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ON A CLASS OF THE UNIVERSALLY INTEGRABLE RANDOM FUNCTIONS

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In the recent papers [4] and [5], Ogawa has developed the theory of a noncausal stochastic integral and proved that this integral naturally contains the symmetric integral of Stratnovich-Fisk as a special case. In this paper we give an extension of his result.

1. Introduction and preliminaries. First of all, we briefly review Ogawa's result. Let $\{W(x, \omega): 0 \leqq x \leqq 1\}$ be a one-dimensional Wiener process on a complete probability space (Ω, F, P), $\phi = \{\phi_n(x)\}$ be a complete orthonormal system (CONS, for short) in $L^2([0, 1])$ and set $K_n(x, y) = \sum_{k=1}^{n} \phi_k(x)\phi_k(y)$. In this section we assume that every random function is $B([0, 1]) \times F$-measurable and satisfies the condition $P\left(\int_0^1 F^2(x, \omega)dx < \infty\right) = 1$. We say that F is ϕ-integrable if

$$\lim_{n\to\infty} \int_0^1 F(x, \omega) \int_0^1 K_n(x, y)d^\circ W(y)dx$$

exists in probability and denote the limit by $\int_0^1 F(x)d_\phi W(x)$. Here $\int_0^1 K_n(x, y)d^\circ W(y)$ stands for the usual Ito (forward) integral. If the limit does not depend on the choice of a CONS ϕ, then we say that F is universally integrable and denote it by ${}^*\!\!\int_0^1 F(x)dW(x)$. Let $G(x, \omega)$ be a causal (i.e., adapted to the family of σ-fields $F_x = \sigma\{W(y): 0 \leqq y \leqq x\}$) random function and $H(x, \omega)$ be a (not necessarily causal) random function whose sample paths are of bounded variation with probability one. We say that the random function $F(x, \omega)$ of the form

$$(1) \qquad F(x, \omega) = H(x, \omega) + \int_0^x G(y, \omega)d^\circ W(y)$$

is a quasi-martingale.

THEOREM (Ogawa [4] and [5]). (1) *Every quasi-martingale is ϕ-integrable if and only if the following condition is satisfied*:

$$(2) \qquad \sup_n \int_0^1 u_n^2(x)dx < \infty\ .$$

Here $u_n(x)=\int_0^x K_n(x, y)dy$. *Furthermore, if* $H(x, \omega)$ *in* (1) *is causal, then the integral* $\int_0^1 F(x)d_\phi W(x)$ *coincides with the symmetric integral of Stratnovich-Fisk* $\int_0^1 F(x)dW(x)\ \left(=\int_0^1 F(x)d^\circ W(x)+(1/2)\int_0^1 G(x)dx\right)$.

(2) *Let* $F(x)$ *be a quasi-martingale of the form* (1) *and assume that* $G(x)$ *is a quasi-martingale again. Then,* $F(x)$ *is universally integrable.*

REMARK 1. It is easily seen that the condition (2) is satisfied for the trigonometric, Haar and Walsh systems. But it is unknown whether (2) is valid for a general CONS in $L^2([0, 1])$.

The purpose of this paper is to show that if we replace the usual Ito integral by the extended Ito integral (E.I.I., for short) in the definition of the quasi-martingale (1), we also obtain an analogous theorem. For the reader's convenience, we recall the basic facts concerning the E.I.I. following [6], which gives a rather systematic survey of recent developments in this theory, mainly due to the Soviet school (such as Daletskii-Paramonova, Shevljakov, Skorokhod etc.).

Let W be a Gaussian random measure on the Borel σ-field $B([0, 1])$ induced by a Wiener process $\{W(x): 0 \leqq x \leqq 1\}$ and $\hat{L}^2([0, 1]^p)=\{f(x_1, \cdots, x_p)\in L^2([0, 1]^p)$; real-valued and symmetric in p variables$\}$ for each positive integer p. We denote by $\int_0^1\cdots\int_0^1 f(x_1, \cdots, x_p)dW(x_1)\cdots dW(x_p)$ the p-th multiple Wiener integral of $f\in\hat{L}^2([0, 1]^p)$ with respect to W. Let $H([0, 1]^k\times\Omega)=L^2([0, 1]^k\times\Omega)$ and $H_p([0, 1]^k\times\Omega)=\left\{\int_0^1\cdots\int_0^1 f(x_1, \cdots, x_k; y_1, \cdots, y_p)dW(y_1)\cdots dW(y_p)\in H([0, 1]^k\times\Omega); f\in L^2([0, 1]^{k+p}), f(x_1, \cdots, x_k; \cdot)\in \hat{L}^2([0, 1]^p)\right\}$ (for convenience, we set $H_0([0, 1]^k\times\Omega)=L^2([0, 1]^k)$). Then, by the theorem of Wiener-Ito expansion, it holds that

