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THE DIOPHANTINE NATURE FOR THE CONVERGENCE OF FORMAL SOLUTIONS

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1. Introduction. In this paper we shall study the diophantine nature of the problem of the convergence of all formal solutions. Concerning the convergence of all formal solutions, Kashiwara-Kawai-Sjöstrand [1] studied the equation $Pu \equiv \sum_{|\alpha|=|\beta|\leq m} a_{\alpha\beta}(x)x^{\alpha}(\partial/\partial x)^{\beta}u = f$ and gave a sufficient condition for the convergence of all formal solutions. Unfortunately this condition is merely sufficient and not necessary.

As for the necessity few results are known. This is mainly because we must treat rather delicate problems of diophantine nature. Concerning this, the first work which clearly showed the diophantine nature of the problem of the convergence of formal solutions was perhaps that of Siegel's in [4]. On the other hand in 1974, Leray [2] studied the diophantine nature of the Goursat problem by using a new diophantine function ρ. Though the problems they studied seem to be quite different, their basic ideas are closely connected. More precisely, their methods to treat the diophantine-type difficulty are the same.

In this paper we shall introduce two diophantine functions σ_{ξ} and ρ which are generalizations of Siegel's condition in [4] and Leray's auxiliary function in [2], respectively. By using these functions we shall give necessary and sufficient conditions for the convergence of formal solutions. We remark that this yields the solvability of the same equation by the usual method. We also give examples showing that we cannot drop any of the assumptions of the main theorem in general. Finally, we point out that the method here is also applicable to the study of C^{∞} (or C^{ω})-hypoellipticity of operators on the torus by slight modification.

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2. Notation and results. Let $x = (x_1, x_2)$ be the variable in $\boldsymbol{C}^2$. For $\eta \in \boldsymbol{R}^2$ and a multi-index $\alpha = (\alpha_1, \alpha_2) \in \boldsymbol{N}^2$, $\boldsymbol{N} = \{0, 1, 2, \cdots\}$, we set $\eta^{\alpha} = \eta_1^{\alpha_1}\eta_2^{\alpha_2}$ and $(x\cdot\partial)^{\alpha} = (x_1\partial_1)^{\alpha_1}(x_2\partial_2)^{\alpha_2}$, where $\partial = (\partial_1, \partial_2)$ and $\partial_j = \partial/\partial x_j$ $(j = 1, 2)$. Let $m \geqq 1$ be an integer and let $\omega \in \boldsymbol{C}^2$. Then we are concerned with the convergence of all formal solutions of the form $u(x) = x^{\omega} \sum_{\eta \in N^2} u_\eta x^\eta/\eta!$ of the equation

$$P(x; \partial)u \equiv \sum_{|\alpha| \leqq m} a_\alpha(x)\partial^\alpha u(x) = f(x)x^\omega , \tag{2.1}$$

where $a_\alpha(x)$ is analytic at the origin and $f(x)$ is a given analytic function. We say that a formal solution $u = x^\omega \sum u_\eta x^\eta/\eta!$ converges if the sum $\sum u_\eta x^\eta/\eta!$ converges and represents an analytic function in x. Let us expand $a_\alpha(x)$ into Taylor series, $a_\alpha(x) = \sum_\gamma a_{\alpha,\gamma} x^\gamma/\gamma!$. Then we define the set M_P by

$$M_P = \{\gamma - \alpha;\ a_{\alpha,\gamma} \neq 0 \text{ for some } \alpha \text{ and } \gamma\} .$$

We assume the following:

(A.1) The set M_P is contained in the half-space $\{\eta \in \boldsymbol{R}^2;\ \eta_1 + \eta_2 \geqq 0\}$ and the set $M_P \cap \{\eta \in \boldsymbol{R}^2;\ \eta_1 + \eta_2 = 0\}$ is contained either in $\{\eta \in \boldsymbol{R}^2;\ \eta_1 + \eta_2 = 0,\ \eta_1 \geqq 0\}$ or in $\{\eta \in \boldsymbol{R}^2;\ \eta_1 + \eta_2 = 0,\ \eta_2 \geqq 0\}$.

Roughly speaking, this condition means that the equation (2.1) is not irregular singular. We denote by Γ_P the smallest closed convex cone with apex at the origin which contains M_P.

Now let us define

$$p(\eta) = \sum_{\alpha, |\alpha| \leqq m} a_{\alpha,\alpha}\eta!/((\eta - \alpha)!\,\alpha!) \tag{2.2}$$

and denote the m-th homogeneous part of $p(\eta)$ by $p_m(\eta)$. We introduce two diophantine functions.

For $\xi \in \boldsymbol{R}^2$, $|\xi| = 1$ and $\varepsilon > 0$, we set $\Gamma(\xi; \varepsilon) = \{\eta \in \boldsymbol{R}^2;\ |\eta/|\eta| - \xi| < \varepsilon\}$. Then we define the quantity $\sigma_{\xi,\varepsilon}$ by

$$\sigma_{\xi,\varepsilon} = \sup\{c \in \boldsymbol{R};\ \liminf_{|\eta| \to \infty,\ \eta \in \Gamma(\xi,\varepsilon) \cap \omega + Z^2} |\eta|^{-c}|p(\eta)| > 0\} , \tag{2.3}$$

where if $\liminf |\eta|^{-c}|p(\eta)| = 0$ for every $c \in \boldsymbol{R}$, we put $\sigma_{\xi,\varepsilon} = -\infty$. Note that $\sigma_{\xi,\varepsilon} \leqq m$, since $p(\eta)$ is of degree m. Since $\sigma_{\xi,\varepsilon}$ increases as ε tends to zero, we set $\sigma_\xi \equiv \lim_{\varepsilon \downarrow 0} \sigma_{\xi,\varepsilon}$. We remark that the function σ_ξ is closely connected with Siegel's condition (cf. [4, (13)]).

Next we define the function ρ following Leray [2];

$$\rho = \liminf_{|\eta| \to \infty,\ \eta \in N^2 + \omega} |p(\eta)|^{1/|\eta|} . \tag{2.4}$$

Note that $0 \leqq \rho \leqq 1$, because $p(\eta)$ is a polynomial. We shall give fundamental properties of σ_ξ and ρ in Section 3.

We define a differential operator $Q(x; \partial) \equiv \sum_{|\beta| \leqq m_0} b_\beta(x)\partial^\beta$ by

$$Q(x; \partial) = P(x; \partial) - \sum_{|\alpha| \leqq m} a_{\alpha,\alpha} x^\alpha \partial^\alpha/\alpha! ,$$

where $m_0 \leqq m$. We assume the following "quasi-ellipticity condition" on P:

(A.2) For every $\xi \in \Gamma_P$ such that $|\xi| = 1$, either $p_m(\xi) \neq 0$ or $\sigma_\xi > m_0$ holds.

Our main result is the following.

THEOREM 2.1. *Suppose that the conditions* (A.1) *and* (A.2) *are satisfied. Then for every f which is holomorphic in a neighborhood of the origin all formal solutions of the equation* (2.1) *converge if and only if $\rho > 0$.*

REMARK 2.1. In Theorem 2.1 we cannot drop any of the assumptions (A.1) and (A.2) in general. We shall give such examples in Section 5.

REMARKS 2.2. (a) In the proof of Theorem 2.1 we can also show the following fact: For an analytic function f, let $\mathrm{MS}_f = \{\eta \in \boldsymbol{N}^2; \partial^\eta f(0) \neq 0\}$. Let $C \subset \boldsymbol{N}^2$ be a finite set and suppose that the conditions (A.1) and (A.2) are satisfied. Then, for every $f(x)$ analytic at the origin such that $\mathrm{MS}_f \subset C + \Gamma_P$, all formal solutions of (2.1) converge if and only if $p(\eta + \omega) \neq 0$ for all $\eta \in \boldsymbol{N}^2$ except a finite number of η's. For example, we may take $f(x)$ to be a polynomial in x.

