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SPECTRAL RIGIDITY OF COMPACT KAEHLER AND CONTACT MANIFOLDS

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Abstract. Complex projective space $\boldsymbol{CP}_n$ with the Fubini-Study metric, and the odd-dimensional constant curvature sphere S^{2n+1} have recently been characterized by the spectrum of the Laplacian on 2-forms. In this paper, $\boldsymbol{CP}_n$ and S^{2n+1} are characterized among the classes of compact Kaehler and Sasakian manifolds, respectively, by the spectrum of the Laplacian on p-forms for any fixed p.

1. Introduction. Let (M, g) be a compact connected Riemannian manifold with complex structure J and Riemannian metric g, and denote by $\Delta = -(dd^* + d^*d)$ the real Laplacian acting on p-forms, where d is the operator of exterior differentiation and d^* is its adjoint with respect to g. Then, for each $p = 0, 1, 2, \cdots, n$, we have the spectrum of Δ:

$$\mathrm{Spec}^p(M, g) = \{0 \geqq \lambda_{1,p} \geqq \lambda_{2,p} \geqq \cdots \geqq \lambda_{k,p} \geqq \cdots \downarrow -\infty\},$$

each eigenvalue being repeated as often as its multiplicity. Hodge theory implies that $0 \in \mathrm{Spec}^p(M, g)$ if and only if the p-th Betti number $b_p(M)$ is not zero, and its multiplicity is then $b_p(M)$. The following theorems were obtained in [4] and [5]. (It is assumed here and in the sequel that M is connected.)

THEOREM A. *Let (M, g) be a compact Kaehler manifold with $\mathrm{Spec}^2(M, g) = \mathrm{Spec}^2(\boldsymbol{CP}_n, g_0)$ where $(\boldsymbol{CP}_n, g_0)$ is complex projective n-space with the Fubini-Study metric g_0. Then, (M, g) is holomorphically isometric with $(\boldsymbol{CP}_n, g_0)$ for all n.*

THEOREM B. *Let (M, g) be a compact Sasakian manifold with $\mathrm{Spec}^2(M, g) = \mathrm{Spec}^2(S^{2n+1}, g_0)$, where (S^{2n+1}, g_0) is the $(2n+1)$-dimensional sphere with constant curvature k_0. Then, g is a metric of constant curvature $k = k_0$.*

Theorem A is the only case known where the geometry of (M, g) is

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completely determined by $\mathrm{Spec}^p(M, g)$ for some fixed p and in all dimensions. However, (M, g) is assumed to be a Kaehler manifold. Similarly, if we restrict ourselves to the class of Sasakian manifolds, that is the class of normal contact Riemannian manifolds, Theorem B may be considered as another example where the geometry of (M, g) is completely determined by $\mathrm{Spec}^p(M, g)$ for some fixed p and in all dimensions.

The crucial point in proving Theorem A is that $b_2(M)$ is one, and in establishing Theorem B that $b_2(M)$ vanishes. This is the reason for taking $p = 2$. This fact concerning $b_2(M)$ is used only to show that M is cohomologically Einstein. For other values of p this may not be the case, but if this is assumed the following results are obtained (see sections 3 and 2 for the definitions of cohomologically Einstein Kaehler and Sasakian manifolds).

THEOREM 1. *Let (M, g) be a compact cohomologically Einstein Kaehler manifold with* $\mathrm{Spec}^p(M, g) = \mathrm{Spec}^p(\boldsymbol{CP}_n, g_0)$ *for p fixed,* $0 \leqq p \leqq 2n$. *Then, (M, g) is holomorphically isometric with $(\boldsymbol{CP}_n, g_0)$ for all n and p with the following possible exceptions*: (i) *n and p satisfy the relation* $p^2 - 2np + n(2n-1)/3 = 0$, *and* (ii) $p = 1$ *or* $2n - 1$, $n = 1, \cdots, 7$.