$$H([0, 1]^k\times\Omega)=\sum_{p=0}^{\infty} H_p([0, 1]^k\times\Omega)\quad\text{(direct sum)}\,.$$

That is, every element $F(x_1, \cdots, x_k)$ of $H([0, 1]^k\times\Omega)$ is represented in the form

$$F(x_1, \cdots, x_k)=\sum_{p=0}^{\infty} F_p(x_1, \cdots, x_k)\,,$$

where $F_p(x_1, \cdots, x_p)=\int_0^1\cdots\int_0^1 f_p(x_1, \cdots, x_k; y_1, \cdots, y_p)dW(y_1)\cdots dW(y_p)$, $f_p\in L^2([0, 1]^{k+p})$ and $f_p(x_1, \cdots x_k; \cdot)\in\hat{L}^2([0, 1]^p)$. In this notation, the inner product $(F, G)_{H([0,1]^k\times\Omega)}$ on the Hilbert space $H([0, 1]^k\times\Omega)$ can be written as follows:

$$\begin{aligned}
&(F, G)_{H([0,1]^k \times \Omega)} \\
&\quad = E\Big[\int_0^1 \cdots \int_0^1 F(x_1, \cdots, x_k) G(x_1, \cdots, x_k) dx_1 \cdots dx_k\Big] \\
&\quad = \sum_{p=0}^{\infty} E\Big[\int_0^1 \cdots \int_0^1 F_p(x_1, \cdots, x_k) G_p(x_1, \cdots, x_k) dx_1 \cdots dx_k\Big] \\
&\quad = \sum_{p=0}^{\infty} p! \int_0^1 \cdots \int_0^1 f_p(x_1, \cdots, x_k; y_1, \cdots, y_p) \\
&\qquad \times g_p(x_1, \cdots, x_k; y_1, \cdots, y_p) dx_1 \cdots dx_k dy_1 \cdots dy_p .
\end{aligned}$$

We also define $H_{\text{finite}}([0, 1]^k \times \Omega)$ by

$$H_{\text{finite}}([0, 1]^k \times \Omega) = \bigcup_{n=0}^{\infty} \sum_{p=0}^{n} H_p([0, 1]^k \times \Omega) .$$

Let $F \in H([0, 1]^k \times \Omega)$. If the series

$$\sum_{p=0}^{\infty} p \int_0^1 \cdots \int_0^1 f_p(x_1, \cdots, x_k; x_{k+1}, y_1, \cdots, y_{p-1}) dW(y_1) \cdots dW(y_{p-1})$$

converges in $H(\Omega)$ for almost every point $(x_1, \cdots, x_{k+1}) \in [0, 1]^{k+1}$, then the sum is called the stochastic derivative of F and denoted by $[DF](x_1, \cdots, x_k; x_{k+1})$. The stochastic derivative of higher order is defined by

$$D^j F = D(D^{j-1} F) \quad (D^0 F = F) ,$$

whenever the right-hand side makes sense. For each nonnegative integer l, let

$H^{(l)}([0, 1]^k \times \Omega) = \{F \in H([0, 1]^k \times \Omega);\ D^j F$ is defined and belongs to $H([0, 1]^{k+j} \times \Omega)$ for $j = 0, 1, \cdots, l\}$

and

$$H^{(\infty)}([0, 1]^k \times \Omega) = \bigcap_{l=0}^{\infty} H^{(l)}([0, 1]^k \times \Omega) .$$

Then, it is obvious that $H^{(l)}([0, 1]^k \times \Omega)$ becomes a Hilbert space with the inner product

$$\begin{aligned}
(F, G)_{H^{(l)}([0,1]^k \times \Omega)} &= \sum_{j=1}^{l} (D^j F, D^j G)_{H([0,1]^{k+j} \times \Omega)} \\
&= \sum_{j=0}^{l} \sum_{p=j}^{\infty} \{p!/(p-j)!\} E\Big[\int_0^1 \cdots \int_0^1 F_p G_p dx_1 \cdots dx_k\Big] \\
&= \sum_{j=0}^{l} \sum_{p=j}^{\infty} \{p!/(p-j)!\} p! \int_0^1 \cdots \int_0^1 f_p g_p dx_1 \cdots dx_k dy_1 \cdots dy_k .
\end{aligned}$$

Evidently, the stochastic derivative D is a bounded linear operator on

$H^{(l)}([0, 1]^k \times \Omega)$ with range in $H^{(l-1)}([0, 1]^{k+1} \times \Omega)$.