(b) We can generalize Theorem 2.1 and the preceeding remark for the Leray-Volevich systems of d ($\geqq 2$) independent variables. We shall briefly sketch necessary modification for a single equation. Further extension to systems is not difficult.

For the sake of simplicity, we assume that $p_m(\eta) \not\equiv 0$. First we note that we can easily extend the definition of the sets M_P, Γ_P and the functions σ_ξ and ρ to the case of d independent variables. We also note that the condition (A.1) is clearly extended to the case of d independent variables. Instead of the condition (A.2) we assume: For every $\theta \in \Gamma_P$ and $\xi \in \boldsymbol{R}^d_+$ such that $|\xi| = 1$, $p_m(\xi) = 0$ and $\sigma_\xi \leqq m_0$, we have $L_\xi(\theta) \neq 0$. Here $L_\xi(\theta)$ is the localization of $p_m(\eta)$ at $\eta = \xi$ defined by, for $\xi, \theta \in \boldsymbol{R}^d$,

$$p_m(\xi + s\theta) = L_\xi(\theta)s^q + O(s^{q+1}) , \tag{2.5}$$

where $q = q(\xi)$ is a nonnegative integer and $L_\xi(\theta) \not\equiv 0$. We note that in case $d = 2$ the condition is exactly equivalent to (A.2). Furthermore in case $d \geqq 3$, we assume the regularity on the roots of $p(\eta)$. We set $S(\eta, t) = t^m p(t^{-1}\eta)$ and take a vector $\tilde{\theta}$ such that $p_m(\tilde{\theta}) \neq 0$ and write $\eta = \zeta_1\tilde{\theta} + \zeta'$. We factor $S(\zeta_1\tilde{\theta} + \zeta', t)$ as a polynomial of ζ_1:

$$S(\zeta_1\tilde{\theta} + \zeta', t) = c \prod_{j=1}^{j_0} (\zeta_1 - \lambda_j(\zeta', t))^{m_j} . \tag{2.6}$$

Then we assume that the roots $\lambda_j(\zeta', t)$ $(j = 1, \cdots, j_0)$ are smooth with respect to ζ' and t. Under these assumptions Theorem 2.1 is valid.

In Theorem 2.1 we assumed the diophantine conditions $\sigma_\xi > m_0$ and $\rho > 0$. However it is difficult to verify them. The following theorem gives a criterion which does not contain diophantine conditions for σ_ξ.

COROLLARY 2.2. *Under the condition* (A.1) *and suppose that* $p_m(\xi) \neq 0$ *for all* $\xi \in \Gamma_P$. *Then, for every f which is holomorphic in a neighborhood of the origin all formal solutions of* (2.1) *converge if and only if* $\rho > 0$.

We remark that we do not assume K-K-S type condition (i.e., $p_m(\xi) \neq 0$ for all $\xi \in \boldsymbol{R}^2_+$, cf. [1]) nor Siegel's type diophantine condition (cf. [4, (13)]). Hence Corollary 2.2 is nontrivial and may be a new type of theorem.

Next we shall introduce the notion of the "diophantine-type ellipticity" and show that the situation is rather simple in this case.

COROLLARY 2.3. *Suppose that* (A.1) *and the following condition are satisfied.*

(A.2)$'$ *Either* $p_m(\xi) \neq 0$ *or* $\sigma_\xi > m_0$ *holds for any* $\xi \in \boldsymbol{R}^2$.
Then all formal solutions of (2.1) *converge for any holomorphic f.*

REMARK 2.3. By the proof of Theorem 2.1 we can also prove the following holomorphic prolongation of solutions, if we assume (A.1) and (A.2)$'$: There exists $R_0 > 0$ with the following property. For any R, $0 < R < R_0$, a formal power series u satisfying that Pu is holomorphic in $D_R \equiv \{x \in \boldsymbol{C}^2;\ |x_1| + |x_2| < R\}$ is holomorphic in D_R.

REMARK 2.4. We set $\phi(x) = |x_1| + |x_2|$ and assume that $m_0 \leqq m - 1$ and that $p_m(\xi) = 0$ for some ξ. Then we easily see that the surface $\phi(x) = R$ $(R > 0)$ is characteristic with respect to P at the point x such that $|x_1| = \xi_1$, $|x_2| = \xi_2$, that is, $p_m(x\cdot(\partial\phi/\partial\bar{x}))|_{|x_j|=\xi_j} = 0$. Hence, for this type of operators, general theory says nothing about the validity of the above holomorphic prolongation. Nevertheless, this is the case if m_0 is sufficiently small so that (A.2)$'$ is satisfied.

3. Fundamental properties of σ_ξ and ρ. In this section we use the same notations as in Section 2. For the sake of simplicity we do not give the proof, unless it is used in the proof of Theorem 2.1.

PROPOSITION 3.1. *The followings are equivalent:* (i) $-\infty \leqq \sigma_\xi \leqq m - 1$, (ii) $p_m(\xi) = 0$. *Especially* $\sigma_\xi = m$ *if and only if* $p_m(\xi) \neq 0$.

For the sake of simplicity we assume that $p_m((1, 0)) \neq 0$. Then we have the factorization

$$S(\eta, t) = t^m p(t^{-1}\eta) = c \prod_{j=1}^{j_0} (\eta_1 - \lambda_j(\eta_2, t))^{m_j}\ , \quad m_j \geqq 1\ .$$

Let $\xi = (\xi_1, \xi_2) \in \boldsymbol{R}^2_+$, $|\xi| = 1$, and let $\omega = (\omega_1, \omega_2)$ be the number given in (2.1). Then we assume the following:

(C.1) For every j such that $\eta_1 + \omega_1 - \lambda_j(\eta_2 + \omega_2, 0) = 0$ there exist a conical neighborhood $\Gamma(\xi)$ of ξ, $C > 0$ and τ_j, $-\infty < \tau_j \leqq 1$, such that, for all $\eta \in \Gamma(\xi) \cap \boldsymbol{N}^2$,

$$|\eta_1 + \omega_1 - \lambda_j(\eta_2 + \omega_2, 1)| \geqq C(1 + |\eta|)^{\tau_j} .$$

(C.2) All the functions $\lambda_j(\eta_2, t)$ with $\xi_1 = \lambda_j(\xi_2, 0)$ are smooth in some neighborhood of $t = 0$ and ξ_2.

PROPOSITION 3.2. *Assume that the conditions* (C.1) *and* (C.2) *are satisfied. Then we have* $\sigma_\xi = m - \sum' m_j$, *where the summation is taken over all* j *such that* $\xi_1 - \lambda_j(\xi_2, 0) = 0$.

Next we show the diophantine property of σ_ξ in case $p(\eta)$ has the form $p(\eta) = p_m(\eta) + R_n(\eta)$, where $n < m$ and where $p_m(\eta)$ is a homogeneous polynomial of degree m and $R_n(\eta)$ is a polynomial of degree $\leqq n$. For $\theta > 0$ we define the multi-valued function F_θ as follows: For $t \in \boldsymbol{C}$,

(3.1) $F_\theta(t)$ is the set of all cluster values of the sequence $\{\mu^\theta(\nu/\mu - t)\}_{\nu,\mu}$ when $\nu, \mu \in \boldsymbol{N}$ and $\nu, \mu \to \infty$.

The fundamental properties of $F_\theta(t)$ are studied in [7]. (cf. Remark 3.1).

PROPOSITION 3.3. *Under the assumptions as above we have*:

(a) *The case* $\xi = (1, 0)$ *or* $(0, 1)$. *Either* $\sigma_\xi = m$ *or* $\sigma_\xi \leqq n$ *holds.*

(b) *The case* $\xi \neq (1, 0)$ *and* $(0, 1)$. *Let* $n < \sigma < m$. *Then* $\sigma_\xi = \sigma$ *if and only if* $p_m(\xi) = 0$ *and the set* $F_{(m-\tau)/n_0}(\xi_1/\xi_2)$ *contains* 0 *for all* τ, $\tau > \sigma$ *and does not contain* 0 *for all* τ, $n < \tau < \sigma$. *Here* $\xi = (\xi_1, \xi_2)$ *and the integer* n_0 *is the multiplicity of the root* $t = \xi_1/\xi_2$ *of the equation* $p_m((t, 1)) = 0$.