THEOREM 2. *Let (M, g) be a compact cohomologically Einstein Sasakian manifold with* $\mathrm{Spec}^p(M, g) = \mathrm{Spec}^p(S^{2n+1}, g_0)$ *for p fixed,* $0 \leqq p \leqq 2n + 1$. *Then, g is a metric of the same constant curvature as g_0 for all n and p with the following possible exceptions*: (i) *n and p satisfy the relation* $p^2 - (2n+1)p + n(2n+1)/3 = 0$, *and* (ii) $p = 1$ *or* $2n$, $n = 2, \cdots, 6$.

REMARK 1. The equations $p^2 - 2pn + n(2n-1)/3 = 0$ and $p^2 - p(2n+1) + n(2n+1)/3 = 0$ may be written in the form

$$p^2 - mp + \frac{m(m-1)}{6} = 0\,, \tag{1.1}$$

where $m = 2n$ and $2n + 1$, respectively. The Diophantine equation (1.1) has infinitely many solutions with m and p positive integers. In fact, we have $p = (1/2)(m \pm [m(m+2)/3]^{1/2})$. If follows that $m(m+2) = 3r^2$, where r is a positive integer. Setting $q = m + 1$, we obtain $q^2 - 3r^2 = 1$. This is the well-known Pell's equation. The positive integer solutions $(q, r) = (q_k, r_k)$, $k = 1, 2, \cdots$, are given by $q_k + \sqrt{3}\, r_k = (2 + \sqrt{3})^k$, $k = 1, 2, \cdots$. It now follows easily that all positive integer solutions of (1.1) have the form $(m, p) = (m_k, p_k)$ or $(m, p) = (m_k, m_k - p_k)$, $k = 1, 2, \cdots$, where

$$m_1 = 6\,,\ p_1 = 1\,,\ m_{k+1} = 5m_k - 6p_k + 1\,,\ p_{k+1} = m_k - p_k\,,\ k = 1, 2, \cdots .$$

REMARK 2. All complete intersection manifolds in CP_{n+r} of dimension $n \geqq 3$ are cohomologically Einstein.

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2. The spectrum. The Minakshisundaram-Pleijel-Gaffney asymptotic formula is given by

$$\sum_{k=0}^{\infty} \exp(\lambda_{k,p} t) = \frac{1}{(4\pi t)^{m/2}} \sum_{i=0}^{N} a_{i,p} t^i + O(t^{N-m/2+1}), \quad t \downarrow 0,$$

where $m = \dim M$. The coefficients $a_{i,p}$, $i = 0, 1, 2$, have been computed by Patodi [8] (see also [1]):

$$a_{0,p} = \binom{m}{p} V, \quad V = \mathrm{vol}(M), \tag{2.1}$$

$$a_{1,p} = \left[\frac{1}{6}\binom{m}{p} - \binom{m-2}{p-1}\right] \int_M \rho \, dV, \tag{2.2}$$

$$a_{2,p} = \int_M (C_1 |R|^2 + C_2 |S|^2 + C_3 \rho^2) dV, \tag{2.3}$$

where $|R|^2 = \sum R^{ijkl} R_{ijkl}$, $|S|^2 = \sum R^{ij} R_{ij}$, R_{ijkl} and R_{ij} being the components of the curvature and Ricci tensors R and S, respectively, and ρ is the scalar curvature. The coefficients C_i, $i = 1, 2, 3$, are given by

$$C_1 = \frac{1}{180}\binom{m}{p} - \frac{1}{12}\binom{m-2}{p-1} + \frac{1}{2}\binom{m-4}{p-2},$$

$$C_2 = -\frac{1}{180}\binom{m}{p} + \frac{1}{2}\binom{m-2}{p-1} - 2\binom{m-4}{p-2},$$

$$C_3 = \frac{1}{72}\binom{m}{p} - \frac{1}{6}\binom{m-2}{p-1} + \frac{1}{2}\binom{m-4}{p-2}.$$

