup (U \cap \partial B)$ are Jordan loops. Denote by A_n and B_n the interiors of α_n and β_n, respectively. Let $A = \{z \in L; |z| < 1\}$ and $C = \{z \in L; 1 < |z| < 2\}$. Let Ω be the interior of $\mathrm{Cl}(A \cup B \cup C)$. The purpose of this section is to prove the following lemma.

LEMMA 3. *There exists a sequence of quasiconformal automorphisms* $\{G_n\}_{n=1}^\infty$ *of* Ω *with* $G_n(z) = z$ *for all* $z \in \partial\Omega$ *which satisfy the following.*

(i) $G_n(l_n) = \partial U \cap \mathrm{Cl}\, A$ *and* $G_n(A_n) = A$.

(ii) $\lim_{n\to\infty} \|\mu[G_n^{-1} | \Omega \cap L]\|_\infty = 0$.

It is known that every quasiconformal mapping between Jordan domains can be extended to a homeomorphism between their closures (Lehto-Virtanen [6, p. 42]). Therefore from now on a quasiconformal mapping of a Jordan domain D onto another means a homeomorphism of $\mathrm{Cl}\, D$ which is quasiconformal in D.

Let f_n be the conformal mapping which maps A_n onto A and which keeps 1, -1 and $-\sqrt{-1}$ invariant. Let R_n be the annulus $\{z \in \boldsymbol{C}; \delta_n < |z| < \delta_n^{-1}\}$. Then by the reflection principle $f_n | A_n \cap R_n$ can be continued analytically to R_n beyond the unit circle and beyond the real line. Thus f_n has a conformal extension to $A_n \cup R_n$, for which by abuse of language we use the same letter f_n. Before proving Lemma 3, we prove Lemmas 4–6 which play essential roles in the proof of Lemma 3.

LEMMA 4. *The sequence $\{f_n\}_{n=1}^{\infty}$ converges to the identity transformation uniformly in R_1.*

PROOF. Each f_n fixes 1, -1 and $-\sqrt{-1}$. Hence $\{f_n\}_{n=m}^{\infty}$ is a normal family in R_m (Lehto-Virtanen [6, p. 73]). By a diagonal argument we obtain a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ which converges uniformly in R_{n_1}, in particular, in R_1 to a conformal mapping f_∞ of $\cup_{i=1}^{\infty} R_{n_i} = \boldsymbol{C} - \{0\}$ ([6, p. 74]). Since f_∞ can be extended to a conformal automorphism of $\hat{\boldsymbol{C}}$ and since f_∞ fixes 1, -1 and $-\sqrt{-1}$, f_∞ is the identity transformation. By the same reasoning as above any other convergent subsequence of $\{f_n\}_{n=1}^{\infty}$ than $\{f_{n_i}\}_{i=1}^{\infty}$ also converges to the identity transformation uniformly in R_1, and so does the sequence $\{f_n\}_{n=1}^{\infty}$ itself. q.e.d.

LEMMA 5. *There exists a quasiconformal mapping g_n of B_n onto B so that $g_n(z) = f_n(z)$ for all $z \in l_n$ and $g_n(z) = z$ for all $z \in \beta_n - l_n$.*

PROOF. Put $q_n(z) = f_n(z)$ if $z \in l_n$ and $=z$ if $z \in \beta_n - l_n$. Then q_n is a homeomorphism of a Jordan loop β_n onto another ∂B. For each point p of β_n we shall show the existence of an open subarc J_p of β_n containing p such that $q_n | J_p$ has a quasiconformal extension to $\hat{\boldsymbol{C}}$. Then by a theorem of Rickman ([8, Theorem 4]) q_n has a quasiconformal extension g_n to $\hat{\boldsymbol{C}}$. Since g_n is sense-preserving, g_n maps B_n onto B.

First let $p \in \beta_n \cap U$. Then $\beta_n \cap U$ is an open subarc of β_n containing p and $q_n | \beta_n \cap U$ has a quasiconformal extension to $\hat{\boldsymbol{C}}$, which is the identity mapping. Secondly, let $p \in l_n - \{\pm 1\}$. Then $l_n - \{\pm 1\}$ is an open subarc of β_n. Since both α_n and ∂A consist of finitely many smooth arcs which meet pairwise at non-zero angles, they are quasicircles (Lehto-Virtanen [6, p. 104]). Hence f_n can be extended to a quasiconformal automorphism $\tilde{f}_n$ of $\hat{\boldsymbol{C}}$ (Ahlfors [1, p. 75]). In particular $q_n | l_n - \{\pm 1\}$ has a quasiconformal extension $\tilde{f}_n$ to $\hat{\boldsymbol{C}}$. Finally, let $p = \pm 1$. Let $b_n \in (\delta_n, 1)$ and let $N_n = \{z \in \boldsymbol{C};\ b_n < p \cdot \operatorname{Re} z < b_n^{-1},\ |\operatorname{Im} z| < 1/2\}$. Then $\beta_n \cap N_n$ is an open subarc of β_n containing p. Set $u_n(z) = f_n(\operatorname{Re} z) + \sqrt{-1} \operatorname{Im} z$ if $b_n < p \cdot \operatorname{Re} z < b_n^{-1}$, $= z - pb_n + f_n(pb_n)$ if $p \cdot \operatorname{Re} z \leqq b_n$, and $= z - pb_n^{-1} + f_n(pb_n^{-1})$ if $p \cdot \operatorname{Re} z \geqq b_n^{-1}$. Then u_n is a quasiconformal extension of $q_n | \beta_n \cap N_n$ to $\hat{\boldsymbol{C}}$. q.e.d.