REMARK 3.1. Using the results of [7] we can say when the set $F_\theta(t)$ contains 0: If $t > 0$ is a rational number or $0 < \theta < 2$, then $F_\theta(t)$ contains 0. If $\theta \geqq 2$ and t is irrational, we expand t in a continued fraction $t = [a_0, a_1, a_2, \cdots]$, where

$$(3.2) \qquad a_0 = [t]\,, \quad \alpha_0 = t - a_0\,, \quad \alpha_1 = 1/\alpha_0\,, \quad a_1 = [\alpha_1], \cdots, a_n = [\alpha_n]\,,$$
$$\frac{1}{\alpha_{n+1}} = \alpha_n - a_n; \cdots .$$

Here $[s]$ denotes the largest integer which does not exceed s. Let us define the integers μ_i $(i = 1, 2, \cdots)$ by $\mu_{i+2} = a_i\mu_{i+1} + \mu_i$, $\mu_1 = 1$, $\mu_2 = 0$. Then in case $\theta > 2$, the set $F_\theta(t)$ contains 0 if and only if $\liminf_{i\to\infty} \mu_i^{\theta-2}/a_{i-1} = 0$. We also remark that there exists a set $E \subset [0, \infty)$ with the Lebesgue measure zero such that $0 \notin F_\theta(t)$ for all $t \notin E$ and $\theta > 2$.

LEMMA 3.4. *Let $\sigma \in \boldsymbol{R}$. Then the set $\{\xi \in \boldsymbol{R}^2; \sigma_\xi \leqq \sigma\}$ is closed.*

PROOF. Suppose that $\xi_i \to \xi$ and that $\sigma_{\xi_i} \leqq \sigma$ $(i = 1, 2, \cdots)$. Then, for every $\tau > \sigma$ and i there exists a sequence $\eta_{i,k} \in \boldsymbol{N}^2$ $(k = 1, 2, \cdots)$ such that $\eta_{i,k}/|\eta_{i,k}| \to \xi_i$ $(k \to \infty)$ for each i and that $\lim_k |\eta_{i,k}|^{-\tau}|p(\eta_{i,k} + \omega)| = 0$. Hence, for every i, $i = 1, 2, \cdots$, we can choose $k = k(i)$ in such a way that

$$\left|\frac{\eta_{i,k(i)}}{|\eta_{i,k(i)}|} - \xi_i\right| < i^{-1}, \qquad |\eta_{i,k(i)}|^{-\tau}|p(\eta_{i,k(i)} + \omega)| < i^{-1}. \tag{3.3}$$

We set $\zeta_i = \eta_{i,k(i)}$. Then it follows from (3.3) that $\zeta_i/|\zeta_i| \to \xi$ and that $|\zeta_i|^{-\tau}|p(\zeta_i + \omega)| \to 0$. This implies that $\sigma_\xi \leqq \sigma$. q.e.d.

LEMMA 3.5. *Suppose $p(\eta) = \eta_1 - \tau\eta_2$. Then $\sigma_\xi > -\infty$, $\xi = (\tau, 1)$ if and only if τ is positive, irrational and not a Liouville number.*

Next we consider the function ρ. We easily see that $0 \leqq \rho \leqq 1$. If $\omega = 0$ and $p(\eta)$ is homogeneous, then the function ρ coincides with that studied by Leray and Pisot [3]. In case $p(\eta)$ is not homogeneous, we assume that $p_m(\xi) \neq 0$ for $\xi = (1, 0)$ and $(0, 1)$ for the sake of simplicity. Let $p(\eta) = c\prod_j (\eta_1 - \lambda_j(\eta_2))^{m_j}$ be the factorization of $p(\eta)$. Then, by using the Puiseaux expansion of λ_j, we see that if $\rho = 0$ then $\lambda_j(\eta_2)$ is real for real η_2. Moreover, the study of ρ is reduced to that of $\liminf |\eta_1 - \lambda_j(\eta_2)|^{1/\eta_2}$. This is, in fact, a diophantine problem. Finally, we give the relation between σ_ξ and ρ.

PROPOSITION 3.6. *If $\sigma_\xi > -\infty$ for all $\xi \in \boldsymbol{R}^2$, then $\rho = 1$.*

4. Proof of the main theorem.

4.1. Preliminary lemmas. Let Γ_P be as in Section 2. Then we have the following:

LEMMA 4.1. *Let Σ be a closed set on the unit sphere $|\xi| = 1$ such that $\Sigma \cap \pm\Gamma_P = \varnothing$. Then there exists $c_0 > 0$ depending only on Σ and Γ_P such that we have $|\theta|/|\zeta| \leqq 16c_0^{-1}\varepsilon$ for every $\xi \in \Sigma$, every small $\varepsilon > 0$ and every ζ $(\neq 0)$ and $\zeta + \theta$ $(\theta \in \pm\Gamma_P)$ in the ε-conical neighborhood of ξ.*

PROOF. Let η $(\neq 0)$ be in the ε-conical neighborhood of ξ and let $\varepsilon < 1/2$. Then we have $|(\eta/|\eta|, \xi) - 1| = |(\eta/|\eta|, \xi) - (\xi, \xi)| \leqq |(\eta/|\eta| - \xi, \xi)| < \varepsilon$. This implies that $(\eta/|\eta|, \xi) > 1 - \varepsilon$. Hence

$$\begin{aligned} |\eta - (\eta, \xi)\xi|/(\eta, \xi) &= |\eta/|\eta| - (\eta/|\eta|, \xi)\xi|/(\eta/|\eta|, \xi) \\ &\leqq (1-\varepsilon)^{-1}\{|\eta/|\eta| - \xi| + |(\eta/|\eta|, \xi) - 1|\} \\ &< 2(\varepsilon + \varepsilon) = 4\varepsilon. \end{aligned} \tag{4.1}$$

By assumption there exists $c_0 > 0$ depending only on Σ and Γ_P such that for all $\xi \in \Sigma$ and $\pm\theta \in \Gamma_P$

$$|\theta - (\theta, \xi)\xi| = |\theta|\,|\theta/|\theta| - (\theta/|\theta|, \xi)\xi| \geqq c_0|\theta| . \tag{4.2}$$

Hence, by (4.1) with $\eta = \zeta$ and (4.2), we obtain

$$|\zeta + \theta - (\zeta + \theta, \xi)\xi| \geqq c_0|\theta| - 4\varepsilon|\zeta| . \tag{4.3}$$

By (4.1) with $\eta = \zeta + \theta$, the left-hand side of (4.3) is bounded by $4\varepsilon(|\zeta| + |\theta|)$. Hence we have $(c_0 - 4\varepsilon)|\theta| \leqq 8\varepsilon|\zeta|$. Therefore if we take ε so small that $c_0 - 4\varepsilon \geqq c_0/2$, we obtain $|\theta|/|\zeta| \leqq 16c_0^{-1}\varepsilon$. q.e.d.