By introducing the Weyl conformal curvature tensor C with components

$$C_{ijkl} = R_{ijkl} - \frac{2}{m-2}(R_{jk} g_{il} - R_{jl} g_{ik} + g_{jk} R_{il} - g_{jl} R_{ik}) + \frac{\rho}{(m-1)(m-2)}(g_{jk} g_{il} - g_{jl} g_{ik}),$$

$a_{2,p}$ may be expressed in the form

$$a_{2,p} = \int_M \left[Q_1 |C|^2 + Q_2\left(|S|^2 - \frac{\rho^2}{m}\right) + Q_3 \rho^2\right] dV, \tag{2.4}$$

where

$$Q_1 = C_1\,, \quad Q_2 = \frac{4}{m-2}C_1 + C_2\,, \quad Q_3 = \frac{2}{m(m-1)}C_1 + \frac{1}{m}C_2 + C_3\,.$$

For Kaehler manifolds, by introducing the Weyl projective curvature tensor W (see [4]) whose components are

$$W^i_{jkl^*} = R^i_{jkl^*} + \frac{1}{n+1}(R_{jl^*}\delta^i_k + R_{kl^*}\delta^i_j)\,,$$

$a_{2,p}$ may be written in the form

$$a_{2,p} = \int_M \left[P_1|W|^2 + P_2\left(|S|^2 - \frac{\rho^2}{m}\right) + P_3\rho^2\right]dV\,, \tag{2.5}$$

where

$$P_1 = C_1\,, \quad P_2 = \frac{8}{m+2}C_1 + C_2\,, \quad P_3 = \frac{8}{m(m+1)}C_1 + \frac{1}{m}C_2 + C_3\,.$$

If $\mathrm{Spec}^p(M, g) = \mathrm{Spec}^p(M', g')$, then $\dim M = \dim M'$, $V = V'$, $b_p(M) = b_p(M')$, and $a_{2,p} = a'_{2,p}$, where the prime indicates corresponding quantities in M'. Moreover, it follows from (2.2) that

$$\int_M \rho dV = \int_{M'} \rho' dV'$$

if $p^2 - mp + m(m-1)/6 \neq 0$.

The following statement was proved in [4].

LEMMA 1. *Let (CP_n, g_0) be complex projective space with the Fubini-Study metric g_0, and (M, g) be a Kaehler-Einstein manifold. Then, if $\rho = \rho_0$, where ρ and ρ_0 are the scalar curvatures of g and g_0, respectively,* $\mathrm{vol}(M, g) \leqq \mathrm{vol}(CP_n, g_0)$ *with equality if and only if (M, g) is isometric with (CP_n, g_0).*

Lemma 1 will be useful in the proof of the following:

LEMMA 2. *Let (M, g) be a compact Kaehler manifold with* $\mathrm{Spec}^p(M, g) = \mathrm{Spec}^p(CP_n, g_0)$ *for a fixed p, $0 \leqq p \leqq 2n$. Assume that $p^2 - 2np + n(2n-1)/3 \neq 0$ and for some $\lambda \in \mathbf{R}$*

$$\int_M (|S|^2 - \lambda\rho^2)dV = \int_{CP_n} (|S'|^2 - \lambda\rho'^2)dV'\,, \tag{2.6}$$

where the prime indicates corresponding quantities in (CP_n, g_0). Then,

(i) *if $\lambda < 1/2n$, (M, g) is holomorphically isometric with (CP_n, g_0) for every n and p satisfying $P_1 \geqq 0$, $P_3 > 0$, and*

(ii) *if $\lambda \geqq 1/2n$, (M, g) is holomorphically isometric with (CP_n, g_0)*

for every n and p satisfying $P_1 \geqq 0$, $(\lambda - 1/2n)P_2 + P_3 > 0$.