We now define the E. I. I. Let $F(x) \in H([0, 1] \times \Omega)$ and set

$$\hat{f}_p(y_1, \cdots, y_{p+1}) = \sum_{\sigma} \{1/(p+1)!\} f_p(y_{\sigma(1)}; y_{\sigma(2)}, \cdots, y_{\sigma(p+1)}) ,$$

where σ runs over all permutations of $\{1, 2, \cdots, p+1\}$. If the series

$$\sum_{p=0}^{\infty} \int_0^1 \cdots \int_0^1 \hat{f}_p(y_1, \cdots, y_{p+1}) dW(y_1) \cdots dW(y_{p+1})$$

converges in $H(\Omega)$, then the sum is denoted by $\dot{\int}_0^1 F(x)dW(x)$ and called the E.I.I. of F.

The following lemmas easily follow from the definition.

LEMMA 1 ([7], [8]). *Let l be a positive integer. Then, for every $F \in H^{(l)}([0, 1] \times \Omega)$, the* E.I.I. *exists and satisfies the inequality*

$$\left\| \dot{\int}_0^1 F(x)dW(x) \right\|_{H^{(l-1)}(\Omega)} \leqq C_l \|F\|_{H^{(l)}([0,1]\times\Omega)} .$$

Here C_l denotes a positive constant which only depends on l.

LEMMA 2 ([6, Corollary 3.3]). *Let $F \in H^{(1)}([0, 1]^2 \times \Omega)$ and $G \in L^2([0, 1])$. Then,*

$$\int_0^1 dxG(x) \dot{\int}_0^1 F(x, y)dW(y) = \dot{\int}_0^1 dW(y) \int_0^1 F(x, y)G(x)dx .$$

LEMMA 3 ([6, Proposition 3.4]). *Let $F \in H^{(2)}([0, 1]^2 \times \Omega)$. Then,*

$$\left[D\left(\dot{\int}_0^1 F(x, y)dW(y)\right)\right](x; z) = F(x, z) + \dot{\int}_0^1 [DF](x, y; z)dW(y) .$$

For $0 \leqq a \leqq b \leqq 1$, we set

$$\dot{\int}_a^b F(x)dW(x) = \dot{\int}_0^1 1_{[a,b]}(x)F(x)dW(x) .$$

It is proved in Shevljakov [7] (see also [6, Proposition 5.2]) that if $F(x)$ is causal, then the E.I.I. $\dot{\int}_0^x F(y)dW(y)$ exists and equals the Ito integral $\int_0^x F(y)d^\circ W(y)$.

The following lemma connects the E.I.I. with the noncausal stochastic integral investigated by Ogawa.

LEMMA 4. *Let $F \in H^{(1)}([0, 1] \times \Omega)$. Then, F is ϕ-integrable if and only if*

$$\lim_{n\to\infty} \int_0^1 \int_0^1 [DF](x; y)K_n(x, y)dxdy$$

exists in probability. In this case, it holds that

$$\int_0^1 F(x)d_\phi W(x) = {}^{\cdot}\!\!\int_0^1 F(x)dW(x) + \lim_{n\to\infty}\int_0^1\int_0^1 [DF](x;y)K_n(x,y)dxdy\ .$$

REMARK 2. Formulas of this kind were obtained in [1], [2] and [8]. See also [6, Theorem 4.1].