Now in (2.1) we assume

$$a_{\alpha,\alpha} \neq 0 \quad \text{for some} \quad \alpha, |\alpha| = m . \tag{4.4}$$

We set $S(\eta, t) = t^m p(t^{-1}\eta)$. If $p_m((1, 0)) \neq 0$, then we have a factorization

$$p(\eta) = c_0 \prod_{j=1}^{j_0} (\eta_1 - \lambda_j(\eta_2))^{m_j} \tag{4.5}$$

for some $c_0 \neq 0$ and $m_j \in \boldsymbol{N}$, $j_0 \in \boldsymbol{N}$. By using (4.5) we have

$$S(\eta, t) = c_0 \prod_{j=1}^{j_0} (\eta_1 - t\lambda_j(t^{-1}\eta_2))^{m_j} . \tag{4.6}$$

If $p_m((1, 0)) = 0$, we take a vector e_0 such that $|e_0| = 1$ and that $p_m(e_0) \neq 0$. Then we make the rotation which maps e_0 to $(1, 0)$. This reduces the general case to the above case. Therefore we have a factorization

$$S(\eta, t) = \prod_{j=1}^{j_0} g_j(\eta, t)^{m_j} \tag{4.7}$$

where $g_j(\eta, t)$ is a continuous function in η and t. Moreover, we have the following:

LEMMA 4.2. *Assume* (4.4) *and let* $\xi_0 \in \boldsymbol{R}^2$, $|\xi_0| = 1$, *satisfy* $p_m(\xi_0) = 0$ *and* $\sigma_{\xi_0} \leqq m_0$, *where* m_0 *is as given in* (A.2). *Then there exists a complex neighborhood* V_{ξ_0} *of* ξ_0 *and* $t_0 > 0$ *such that for* $1 \leqq j \leqq j_0$, $\xi \in V_{\xi_0}$, $\theta \in \Gamma_P$, $|\theta| = 1$ *and* t *with* $|t| < t_0$, *the limit*

$$C_j(\xi, \theta, t) = \lim_{s\to 0} s^{-1}\{g_j(\xi + s\theta, t) - g_j(\xi, t)\} \tag{4.8}$$

exists uniformly with respect to (ξ, θ, t) *in* $V_{\xi_0} \times \Gamma_P \cap \{|\theta| = 1\} \times \{t; |t| < t_0\}$.

PROOF. We shall prove only the case $p_m((1, 0)) \neq 0$ since the other case can be proved in a similar manner. It follows that $\xi_0 \neq (1, 0)$ and that $\xi_2 \neq 0$ for $\xi = (\xi_1, \xi_2) \in V_{\xi_0}$ if V_{ξ_0} is sufficiently small. Hence we have $\xi_2 + s\theta_2 \neq 0$ for $\theta = (\theta_1, \theta_2)$, $|\theta| = 1$, if s is small. It follows that $t^{-1}(\xi_2 +$

$s\theta_2) \to \infty$ as $t \to 0$. Now we expand $t\lambda_j(t^{-1}\xi_2 + st^{-1}\theta_2)$ into Puiseaux series

$$(4.9) \qquad t\lambda_j(t^{-1}\xi_2 + t^{-1}s\theta_2) = (\xi_2 + s\theta_2)\lambda_{j,0} + t\sum_{k=1}^{\infty} c_k(t^{-1}\xi_2 + t^{-1}s\theta_2)^{1-k/q} ,$$

where $q \geqq 1$ is an integer and $\lambda_{j,0}$ and c_k are constants. Here we have used the fact that $t\lambda_j(t^{-1}\xi_2 + t^{-1}s\theta_2) \to (\xi_2 + s\theta_2)\lambda_{j,0}$ as $t \to 0$. By Taylor's formula we have

$$\begin{aligned} &g_j(\xi + s\theta, t) - g_j(\xi, t) \\ &\quad = s(\theta_1 + \lambda_{j,0}\theta_2) + t\sum_{k=1}^{\infty} c_k\{(t^{-1}(\xi_2 + s\theta_2))^{1-k/q} - (t^{-1}\xi_2)^{1-k/q}\} \\ &\quad = s(\theta_1 + \lambda_{j,0}\theta_2) + t\sum_{k} c_k(1 - k/q)(t^{-1}\xi_2 + t^{-1}\sigma_k s\theta_2)^{-k/q} \\ &\quad = s\{\theta_1 + \lambda_{j,0}\theta_2 + \theta_2\sum_{k=1}^{\infty} c_k(1 - k/q)(t^{-1}\xi_2)^{-k/q}\} \\ &\qquad + s\theta_2\sum_{k=1}^{\infty} c_k(1 - k/q)\{(t^{-1}\xi_2 + t^{-1}\sigma_k s\theta_2)^{-k/q} - (t^{-1}\xi_2)^{-k/q}\} , \end{aligned}$$

where $0 < \sigma_k < 1$. Applying Taylor's formula to the last term of the right-hand side again, we see that the second term is $O(s^2)$. Hence we get (4.8). The remaining part is clear. q.e.d.

Using the same notation as in Lemma 4.2 we have:

LEMMA 4.3. *Assume* (A.1), (A.2) *and* (4.4). *Suppose* $g_j(\xi_0, 0) = 0$. *Then there exist* $K_0 > 0$ *and* $0 < t_0' < t_0$ *such that*

$$(4.10) \qquad |C_j(\xi, \theta, t)| \geqq K_0$$

for all $(\xi, \theta, t) \in V_{\xi_0} \times \Gamma_P \cap \{\theta; |\theta| = 1\} \times \{t; |t| < t_0'\}$.

REMARK. Though it is not necessary in this paper, we can also prove that under (A.1) and (4.4) the condition (A.2) is equivalent to (4.10).

PROOF OF LEMMA 4.3. By Lemma 4.2, $C_j(\xi, \theta, t)$ is continuous. Hence, in order to prove (4.10) it is sufficient to show that $C_j(\xi_0, \theta, 0) \neq 0$. In view of the expression for C_j in the proof of Lemma 4.2, this is equivalent to $\theta_1 + \lambda_{j,0}\theta_2 \neq 0$.

By (A.2) and the definition of ξ_0 in Lemma 4.2, we have $\xi_0 \notin \Gamma_P$. On the other hand, since $g_j(\xi_0, 0) = 0$, it follows that $\xi_0^1 + \lambda_{j,0}\xi_0^2 = 0$, where $\xi_0 = (\xi_0^1, \xi_0^2)$. This implies that $\theta_1 + \lambda_{j,0}\theta_2 \neq 0$ by (A.1). q.e.d.

4.2. Proof of the necessity of Theorem 2.1. Suppose that all formal solutions of the equation (2.1) converge and that $\rho = 0$. First we shall show that $p(\eta + \omega)$ does not vanish except for a finite number of η's in N^2. Suppose that this is not the case. Then we shall show that (2.1) with $f=0$ has infinitely many linearly independent formal solutions. By substituting the expansions

$$(4.11)\qquad u(x)=\sum_{\eta} u_{\eta}\frac{x^{\eta+\omega}}{\eta!}\,,\quad f(x)=\sum_{\eta} f_{n}\frac{x^{\eta}}{\eta!}\,,\quad a_{\alpha}(x)=\sum_{\gamma} a_{\alpha,\gamma}\frac{x^{\gamma}}{\gamma!}$$

into (2.1) and by comparing the coefficients of $x^{\eta+\omega}$, we get

$$(4.12)\qquad f_{\eta}=p(\eta+\omega)u_{\eta}+\sum_{\eta=\gamma-\alpha+\delta,\alpha,\delta} a_{\alpha,\gamma}\frac{(\delta+\omega)!}{(\delta+\omega-\alpha)!}\frac{\eta!}{\gamma!\,\delta!}u_{\delta}\,.$$

In case $a_{\alpha,\alpha}=0$ for all α in (2.1), we must have $\sigma_{\xi}=-\infty$ by the definition of σ_{ξ}. Hence it follows from (A.2) that $m_0=-\infty$, that is, $Q\equiv 0$. The assertion is trivial in this case. Therefore we may assume that $p(\eta)\not\equiv 0$.

Let us take $\zeta\in \boldsymbol{N}^2$ such that $p(\zeta+\omega)=0$. We shall show that $p(\eta+\omega)\neq 0$ for all $\eta\in\zeta+\Gamma_P\cap \boldsymbol{N}^2$ if $|\eta|$ is sufficiently large. Suppose that there exist distinct $\eta_n\in\Gamma_P\cap\boldsymbol{N}^2$ $(n=1,2,\cdots)$ such that $p(\zeta+\eta_n+\omega)=0$ $(n=1,2,\cdots)$. Replacing $\{\eta_n\}$ by its subsequence, we may assume that the sequence $\{\eta_n/|\eta_n|\}$ and $\{(\zeta+\eta_n)/|\zeta+\eta_n|\}$ converge to the same point $\xi\in\boldsymbol{R}^2_+$, $|\xi|=1$, as $n\to\infty$. Since $\eta_n/|\eta_n|\in\Gamma_P$ and since the set Γ_P is closed, it follows that $\xi\in\Gamma_P$. On the other hand, by the definition of σ_ξ and $\zeta+\eta_n$ we have $\sigma_\xi=-\infty$. This contradicts (A.2).