PROOF. Since $W' = 0$ and $|S'|^2 = \rho'^2/2n$, formula (2.5) yields

$$\int_M \left[P_1|W|^2 + P_2\left(|S|^2 - \frac{\rho^2}{2n}\right) + P_3(\rho^2 - \rho'^2)\right]dV = 0\,, \tag{2.7}$$

which for some constant μ may be written in the form

$$\int_M \Big[P_1|W|^2 + \mu P_2\left(|S|^2 - \frac{\rho^2}{2n}\right) + (1-\mu)P_2\Big((|S|^2 - \lambda\rho^2) + \left(\lambda - \frac{1}{2n}\right)\rho^2\Big) + P_3(\rho^2 - \rho'^2)\Big]dV = 0\,. \tag{2.8}$$

By (2.6), this becomes

$$\int_M \Big[P_1|W|^2 + \mu P_2\left(|S|^2 - \frac{\rho^2}{2n}\right) + \left(\left(\lambda - \frac{1}{2n}\right)P_2 + P_3 - \left(\lambda - \frac{1}{2n}\right)\mu P_2\right)(\rho^2 - \rho'^2)\Big]dV = 0 \tag{2.9}$$

since $|S'|^2 = \rho'^2/2n$. If $\lambda \geqq 1/2n$, we take $\mu = 0$. Formula (2.9) then becomes

$$\int_M \left[P_1|W|^2 + \left(\left(\lambda - \frac{1}{2n}\right)P_2 + P_3\right)(\rho^2 - \rho'^2)\right]dV = 0\,.$$

Since $\int_M \rho dV = \int_{M'} \rho' dV'$, Schwarz's inequality yields

$$\int_M (\rho^2 - \rho'^2)dV \geqq 0$$

with equality if and only if $\rho = \rho'$. The conditions on the P_i in (ii) give rise to $\int_M (\rho^2 - \rho'^2)dV = 0$, so $\rho = \rho'$. Hence, by (2.6)

$$\begin{aligned}\int_M \left(|S|^2 - \frac{\rho^2}{2n}\right)dV &= \int_M \left[(|S|^2 - \lambda\rho^2) + \left(\lambda - \frac{1}{2n}\right)\rho^2\right]dV \\ &= \int_{CP_n} \left[(|S'|^2 - \lambda\rho'^2) + \left(\lambda - \frac{1}{2n}\right)\rho'^2\right]dV' \\ &= \int_{CP_n} \left(|S'|^2 - \frac{\rho'^2}{2n}\right)dV' = 0\,.\end{aligned}$$

But, $|S|^2 \geqq \rho^2/2n$, so $|S|^2 = \rho^2/2n$, that is g is an Einstein metric. Applying Lemma 1, it follows that (M, g) is isometric with $(\boldsymbol{CP}_n, g_0)$.

If $\lambda < 1/2n$ and $P_2 \neq 0$, take μ to be of the same sign as P_2, and $|\mu|$ to be so large that $(\lambda - 1/2n)P_2 + P_3 - (\lambda - 1/2n)\mu P_2 > 0$. Then, by

(2.9), $\rho = \rho'$ and $|S|^2 - \rho^2/2n = 0$. Again, by Lemma 1, (M, g) is isometric with $(\boldsymbol{CP}_n, g_0)$. Finally, let $\lambda < 1/2n$ and $P_2 = 0$. Then, from (2.7) and (2.6), g is an Einstein metric, so again by Lemma 1 we obtain the desired conclusion.