LEMMA 6. *There exists a quasiconformal automorphism h_n of C so that $h_n(z) = f_n(z)$ for $z \in \partial C \cap \partial \Delta$ and $=z$ for $z \in \partial C \cap \Delta^*$ and that $\lim_{n\to\infty} \|\mu[h_n]\|_\infty = 0$.*

PROOF. For $\theta \in [-\pi, 0]$ define $\psi_n(\theta) \in [-\pi, 0]$ as $f_n(\exp(\sqrt{-1}\theta)) = \exp(\sqrt{-1}\psi_n(\theta))$. Set $h_n(\rho \exp(\sqrt{-1}\theta)) = \rho \exp[\sqrt{-1}\{(\rho-1)\theta + (2-\rho)\psi_n(\theta)\}]$, where $\rho \in [1, 2]$ and $\theta \in [-\pi, 0]$. Then h_n is a homeomorphism of $\mathrm{Cl}\, C$ onto itself with $h_n(z) = f_n(z)$ for $z \in \partial C \cap \partial \Delta$ and $h_n(z) = z$ for $z \in \partial C \cap \Delta^*$. For $z = \rho \exp(\sqrt{-1}\theta) \in C$ it holds that

$$\begin{aligned}|\mu[h_n](z)| &= |[\rho(h_n)_\rho(z) + \sqrt{-1}(h_n)_\theta(z)]/[\rho(h_n)_\rho(z) - \sqrt{-1}(h_n)_\theta(z)]| \\ &= |[(2-\rho)\{1 - \psi_n'(\theta)\} + \sqrt{-1}\rho\{\theta - \psi_n(\theta)\}] \\ &\quad \times [\rho + (2-\rho)\psi_n'(\theta) + \sqrt{-1}\rho\{\theta - \psi_n(\theta)\}]^{-1}| \,.\end{aligned}$$

By Lemma 4 $\lim_{n\to\infty} \psi_n(\theta) = \theta$ and $\lim_{n\to\infty} \psi_n'(\theta) = 1$ uniformly on $(-\pi, 0)$. Hence we see $\lim_{n\to\infty} \|\mu[h_n]\|_\infty = 0$. q.e.d.

PROOF OF LEMMA 3. Define $G_n(z) = f_n(z)$ if $z \in \mathrm{Cl}\, A_n$, $= g_n(z)$ if $z \in \mathrm{Cl}\, B_n$ and $= h_n(z)$ if $z \in \mathrm{Cl}\, C$. Then Lemma 3 follows from Lemmas 5 and 6. q.e.d.

6. Proof of Lemma 2. Let r and s be real numbers with $r > 1$ and $0 < s < r + r^{-1}$. Let T be the vertical line in $\hat{C}$ passing through s. Then E_r and T intersect at exactly two points $\zeta \in U$ and $\bar{\zeta} \in L$. Let I be the bounded closed subarc of T joining ζ to $\bar{\zeta}$. Let P be the component of $\hat{C} - T$ containing the origin. Denote by J the Jordan loop $(E_r \cap P) \cup I$. Let Q be the interior of the circle with the diameter I. Note that both T and P depend on s, and ζ, I, J and Q all depend on both r and s.

PROOF OF LEMMA 2. Fix an $s \in (0, r + r^{-1})$ sufficiently near to $r + r^{-1}$ so that

$$\operatorname{diam} Q \leqq d/2 \tag{6.1}$$

and

$$\delta(J, E_r) \leqq d/2 \,, \tag{6.2}$$

where $\operatorname{diam} Q$ denotes the Euclidean diameter of Q.