In what follows we assume that $M_P\cap\{\eta\in\boldsymbol{R}^2;\ \eta_1+\eta_2=0\}\subset\{\eta=(\eta_1,\eta_2)\in\boldsymbol{R}^2;\ \eta_1\geqq 0\}$ in (A.1) for the sake of simplicity. The proof is similar in the other case. Let $\eta_1=(\eta_1^1,\eta_2)^1$ be the point in $\zeta+\Gamma_P\cap\boldsymbol{N}^2$ such that the length $|\eta_1|$ and the first coordinate η_1^1 are the largest among $\eta\in\zeta+\Gamma_P\cap\boldsymbol{N}^2$ satisfying $p(\eta+\omega)=0$. Then we have $p(\eta+\omega)\neq 0$ for all $\eta\in(\eta_1+\Gamma_P)\cap\boldsymbol{N}^2\setminus\{\eta_1\}$. Indeed, let us assume that $p(\eta'+\omega)=0$ for some $\eta'\in(\eta_1+\Gamma_p)\cap\boldsymbol{N}^2\setminus\{\eta_1\}$. Since $\eta_1-\zeta\in\Gamma_P$ and since Γ_P is a convex cone, it follows that $\eta_1+\Gamma_P=\zeta+\eta_1-\zeta+\Gamma_P\subset\zeta+\Gamma_P$. In view of (A.1) and the definition of η_1 this implies that $|\eta'|=|\eta_1|$. On the other hand, it follows from (A.1) that the first coordinate of η' is larger than η_1^1, a contradiction to the choice of η_1. Repeating this arguement we can choose $\eta_k\in\boldsymbol{N}^2$ $(k=1,2,\cdots)$ in such a way that for $k=1,2,\cdots$,

$$|\eta_k|<|\eta_{k+1}|\,,\quad p(\eta_k+\omega)=0\,,\quad p(\eta+\omega)\neq 0$$
$$\text{for all}\quad \eta\in(\eta_k+\Gamma_P)\cap\boldsymbol{N}^2\setminus\{\eta_k\}\,.$$

We note that, in view of the definition of Γ_P and M_P we may take the summation in (4.12) over $\delta\in\eta-\Gamma_P$. Let η_0 be one of η_k's and let u_{η_0} be a non-zero number. Since $p(\eta+\omega)$ does not vanish for all $\eta\in\eta_0+\Gamma_P\setminus\{\eta_0\}$ we can determine u_η for $\eta\in\eta_0+\Gamma_P\setminus\{\eta_0\}$ inductively by (4.12).

We set $u_\eta = 0$ for $\eta \notin \eta_0 + \Gamma_P$. Then we easily see that the formal sum $u(x) = \sum u_\eta x^{\eta+\omega}/\eta!$ is the formal solution of (2.1) for $f = 0$. Since η_0 is arbitrary, we get infinitely many formal solutions. A linear combination of these formal solutions gives rise to a formal solution of (2.1) which does not converge in any neighborhood of the origin. This contradicts the assumption.

Next let us assume that $\rho = 0$. By definition there exist $\eta_n \in \boldsymbol{N}^2$ $(n = 1, 2, \cdots)$ such that $|p(\eta_n + \omega)| \leqq n^{-|\eta_n|}$ $(n = 1, 2, \cdots)$. Replacing $\{\eta_n\}$ by a subsequence, if necessary, we may assume that $p(\eta + \omega) \neq 0$ for $\eta \in (\eta_n + \Gamma_P) \cap \boldsymbol{N}^2$ $(n = 1, 2, \cdots)$ and that the sequence $\{\eta_n/|\eta_n|\}$ converges to some $\xi \in \boldsymbol{R}^2_+$, $|\xi| = 1$. In view of the definition of σ_ξ and $\{\eta_n\}$, we have $\sigma_\xi = -\infty$. Hence $\xi \notin \Gamma_P$ by (A.2). We also note that $\xi \notin -\Gamma_P$ in view of (A.1). Since Γ_P is closed, there exists $\varepsilon > 0$ such that $\Gamma(\xi; \varepsilon) \cap \pm\Gamma_P = \emptyset$ where $\Gamma(\xi; \varepsilon)$ denotes the ε-conical neighborhood of ξ. Now let $n(1)$ be an integer such that $\eta_n \in \Gamma(\xi; \varepsilon/2)$ for $n \geqq n(1)$. If we choose $n(2)$ $(\geqq n(1))$ sufficiently large, then we have, for $n \geqq n(2)$,

$$\left|\frac{\eta_n - \eta_{n(1)}}{|\eta_n - \eta_{n(1)}|} - \xi\right| \leqq \left|\frac{\eta_n}{|\eta_n|}\frac{|\eta_n|}{|\eta_n - \eta_{n(1)}|} - \xi\right| + \frac{|\eta_{n(1)}|}{|\eta_n - \eta_{n(1)}|} < \varepsilon .$$

This implies that $\eta_n - \eta_{n(1)} \in \Gamma(\xi; \varepsilon)$. Hence we have $\eta_n - \eta_{n(1)} \notin \pm\Gamma_P$ for all $n \geqq n(2)$. By repeating this argument, we can choose $n(k)$ $(k = 1, 2, \cdots)$ in such a way that

$$\eta_n - \eta_{n(k)} \notin \pm\Gamma_P \quad \text{for all} \quad n \geqq n(k+1) . \tag{4.13}$$

If we set $k=l$ and $n = n(\nu)$ for $l < \nu$ in (4.13), we have $\eta_{n(l)} \notin \eta_{n(\nu)} - \Gamma_P$ for $l < \nu$. On the other hand, by setting $k = \nu$ and $n = n(l)$ $(l \geqq \nu + 1)$ in (4.13), we have $\eta_{n(l)} \notin \eta_{n(\nu)} - \Gamma_P$ for $l > \nu$. Hence

$$\eta_{n(l)} \notin \eta_{n(k)} - \Gamma_P \quad \text{if} \quad l \neq k . \tag{4.14}$$

Now we can construct a divergent formal solution. We set $f_\eta = 1$ if $\eta = \eta_{n(k)}$, $k = 1, 2, \cdots$, and $f_\eta = 0$ otherwise and define a holomorphic function $f(x)$ by $f(x) = \sum_\eta f_\eta |\eta|!\ x^\eta/\eta!$. By solving (4.1) recurrently we construct a formal solution $u(x) = \sum_\eta u_\eta x^{\eta+\omega}/\eta!$ of (2.1) for this f under the condition that $u_\eta = 0$ for all $\eta \notin \cup_{k=1}^\infty ((\eta_{n(k)} + \Gamma_P) \cap \boldsymbol{N}^2)$. This is possible, since $p(\eta + \omega)$ does not vanish by the definition of η_n and since u_η in (4.12) is determined by f_δ $(|\delta| \leqq |\eta|)$ and u_δ $(\delta \in \eta - \Gamma_P)$. Then it follows from (4.14) that $u_\eta = p(\eta + \omega)^{-1}|\eta|!$ for $\eta = \eta_{n(k)}$, $k = 1, 2, \cdots$. Recalling the definition of η_n, we have $|u_\eta| \geqq |\eta|!\ n(k)^{|\eta|}$ for $\eta = \eta_{n(k)}$, $k = 1, 2, \cdots$. Since $n(k) \to \infty$ as $k \to \infty$, this implies that the formal solution does not converge in any neighborhood of the origin, a contradiction.