LEMMA 3. *Let (M, g) be a compact Riemannian manifold with* $\mathrm{Spec}^p(M, g) = \mathrm{Spec}^p(S^m, g_0)$ *for a fixed p, $0 \leqq p \leqq m$, where S^m is the m-dimensional sphere with metric of constant curvature k_0. Assume that $p^2 - mp + m(m-1)/6 \neq 0$ and for some $\lambda \in \boldsymbol{R}$*

$$\int_M (|S|^2 - \lambda\rho^2)dV = \int_{S^m} (|S'|^2 - \lambda\rho'^2)dV' ,$$

where the prime indicates corresponding quantities in (S^m, g_0). Then,

(i) *if $\lambda < 1/m$, g is a metric of constant curvature k_0 for every m and p satisfying $Q_1 \geqq 0$, $Q_3 > 0$, and*

(ii) *if $\lambda \geqq 1/m$, g is a metric of constant curvature k_0 for every m and p satisfying $Q_1 \geqq 0$, $(\lambda - 1/m)Q_2 + Q_3 > 0$.*

The proof is similar to that of Lemma 2.

Let (M, g) be a Kaehler manifold, J its almost complex structure, and Ω its fundamental 2-form. Consider the 2-form $\tilde{S}$ given by $\tilde{S}(X, Y) = S(X, JY)$. M is said to be *cohomologically Einstein* if $[\tilde{S}] = a[\Omega]$ for some $a \in \boldsymbol{R}$, where $[\tilde{S}]$ and $[\Omega]$ are the cohomology classes of $H^2(M, \boldsymbol{R})$ represented by $\tilde{S}$ and Ω, respectively.

LEMMA 4 (Ogiue [6]). *Let (M, g) be a cohomologically Einstein Kaehler manifold. Then,*

$$\int_M \left(|S|^2 - \frac{\rho^2}{2}\right)dV + \frac{n-1}{2nV}\left(\int_M \rho dV\right)^2 = 0 .$$

3. Contact manifolds. An m $(=2n+1)$-dimensional C^∞ manifold is called a contact manifold if it carries a global 1-form η, called the contact form, with the property $\eta \wedge (d\eta)^n \neq 0$ everywhere. The classical example is the bundle of unit tangent vectors to an oriented $(n+1)$-dimensional manifold. An odd-dimensional sphere possesses a contact structure which is not of this type. J. Martinet showed that every compact 3-manifold carries a contact structure. A compact Hodge manifold B has a contact manifold canonically associated with it as a circle bundle with B as base space. Thus, the class of contact manifolds is quite extensive.

An almost contact structure (ϕ, X_0, η) on a $(2n+1)$-dimensional C^∞ manifold M is given by an affine collineation ϕ, a vector field X_0, and a 1-form η satisfying

$$\eta(X_0) = 1\,, \quad \phi X_0 = 0 \quad \text{and} \quad \phi^2 = -I + \eta \otimes X_0\,.$$

In this case, a Riemannian metric g can be found with

$$\eta = g(X_0, \cdot) \quad \text{and} \quad g(\phi X, Y) = -g(X, \phi Y)$$

for any vector fields X and Y.

A contact manifold with contact form η has an underlying almost contact Riemannian structure (ϕ, X_0, η, g) such that $g(X, \phi Y) = d\eta(X, Y)$. If the almost complex structure J on $M \times \boldsymbol{R}$ defined by $J(X, fd/dt) = (\phi X - fX_0, \eta(X)d/dt)$ is integrable, the almost contact structure is said to be *normal*. In this case, the unit vector field X_0 is a Killing field. Moreover, $g(R(X, X_0)Y, X_0) = g(\phi X, \phi Y)$ and

$$S(X, X_0) = 2n\eta(X)\,. \tag{3.1}$$

The standard contact Riemannian structure on an odd-dimensional sphere is normal.

Set $\tilde{S}(X, Y) = S(X, \phi Y)$. Then, $\tilde{S}$ is a skew symmetric bilinear form on M. An almost contact manifold is said to be *cohomologically Einstein* if $[\tilde{S}] = a[\Phi]$, where $\Phi(X, Y) = g(X, \phi Y)$ and $a \in \boldsymbol{R}$. If the almost contact structure underlies a contact structure then $\Phi = d\eta$, and so $[\tilde{S}] = 0$.

A normal contact Riemannian manifold is sometimes called a *Sasakian manifold*.