First we construct $\{\sigma_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$. Let τ_n be a Möbius transformation such that $\tau_n(P) = U$ and $\tau_n(Q) = \hat{C} - \mathrm{Cl}\, V_n$, where V_n is the open ball $\{z \in C;\ |z| < \delta_n\}$ defined at the beginning of §5. Then $j_n = \tau_n(E_r \cap \mathrm{Cl}\, P)$ is a smooth closed Jordan arc in $\mathrm{Cl}\, V_n$ joining $-\delta_n$ to δ_n. Let $\{G_n\}_{n=1}^\infty$ be the sequence of quasiconformal automorphisms of Ω in Lemma 3. Let D_0 be a Dirichlet fundamental region for Γ in Δ. Since Γ is of the second kind, D_0 has free sides. Let D be the union of D_0, the region obtained from D_0 by reflection in $\partial\Delta$ and the free sides of D_0. Let σ be a Möbius transformation such that $\sigma(U) = \Delta$ and $\sigma(\mathrm{Cl}\, \Omega) \subset D$. Define

$$F_n = \begin{cases} \gamma \circ \sigma \circ G_n \circ \sigma^{-1} \circ \gamma^{-1} & \text{in } \ \gamma \circ \sigma(\Omega) \ \text{ for all } \ \gamma \in \Gamma \\ \text{the identity mapping in } & \hat{C} - \bigcup_{\gamma \in \Gamma} \gamma \circ \sigma(\Omega) \end{cases} \tag{6.3}$$

and $\sigma_n = \sigma \circ \tau_n$. Then F_n is a homeomorphism of $\hat{C}$ onto itself which is quasiconformal off $\partial\Delta$. Hence F_n is a quasiconformal automorphism of $\hat{C}$ (Lehto-Virtanen [6, p. 45]).

Secondly, we prove (3.1), (3.2) and (3.3). By (6.3) we see $F_n \circ \gamma = \gamma \circ F_n$ for all $\gamma \in \Gamma$. Since $j_n - \{-\delta_n, \delta_n\} = \tau_n(P \cap E_r) \subset \tau_n(P \cap (\hat{C} - \mathrm{Cl}\, Q)) = U \cap V_n$ and since $\tau_n(\infty) \in \tau_n(T - I) \subset \tau_n(T \cap (\hat{C} - \mathrm{Cl}\, Q)) = \partial U \cap V_n$, the point $\tau_n(\infty)$ belongs to A_n. Then by Lemma 3(i) and (6.3) we see $F_n \circ \sigma_n(\infty) = F_n \circ \sigma \circ \tau_n(\infty) \in F_n \circ \sigma(A_n) = \sigma \circ G_n(A_n) \subset \sigma(L) = \Delta^*$. Since by (6.3) $\|\mu[F_n^{-1}|\Delta^*]\|_\infty = \|\mu[F_n^{-1}|\Delta^* \cap D]\|_\infty = \|\mu[G_n^{-1}|\Omega \cap L]\|_\infty$, Lemma 3(ii) shows $\lim_{n\to\infty} \|\mu[F_n^{-1}|\Delta^*]\|_\infty = 0$.

Finally, we prove (3.4). It follows from Lemma 3(i) and (6.3) that

$$\begin{aligned}\sigma_n^{-1} \circ F_n^{-1}(\partial\Delta) &= \tau_n^{-1} \circ \sigma^{-1} \circ F_n^{-1}(\partial\Delta) \subset \tau_n^{-1}(l_n \cup (\hat{C} - \Omega)) \\ &\subset \tau_n^{-1}(j_n \cup (\hat{C} - \mathrm{Cl}\, V_n)) = (E_r \cap \mathrm{Cl}\, P) \cup Q \subset J \cup Q\,.\end{aligned}$$

Hence by (6.1) $\delta(\sigma_n^{-1} \circ F_n^{-1}(\partial\Delta), J) \leqq d/2$. This together with (6.2) yields that $\delta(\sigma_n^{-1} \circ F_n^{-1}(\partial\Delta), E_r) \leqq d$. Now we complete the proof of Lemma 2 and hence that of Theorem.

References

[1] L. V. Ahlfors, Lectures on Quasiconformal Mappings, Van Nostrand, Princeton, N. J., 1966.

[2] L. V. Ahlfors and L. Bers, Riemann's mapping theorem for variable metrics, Ann. of Math. (2) 72 (1960), 385-404.

[3] L. Bers, On boundaries of Teichmüller spaces and on kleinian groups. I, Ann. of Math. (2) 91 (1970), 570-600.

[4] T. Chu, On the outradius of finite-dimensional Teichmüller spaces, Discontinuous Groups and Riemann Surfaces, Ann. of Math. Studies, No. 79, Princeton Univ. Press, Princeton, N. J., 1974, pp. 75-79.

[5] C. J. Earle, On holomorphic cross-sections in Teichmüller spaces, Duke Math. J. 36 (1969), 409-415.

[6] O. Lehto and K. I. Virtanen, Quasiconformal Mappings in the Plane, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[7] Z. Nehari, The Schwarzian derivative and schlicht functions, Bull. Amer. Math. Soc. 55 (1949), 545-551.

[8] S. Rickman, Quasiconformally equivalent curves, Duke Math. J. 36 (1969), 387-400.

[9] H. Sekigawa, The outradius of the Teichmüller space, Tôhoku Math. J. (2) 30 (1978), 607-612.

[10] H. Sekigawa and H. Yamamoto, Outradii of Teichmüller spaces of finitely generated Fuchsian groups of the second kind, J. Math. Kyoto Univ. 26 (1986), 23-30.

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