4.3. Sufficiency. We now assume $\rho > 0$ and show that all formal solutions converge. We first show that we may assume (4.4) without loss of generality. Indeed, if $\Gamma_P = \{0\}$, then $P = \sum_\alpha a_{\alpha,\alpha}(\alpha!)^{-1}x^\alpha\partial^\alpha$. In this case, the theorem can be verified by simple computation. On the other hand, if $\Gamma_P \neq \{0\}$ and $p_m(\xi) = 0$ for some $\xi \in \Gamma_P$, we have $\sigma_\xi > m_0$ by (A.2). Let n be the maximum of $|\alpha|$ for α satisfying $a_{\alpha,\alpha} \neq 0$. Then $\sigma_\xi \leqq n$, hence we have $m_0 < n$. In view of the definition of m_0, this means that the order of P is n. Hence we may assume (4.4) from now on.

By substituting the expansions of u, f and a_α into (2.1) we get (4.12). If $p(\zeta + \omega) \neq 0$ we set

$$(4.15)\quad A_{\alpha,\zeta,\delta} = -p(\zeta+\omega)^{-1}a_{\alpha,\zeta-\delta+\alpha}\frac{(\delta+\omega)!}{(\delta+\omega-\alpha)!}\,\frac{\zeta!}{(\zeta-\delta+\alpha)!\ \delta!}\,.$$

By (A.1) we see that every δ in the summation (4.12) satisfies $|\eta| > |\delta|$ or $\eta_1 - \delta_1 \geqq 0$ (resp. $\leqq 0$), where $\eta = (\eta_1, \eta_2)$, $\delta = (\delta_1, \delta_2)$. We also note that if $a_{\alpha,\gamma} \neq 0$ and $\alpha \neq \gamma$ in (4.11), we have $|\alpha| \leqq m_0$ by the definition of m_0.

On the other hand, we see from the condition $\rho > 0$ that $p(\zeta + \omega)$ does not vanish except for a finite number of ζ in N^2. Hence, by using (4.12) repeatedly, we have

$$(4.16)\quad u_\eta = -p(\eta+\omega)^{-1}f_\eta + \textstyle\sum^1 \Xi(\eta, \delta^1, \cdots, \delta^\nu; \alpha^1, \cdots, \alpha^\nu)p(\delta+\omega)^{-1}f_\delta \\ + \textstyle\sum^2 \Xi(\eta, \delta^1, \cdots, \delta^\nu; \alpha^1, \cdots, \alpha^\nu)u_\delta\,,$$

where

$$(4.17)\quad \Xi \equiv \Xi(\eta, \delta^1, \cdots, \delta^\nu; \alpha^1, \cdots, \alpha^\nu) = A_{\alpha^\nu,\eta,\delta^\nu}\cdots A_{\alpha^1,\delta^2,\delta^1}\,.$$

The summation $\sum^1$ in (4.16) is taken over all the pairs $(\delta^1, \cdots, \delta^\nu; \alpha^1, \cdots, \alpha^\nu)$ of multi-indices satisfying

$$(4.18)\quad |\eta| \geqq |\delta^\nu| \geqq \cdots \geqq |\delta^1|\,,\quad \delta^{\lambda+1} - \delta^\lambda \in M_P\setminus\{0\}\,,\quad \delta^1 = \delta\,,\quad \delta^{\nu+1} = \eta\,; \\ |\alpha^\lambda| \leqq m_0\,,\quad p(\delta^\lambda+\omega) \neq 0 \qquad (\lambda = 1, \cdots, \nu+1)$$

The summation $\sum^2$ is taken over all the pairs of multi-indices satisfying the conditions obtained from (4.18) with the condition $p(\delta^1 + \omega) \neq 0$ replaced by $p(\delta^1 + \omega) = 0$.

We write the sum $\sum^1$ in (4.16) in the form

$$(4.19)\quad \textstyle\sum^1 = \sum_{n=1}^{q}\sum_{\nu=1}^{q-n}\sum_{n=n(1)<n(2)<\cdots<n(\nu)<q}\sum^{1'}\,,$$

where q and n are integers such that $q = |\eta| + 1$, $n = |\delta| + 1$. Here the summation $\sum_{n=n(1)<n(2)<\cdots<n(\nu)<q}$ is taken over all the combinations and $\sum^{1\prime}$ denotes the summation over all the pairs of multi-indices $(\delta^1, \cdots, \delta^\nu; \alpha^1, \cdots, \alpha^\nu)$ satisfying (4.18) and the condition $n(\lambda)=|\delta^\lambda|+1$ $(\lambda=1,\cdots,\nu)$.

Let $a_{\alpha,\gamma}$ be given by (4.11) and assume that $\gamma \neq \alpha$. Then it follows from (A.1) and Cauchy's formula that for any small $R>0$ there exists $K_1>0$ such that

$$(4.20) \qquad |a_{\alpha,\gamma}| \leqq K_1 R^{1-|\gamma|}|\gamma|!(1+|\gamma|)^{-3} \quad \text{for all} \quad \alpha, \gamma .$$

In order to estimate $A_{\alpha,\zeta,\delta}$ in (4.15) we first note that $|\zeta-\delta+\alpha| = |\zeta+\alpha|-|\delta| = |\zeta|+|\alpha|-|\delta|$, since $\zeta+\alpha \geqq \delta \geqq 0$ by definition. On the other hand, since we have $|\zeta| \geqq |\delta|$ by (4.18), we get

$$(4.21) \qquad \left|\frac{(\delta+\omega)!}{(\delta+\omega-\alpha)!}\,\frac{\zeta!}{(\zeta-\delta+\alpha)!\ \delta!}\right| \leqq K_2\frac{(|\zeta|+|\alpha|)!}{(|\zeta|-|\delta|+|\alpha|)!\ |\delta|!}$$

for some $K_2>0$ independent of ζ and δ. Hence, by (4.20) and $|\alpha| \leqq m_0$ in (4.18), we get

$$(4.22) \qquad |A_{\alpha,\zeta,\delta}| \leqq K_1K_2R^{1-|\zeta|+|\delta|}(|\zeta|+1)^{m_0}|\zeta|!\ (1+|\zeta|-|\delta|)^{-3}(|\delta|!\ |p(\zeta+\omega)|)^{-1} .$$

Let Ξ be given by (4.17). Then we shall show the estimate

$$(4.23) \qquad |\Xi| \leqq K_3(r_1R)^\nu r_2^q R^{n-q}(q-1)!\ ((n-1)!)^{-1}\prod_{\lambda=1}^{\nu}(n(\lambda+1)-n(\lambda)+1)^{-3} ,$$

where the constants $K_3>0$, $r_1>0$ and $r_2>0$ are independent of n, q, R satisfying $q=|\eta|+1$, $n=|\delta|+1$. In order to prove this let us first assume that all δ^λ's $(1 \leqq \lambda \leqq \nu+1)$ in (4.18) are in some small conical neighborhood of ξ_0 satisfying $|\xi_0|=1$, $p_m(\xi_0)=0$ and $\sigma_{\xi_0} \leqq m_0$. Let $g_j(\eta, t)$ be given by (4.7). Without loss of generality we may assume that $g_j(\xi_0, 0)=0$ if $1 \leqq j \leqq j_1$ and $\neq 0$ if $j>j_1$ for some $1 \leqq j_1 \leqq j_0$.

If $j>j_1$ we get, by the definition of g_j, that

$$(4.24) \qquad |\eta|^{-1}|g_j(\eta+\omega, 1)| = \left|g_j\!\left(\frac{\eta+\omega}{|\eta|},\ \frac{1}{|\eta|}\right)\right| \geqq K_4$$

for some $K_4>0$ independent of j when η moves in a sufficiently small conical neighborhood of ξ_0 and $|\eta|$ is large.