2. Main result. We now state our theorem.

THEOREM. (1) *Let $H\in H^{(3)}([0,1]\times\Omega)$ and set $G(x) = {}^{\cdot}\!\!\int_0^x H(y)dW(y)$. Then, $F(x) = {}^{\cdot}\!\!\int_0^x G(y)dW(y)$ is universally integrable and satisfies the equality*

$$(3)\qquad {}^{*}\!\!\int_0^a F(x)dW(x) = {}^{\cdot}\!\!\int_0^a F(x)dW(x) + (1/2)\int_0^a G(x)dx + \int_0^a dx\ {}^{\cdot}\!\!\int_0^x [DG](z;x)dW(z)\ .$$

(2) *Let $\phi=\{\phi_n\}$ be a* CONS *in $L^2([0,1])$ and assume that the condition* (2) *is fulfilled. Then, for every $G\in H^{(2)}([0,1]\times\Omega)$, $F(x) = {}^{\cdot}\!\!\int_0^x G(y)dW(y)$ is ϕ-integrable and satisfies the equality*

$$\int_0^a F(x)d_\phi W(x) = {}^{\cdot}\!\!\int_0^a F(x)dW(x) + (1/2)\int_0^a G(x)dx + \int_0^a dx\ {}^{\cdot}\!\!\int_0^x [DG](z,x)dW(z)\ .$$

(3) *Let $G\in H^{(2)}([0,1]\times\Omega)$ and assume that the sample paths of G are of bounded variation with probability one. Then, $F(x) = {}^{\cdot}\!\!\int_0^x G(y)dW(y)$ is universally integrable and satisfies the equality* (3).

To prove the theorem, we begin with several lemmas.

LEMMA 5. *Let $G\in H^{(2)}([0,1]\times\Omega)$. Then,*

$$(4)\qquad \int_0^a dx\ {}^{\cdot}\!\!\int_0^x [DG](z;x)dW(z) = \lim_{n\to\infty}\int_0^a dx\int_0^1 {}^{\cdot}\!\!\int_0^x [DG](z;y)dW(z)K_n(x,y)dy$$

in $H(\Omega)$ for each $a\in[0,1]$.

PROOF. By Lemma 2, we have

$$\int_0^a dx\ {}^{\cdot}\!\!\int_0^x [DG](z;x)dW(z) = {}^{\cdot}\!\!\int_0^a dW(z)\int_z^a [DG](z;x)dx$$

and

$$\begin{aligned}\int_0^a dx\int_0^1 {}^{\cdot}\!\!\int_0^x &[DG](z;y)dW(z)K_n(x,y)dy\\ &= {}^{\cdot}\!\!\int_0^a dW(z)\int_z^a dx\int_0^1 [DG](z;y)K_n(x,y)dy\ .\end{aligned}$$

Set

$$H(z) = 1_{[0,a]}(z) \int_z^a [DG](z; x)dx$$

and

$$H^{(n)}(z) = 1_{[0,a]}(z) \int_z^a dx \int_0^1 [DG](z; y)K_n(x, y)dy .$$

By Lemma 1, it suffices to show that $H^{(n)}$ converges to H in $H^{(1)}([0, 1]\times\Omega)$ as $n\to\infty$. By Parseval's equality, $H^{(n)}$ converges to H $dz\times dP$-almost everywhere and $\{H^{(n)}(z)\}^2$ is bounded by $\int_0^1 \{[DG](z; y)\}^2 dy \in H([0, 1]\times\Omega)$, so that, by Lebesgue's dominated convergence theorem, $H^{(n)}$ converges to H in $H([0, 1]\times\Omega)$ as $n\to\infty$. Similarly, noting that

$$[DH](z; u) = 1_{[0,a]}(z) \int_z^a [D^2G](z; x, u)dx$$

and

$$[DH^{(n)}](z; u) = 1_{[0,a]}(z) \int_z^a dx \int_0^1 [D^2G](z; y, u)K_n(x, y)dy ,$$

we can show that $DH^{(n)}$ converges to DH in $H([0, 1]^2\times\Omega)$ as $n\to\infty$. Therefore, $H^{(n)}$ converges to H in $H^{(1)}([0, 1]\times\Omega)$ as $n\to\infty$.

LEMMA 6. *For each* $H\in H^{(3)}([0, 1]\times\Omega)$, *set* $G(x) = {}^{\bullet}\!\int_0^x H(y)dW(y)$. *Then,* $G\in H^{(2)}([0, 1]\times\Omega)$ *and*

$$(1/2)\int_0^a G(x)dx = \lim_{n\to\infty}\int_0^a dx\int_0^x G(y)K_n(x, y)dy \tag{5}$$

in $H(\Omega)$ *for each* $a\in[0, 1]$.