LEMMA 5. *Let (M, g) be a compact cohomologically Einstein Sasakian manifold. Then, there exists a 1-form α on M such that $\tilde{S} = d\alpha$ and* $\alpha(X_0) = \text{const}$.

PROOF. Since $\tilde{S}$ is exact, we set $\tilde{S} = d\beta$. Let H denote the isometry group preserving $\tilde{S}$. Then, H is a compact Lie group. Let H_0 be the 1-parameter group of diffeomorphisms of M generated by X_0. Then, since X_0 is a Killing field, H_0 is a group of isometries. Moreover, since $i(X_0)\tilde{S} = 0$, $L_{X_0}\tilde{S} = (i(X_0)d + di(X_0))\tilde{S} = 0$, where L_X and $i(X)$ are the Lie derivative and interior product by X, respectively. The elements of H_0 therefore preserve $\tilde{S}$ and so $H_0 \subset H$. Set $\alpha = \int_H h^*(\beta)dh$, where h is an arbitrary element of H, and dh is the invariant measure on H normalized by the condition $\int_H dh = 1$. Then,

$$d\alpha = \int_H h^*(d\beta)dh = \int_H h^*(\tilde{S})dh = \int_H \tilde{S}dh = \tilde{S}\,.$$

Clearly, $h^*(\alpha) = \alpha$ for any $h \in H$, so $L_{X_0}\alpha = 0$. Since $di(X_0)\alpha = L_{X_0}\alpha - i(X_0)d\alpha = 0$, we conclude that $\alpha(X_0) = i(X_0)\alpha = \text{const}$.

LEMMA 6. *Let (M, g) be a compact cohomologically Einstein Sasakian manifold of dimension $2n+1$. Then,*

$$\int_M \Big(|S|^2 - \frac{\rho^2}{2} + 2\rho\Big) dV + \frac{n-1}{2nV}\Big(\int_M \rho dV\Big)^2 = 2n(2n+1)V .$$

PROOF. The following relations may be found in [2]:

$$\nabla_X X_0 = -\phi X , \tag{3.2}$$

$$(\nabla_X \eta)(Y) = \Phi(X, Y) , \tag{3.3}$$

$$(\nabla_X \phi) Y = g(X, Y) X_0 - \eta(Y) X , \tag{3.4}$$

$$(\nabla_X \Phi)(Y, Z) = \eta(Y) g(X, Z) - \eta(Z) g(X, Y) , \tag{3.5}$$

for any vector fields X, Y and Z on M. By direct computation, (3.1)-(3.5) give rise to

$$d^*\Phi = 2n\eta , \tag{3.6}$$

$$d^*\Phi^2 = 4(n-1)\eta \wedge \Phi , \tag{3.7}$$

$$i(\Phi)\tilde{S} = \frac{1}{2}(\rho - 2n) , \tag{3.8}$$

$$i(\Phi^2)\tilde{S}^2 = \frac{1}{2}(\rho^2 - 2|S|^2 - 4n\rho + 12n^2) , \tag{3.9}$$

$$i(\tilde{S})(\eta \wedge \Phi) = \frac{1}{2}(\rho - 2n)\eta , \tag{3.10}$$

where i is the adjoint of exterior multiplication that is, if $\langle\ ,\ \rangle$ denotes the local scalar product with respect to the Riemannian metric g, $\langle i(\alpha)\beta, \gamma\rangle = \langle \beta, \alpha \wedge \gamma\rangle$, where α, β and γ are forms of degrees p, q and $q-p$, respectively. Denote by $(\alpha, \beta) = \int_M \langle \alpha, \beta\rangle dV$ the global scalar product. By (3.6), (3.8) and Lemma 5, $(1/2)\int_M (\rho - 2n) dV = (i(\Phi)\tilde{S}, 1) = (\tilde{S}, \Phi) = (d\alpha, \Phi) = (\alpha, d^*\Phi) = 2n(\alpha, \eta) = 2n\alpha(X_0)V$. Thus,