In case $j \leqq j_1$, let λ_j $(1 \leqq j \leqq \nu+1)$ be such that

$$(4.25) \qquad |g_j(\delta^{\lambda_j}+\omega, 1)| = \min_{1\leqq\lambda\leqq\nu+1}|g_j(\delta^\lambda+\omega, 1)| .$$

For the sake of simplicity we write $\delta^{\lambda_j}=\delta^0$ and determine the vector A^λ by $A^\lambda=\delta^\lambda-\delta^0$ for $\lambda \neq \lambda_j$. Note that $A^\lambda \in \pm\Gamma_p$ and $A^\lambda \neq 0$ by (4.18). We set

$$\xi = \frac{\delta^0 + \omega}{|\delta^0|}, \quad s = \frac{|A^\lambda|}{|\delta^0|}, \quad \theta = \frac{A^\lambda}{|A^\lambda|}, \quad t = \frac{1}{|\delta^0|} \tag{4.26}$$

if $A^\lambda \in \Gamma_P$. In case $-A^\lambda \in \Gamma_P$, we replace A^λ and s in (4.26) by $-A^\lambda$ and $-s$, respectively. By Lemmas 4.2 and 4.3 we have

$$\left|g_j\left(\frac{\delta^0 + \omega}{|\delta^0|} + \frac{A^\lambda}{|\delta^0|}, \frac{1}{|\delta^0|}\right) - g_j\left(\frac{\delta^0 + \omega}{|\delta^0|}, \frac{1}{|\delta^0|}\right)\right| \geqq \frac{K_0 |A^\lambda|}{2|\delta^0|} \tag{4.27}$$

if $|A^\lambda|/|\delta^0|$ is sufficiently small and $|\delta^0|$ is large. We note that this condition is really satisfied by Lemma 4.1 if we take a small enough conical neighborhood of ξ_0. Now in case A^λ satisfies

$$K_0 |A^\lambda|/4 \geqq |g_j(\delta^0 + \omega, 1)| \tag{4.28}$$

we get from (4.27) and the homogeneity $g_j(c\eta, ct) = cg_j(\eta, t)$ of g_j, that

$$|g_j(\delta^0 + \omega + A^\lambda, 1)| \geqq K_0 |A^\lambda|/4 . \tag{4.29}$$

This inequality is still true in case A^λ does not satisfy (4.28), since we have the following inequality by (4.25):

$$|g_j(\delta^0 + \omega + A^\lambda, 1)| \geqq |g_j(\delta^0 + \omega, 1)| .$$

On the other hand, it follows from (A.1) that the set Γ_P is a proper cone. Hence we can take a vector $\vec{e}$ with positive integral components and $c_1 > 0$ such that

$$c_1^{-1} |\alpha| \geqq \vec{e} \cdot \alpha \geqq c_1 |\alpha| \quad \text{for all} \quad \alpha \in \Gamma_P .$$

Hence if $\lambda > \lambda_j$ we have

$$|A^\lambda| \geqq c_1 \vec{e} \cdot A^\lambda = c_1 \vec{e} \cdot (\delta^\lambda - \delta^{\lambda_j}) = c_1 \sum_{k=\lambda_j}^{\lambda-1} \vec{e} \cdot (\delta^{k+1} - \delta^k) .$$

Since $\delta^{k+1} - \delta^k \in M_P \setminus \{0\}$ by (4.18), we see that $\vec{e} \cdot (\delta^{k+1} - \delta^k)$ is a positive integer. This implies that $|A^\lambda| \geqq c_1 |\lambda - \lambda_j|$. We have the same estimate in case $\lambda \leqq \lambda_j$. Substituting this estimate into (4.29) and noting that $\delta^\lambda = A^\lambda + \delta^0$, we have

$$|g_j(\delta^\lambda + \omega, 1)| \geqq K_0 c_1 |\lambda - \lambda_j|/4 . \tag{4.30}$$

Now it follows from (4.17), (4.22) and the condition $n(\lambda) = |\delta^\lambda| + 1$ that

$$\begin{aligned} |\Xi| \leqq (K_1 K_2 R)^\nu R^{n-q} (q-1)! \left(\prod_{\lambda=1}^{\nu} (n(\lambda+1) - n(\lambda) + 1)^{-s}\right) \\ \times \left((n-1)! \prod_{\lambda=2}^{\nu+1} (|\delta^\lambda| + 1)^{-m_0} |p(\delta^\lambda + \omega)|\right)^{-1} . \end{aligned} \tag{4.31}$$

By (4.7), (4.24) and $\sum m_j = m \geqq m_0$ we have

$$\left|\prod_{\lambda=2}^{\nu+1}|\delta^\lambda|^{-m_0}p(\delta^\lambda+\omega)\right| \geqq \prod_{\lambda=2}^{\nu+1}\left(\prod_{j=1}^{j_1-1} I_{\lambda,j}^{m_j}\prod_{j=j_1}^{j_0} I_{\lambda,j}^{m_j}\right) \tag{4.32}$$
$$\geqq K_5^\nu \prod_\lambda \prod_{j=j_1}^{j_0} I_{\lambda,j}^{m_j} = K_5^\nu \prod_j \left(I_{\lambda_j,j}^{m_j}\prod_{\lambda\neq\lambda_j} I_{\lambda,j}^{m_j}\right)$$

for some $K_5>0$, where $I_{\lambda,j}=|\delta^\lambda|^{-1}|g_j(\delta^\lambda+\omega,1)|$.

On the other hand, in terms of (4.30) and $|\delta^\lambda|\leqq|\eta|\leqq q$ we have, for some $K_6>0$,

$$\prod_j\prod_{\lambda\neq\lambda_j} I_{\lambda,j}^{m_j} \geqq \prod_j\prod_{\lambda\neq\lambda_j}((K_0c_1/4)|\delta^\lambda|^{-1}|\lambda-\lambda_j|)^{m_j} \tag{4.33}$$
$$\geqq K_6^\nu\Big(\prod_j\prod_{\lambda\neq\lambda_j}(|\delta^\lambda|/|\lambda-\lambda_j|)\Big)^{-m_j} \geqq K_6^\nu\Big(\prod_j\prod_{\lambda\neq\lambda_j} q/|\lambda-\lambda_j|\Big)^{-m_j}$$
$$\geqq K_6^\nu\Big(\prod_j\prod_{\mu=1}^{\nu+1} q/\mu\Big)^{-2m_j} \geqq K_6^\nu\prod_j e^{-2qm_j} \geqq K_6^\nu e^{-2qm}\,.$$

It follows from the assumtion $\rho>0$ and (4.7) with $t=1$ that

$$\liminf_{|\delta^\lambda|\to\infty}|I_{\lambda,j}|^{1/|\delta^\lambda|}>0 \quad \text{for all} \quad j\leqq j_0\,.$$

This implies that the term $\prod_j I_{\lambda_j,j}^{m_j}$ is bounded from below by K_7^q for some $K_7>0$ independent of q and δ^λ. Therefore we get (4.23) from (4.31), (4.32) and (4.33). We remark that the estimate (4.33) is valid for any sequence $\{\delta^\lambda\}$, if it is in a small conical neighborhood of ξ_0. Since the set of ξ satisfying $\sigma_\xi\leqq m_0$, $p_m(\xi)=0$ and $|\xi|=1$ is compact (cf. Lemma 3.4) we can cover the set by a finite number of open sets in each of which the estimate (4.33) is valid. Hence (4.33) is valid for any $\{\delta^\lambda\}$ which is in a small neighborhood of the set $\{\xi;\sigma_\xi\leqq m_0,\ p_m(\xi)=0\}$. On the other hand, if $\{\delta^\lambda\}$ is contained in the set $\{\xi;\sigma_\xi>m_0, \text{ or } p_m(\xi)\neq0\}$, we can easily see that $(|\eta|+1)^{-m_0}|p(\eta+\omega)|\geqq c_2>0$ for some $c_2>0$ independent of η. Hence in view of (4.31) we get (4.23).