PROOF. From Lemma 1, it immediately follows that $G\in H^{(2)}([0, 1]\times\Omega)$. We show the equality (5). By Lemma 2 and the symmetric property of $K_n(x, y)$,

$$(1/2)\int_0^a G(x)dx = (1/2)\ {}^{\bullet}\!\int_0^a dW(z)H(z)\int_z^a dz$$

and

$$\begin{aligned}\int_0^a dx\int_0^x G(y)K_n(x, y)dy &= {}^{\bullet}\!\int_0^a dW(z)H(z)\int_z^a dy\int_y^a K_n(x, y)dx \\ &= (1/2)\ {}^{\bullet}\!\int_0^a dW(z)H(z)\int_z^a dy\int_z^a K_n(x, y)dx .\end{aligned}$$

Because $\int_z^a dy\int_z^a K_n(x, y)dx$ converges to $\int_z^a dx$ uniformly in z as $n\to\infty$,

$1_{[0,a]}(z)H(z)\int_z^a dy \int_z^a K_n(x, y)dx$ converges to $1_{[0,a]}(z)H(z)\int_z^a dx$ in $H^{(1)}([0, 1]\times\Omega)$. Therefore, by Lemma 1, we obtain the equality (5).

LEMMA 7. *The condition* (2) *is satisfied if and only if the following equality is valid for every* $g \in L^2([0, 1])$:

$$(1/2)\int_0^1 g(x)dx = \lim_{n\to\infty}\int_0^1 dx \int_0^x g(y)K_n(x, y)dy\ . \tag{6}$$

In this case,

$$\left|\int_0^1 dx \int_0^x g(y)K_n(x, y)dy\right|^2 \leqq \text{const.} \int_0^1 g^2(x)dx\ . \tag{7}$$

PROOF. It is known that the condition (2) is equivalent to the equality

$$(1/2)\int_0^1 g(x)dx = \lim_{n\to\infty}\int_0^1 dx \int_0^x g(x)K_n(x, y)dy$$

([3, p. 342, Exercise 23], see also [5, Proposition 1]). The equality (6) easily follows from this fact. We now prove the inequality (7):

$$\begin{aligned}
&\left|\int_0^1 dx \int_0^x g(y)K_n(x, y)dy\right|^2 \\
&\quad= \left|\int_0^1 dx \int_0^1 g(y)K_n(x, y)dy - \int_0^1 dx \int_x^1 g(y)K_n(x, y)dy\right|^2 \\
&\quad\leqq 2\left\{1 + \sup_n \int_0^1 u_n^2(x)dx\right\}\int_0^1 g^2(x)dx\ .
\end{aligned}$$

COROLLARY. *If the condition* (2) *is fulfilled, then the equality* (5) *holds for every* $G \in H([0, 1]\times\Omega)$.

PROOF OF THEOREM. In any case, it is easily seen that $F \in H^{(1)}([0, 1]\times\Omega)$ by Lemma 1, and that

$$[DF](x; y) = 1_{[0,x]}(y)G(y) + {}^{\bullet}\!\!\int_0^x [DG](z; y)dW(z)$$

by Lemma 3. Therefore, by Lemma 4, it suffices to show that the equalities (4) and (5) hold. In the cases of (1) and (2), it has already been proved in Lemmas 5, 6 and Corollary. In the case of (3), we have only to note that the equality (6) holds for every function g of bounded variation without any condition on $\phi = \{\phi_n\}$.

REMARK 3. (1) If $G \in H^{(2)}([0, 1]\times\Omega)$ is causal, then it easily follows that

$$\int_0^a dx\ {}^{\bullet}\!\!\int_0^x [DG](z; x)dW(z) = 0$$

for each $a \in [0, 1]$. Therefore, our theorem coincides with that of Ogawa. However, Ogawa's theorem is not completely contained in our result, because there is a causal random function in $H([0, 1]\times \Omega)$ which does not belong to $H^{(1)}([0, 1]\times \Omega)$ (see [6, p. 139, Example]).

(2) It is easy to construct a random function $G \in H^{(2)}([0, 1]\times \Omega)$ such that $\int_0^a dx \int_0^x [DG](z; x)dW(z)$ does not equal zero. Therefore, our theorem shows the existence of a noncausal random function which is ϕ-integrable, though the E.I.I. only deals with a random function in $H([0, 1]\times \Omega)$.

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