We shall show that the number of pairs $(\delta^\lambda,\alpha^\lambda;\lambda=1,\cdots,\nu)$ satisfying (4.18) and $n(\lambda)=|\delta^\lambda|+1$ is bounded by $c_3d_0^\nu\prod_{\lambda=1}^\nu(n(\lambda+1)-n(\lambda)+1)$ for some $c_3>0$ and $d_0>0$ independent of $n(\lambda)$, δ^λ, α^λ. In order to prove this let us first count the possible number of δ^ν's when $\eta=\delta^{\nu+1}$ is fixed. We set $\gamma^\nu=\eta-\delta^\nu$. Then we may count the number of γ^ν's instead of that of δ^ν's. Noting that $|\gamma^\nu|=n(\nu+1)-n(\nu)$ and that γ^ν is contained in a proper cone Γ_P, such number is bounded by $c_5(n(\nu+1)-n(\nu)+1)$. Then we fix δ^ν and count the possible number of $\delta^{\nu-1}$ in a similar way. Repeating this arguement ν times, we see that the possible number of pairs $(\delta^1,\cdots,\delta^\nu)$ is at most $\prod_{\lambda=1}^\nu(n(\lambda+1)-n(\lambda)+1)$. On the other hand, the number of pairs $(\alpha^1,\cdots,\alpha^\nu)$ such that $|\alpha^\lambda|\leqq m$ is at most d_0^ν for some d_0.

Now we can easily show that (cf. [6, p. 57])

$$\sum_{n=n(1)<n(2)<\cdots<n(\nu)<q} \prod_{\lambda=1}^{\nu} (n(\lambda+1)-n(\lambda)+1)^{-2} \leqq 2^{2\nu}(q-n+1)^{-2} . \tag{4.34}$$

On the other hand, by the analyticity of f and $\rho>0$ we have, for some $R>0$,

$$|p(\delta+\omega)^{-1}f_\delta| \leqq |\delta|!\, R^{-|\delta|} \quad \text{for all} \quad \delta \in N^2 \cdot \tag{4.35}$$

Hence it follows from (4.16), (4.19), (4.23), (4.34) and (4.35) that

$$\begin{aligned} |\textstyle\sum^1 \Xi p(\delta+\omega)^{-1}f_\delta| &\leqq \sum_{n=1}^{q}\sum_{\nu=1}^{q-n} \sum_{n=n(1)<\cdots<q} \textstyle\sum^{1'} |\Xi| \\ &\times |\delta|!\, R^{-|\delta|} \leqq \sum_{n=1}^{q}\sum_{\nu=1}^{q-n} \sum_{n=n(1)<\cdots<q} (n-1)!\, R^{1-n} K_3 c_3 d_0^\nu \\ &\times (r_1R)^\nu r_2^q R^{n-q}(q-1)!\, ((n-1)!)^{-1} \prod_{\lambda=1}^{\nu} (n(\lambda+1)-n(\lambda)+1)^{-2} \\ &\leqq \sum_{n=1}^{q}\sum_{\nu=1}^{q-n} R^{1-q}K_3(r_1R2^2d_0)^\nu (q-1)!\, r_2^q c_3 . \end{aligned} \tag{4.36}$$

If we take R so small that $2^2Rr_1d_0<1$, we see that the right-hand side of (4.36) is $O(q!\, r_2^qR^{-q})$ as $q\to\infty$. Since the term $\sum^2 \Xi u_\delta$ in (4.16) has the same form as the first term, we can show that it has the same estimate. Consequently, we have proved that the formal solution converges. This proves the sufficiency.

5. **Examples.**

In this section we shall give examples which shows that we cannot omit the assumption (A.1) and (A.2) in Theorem 2.1 in general.

EXAMPLE 1. Let $a \geqq 0$ and consider the equation

$$((x_1\partial_1+1)^2 + ax_1(x_1\partial_1)^2 - \partial_1)((x_2\partial_2)^2+1)u = -x_1 . \tag{5.1}$$

We can easily see that $M_P \cap \{\eta \in \boldsymbol{R}^2;\, \eta_1+\eta_2<0\} \neq \varnothing$. Moreover, since $p(\eta)$ does not vanish by definition (cf. §2), the equation (5.1) satisfies (A.2) and $\rho>0$. We shall show that (5.1) has a divergent formal solution.

We set $((x_2\partial_2)^2+1)u=v$. Then we see that the formal power series v converges if and only if u converges. On the other hand, by substituting the expansion $v=\sum_{n=0}^{\infty} v_n(x_2)x_1^n$ of v into (5.1) and by comparing the coefficients of x_1^n we have

$$v_{n+1}(x_2) = (n+1)v_n(x_2) + a(n-1)^2\frac{v_{n-1}(x_2)}{(n+1)} + \delta_{n,1} \tag{5.2}$$

for $n=1,2,\cdots$, where $\delta_{n,1}$ is Kronecker's delta.

Since $a\geqq 0$, it follows from (5.2) that $v_{n+1} \geqq (n+1)v_n$. We set $v_0=v_1=0$. Then we have that $v_{n+1}\geqq (n+1)!/2$. This implies that the formal solution v does not converge.

EXAMPLE 2. Next we shall give an example which shows that we cannot drop the latter half of (A.1). The following example is due to Leray [2], [5]. Let us consider

$$(\partial_1^2 + \varepsilon\partial_1\partial_2 + \partial_2^2)(x_1 x_2 u) = f(x) , \tag{5.3}$$

where ε ($\neq 0$) is a complex constant. We can easily see that $M_P = \{(1, -1), (-1, 1), (0, 0)\}$, $p_2(\eta) = \varepsilon\eta_1\eta_2$. Hence (5.3) satisfies the former half of (A.1), (A.2) and the condition $\rho > 0$ but does not satisfy the latter half of (A.1).

If we set $v(x) = x_1 x_2 u(x)$, then (5.3) is equivalent to the Goursat problem for $v(x)$ with the boundary conditions $v(0, x_2) \equiv v(x_1, 0) \equiv 0$. Leray showed that for an appropriate choice of ε and f, (5.3) has a formal solution not convergent in any neighborhood of the origin.

EXAMPLE 3. We shall show that we cannot drop (A.2) in general. For this purpose let us consider the equation

$$Pu \equiv (x_1\partial_1 + x_2\partial_2 + 1)u + x_1(x_1\partial_1 + x_2\partial_2 + 1)^2 u = f(x) . \tag{5.4}$$

We easily see that $\sigma_\xi = 1$ for all $\xi \in \boldsymbol{R}_+^2$ and that $p_2(\xi) \equiv 0$, $\Gamma_P = \{t(1, 0); t \geqq 0\}$. Hence (5.4) does not satisfy (A.2). Note that (5.4) satisfies (A.1) and $\rho > 0$. On the other hand, by the method of indeterminate coefficients we easily see that (5.4) has a divergent formal solution for an appropriate choice of f.

EXAMPLE 4. In (5.4) the degree of the "top term" $x_1\partial_1 + x_2\partial_2 + 1$ is less than that of the "perturbation term" $x_1(x_1\partial_1 + x_2\partial_2 + 1)^2$. We shall show that we cannot drop (A.2), if we do not assume this.

Let $s \in \boldsymbol{N}$ and let m and n be positive integers such that $m \geqq 4n$, $s \leqq m$. Take a positive irrational number τ such that (cf. [3])

$$\liminf_{q\to\infty, q\in N} (\min_{p\in Z} |p - \tau q|^{1/q}) = 1 \tag{5.5}$$

and consider the equation

$$(x_1\partial_1 + x_2\partial_2)^{m-2n}(x_1\partial_1 - \tau x_2\partial_2)^{2n} u = (x_1 + x_2)(x_1\partial_1 + x_2\partial_2 + 1)^s u + f(x) . \tag{5.6}$$

This satisfies (A.1) and we easily see that $\sigma_{(\tau,1)} \leqq m - 4n$, $\sigma_\xi = m$ if $\xi \neq (\tau, 1)$, because there exist infinitely many positive integers p and q such that $|p/q - \tau| < q^{-2}$. This implies that (5.6) satisfies (A.1) and does not satisfy (A.2) if $s \geqq m - 4n$. By using the method of indeterminate coefficients, we can easily prove that if $s > m - 2n$, (5.6) with $f = 0$ has a formal solution not convergent in any neighborhood of the origin. On the other hand, a rather complicated estimate shows that if $s \leqq m - 4n$, all formal solutions of (5.6) converge for any holomorphic f.

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