$$\alpha(X_0) = \frac{1}{4nV}\int_M (\rho - 2n) dV . \tag{3.11}$$

By (3.7), (3.9)-(3.11) and Lemma 5, $(1/2)\int_M (\rho^2 - 2|S|^2 - 4n\rho + 12n^2) dV = (i(\Phi^2)\tilde{S}^2, 1) = (\tilde{S}^2, \Phi^2) = (d(\alpha \wedge \tilde{S}), \Phi^2) = (\alpha \wedge \tilde{S}, d^*\Phi^2) = 4(n-1)(\alpha \wedge \tilde{S}, \eta \wedge \Phi) = 4(n-1)(\alpha, i(\tilde{S})(\eta \wedge \Phi)) = 2(n-1)\alpha(X_0)\int_M (\rho - 2n) dV = ((n-1)/2nV)\Big[\int_M (\rho - 2n) dV\Big]^2$, from which the lemma follows.

4. Proofs of Theorems 1 and 2. By Lemma 4,

$$\int_M \left(|S|^2 - \frac{\rho^2}{2}\right) dV = \int_{CP_n} \left(|S'|^2 - \frac{\rho'^2}{2}\right) dV' ,$$

so by Lemma 2, (M, g) is holomorphically isometric with (CP_n, g_0) for all n and p satisfying $P_1 \geqq 0$ and $(1/2 - 1/2n)P_2 + P_3 > 0$. We shall need the following:

LEMMA 7. *$P_1(n, p) \geqq 0$ for all n and p, $0 \leqq p \leqq 2n$, with the possible exception of $p = 1$, $p = 2n - 1$ for $n = 1, \cdots, 7$.*

PROOF. $P_1(n, 0) = P_1(n, 2n) = 1/180 > 0$, and

$$P_1(n, 1) = P_1(n, 2n - 1) = \frac{2n - 15}{180} > 0 , \quad n \geqq 8 .$$

For $n \geqq 4$ and $5 \leqq p \leqq 2n - 2$,

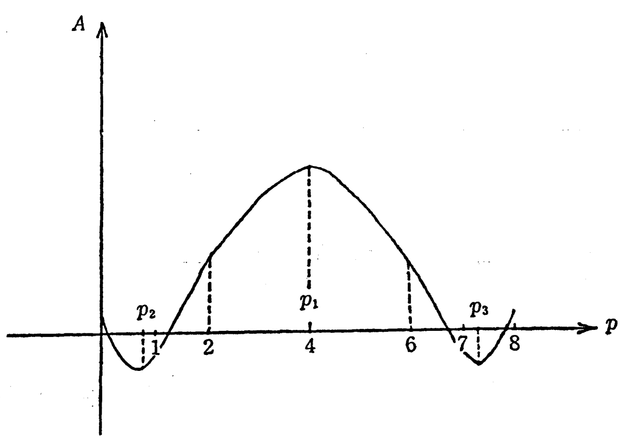
$$P_1(n, p) = \frac{(2n-4)(2n-5) \cdots (2n-p+1)}{180p!} A(n, p) ,$$

where

$$\begin{aligned} A(n, p) = {} & 2n(2n-1)(2n-2)(2n-3) \\ & - 30(2np - p^2)[2n^2 + n - 3(2np - p^2)] . \end{aligned}$$

Fix n and consider $A(n, p)$ as a function of the continuous variable p. Then,

$$\frac{dA}{dp} = -60(n - p)[2n^2 + n - 6(2np - p^2)] .$$



FIGURE

The critical points of $A(n, p)$ are $p_1 = n$, $p_2 = n - [(2/3)n^2 - (1/6)n]^{1/2}$, and $p_3 = n + [(2/3)n^2 - (1/6)n]^{1/2}$. Since $A(n, p_1) = 2n(2n-1)(23n^2 - 16n + 6) > 0$ for $n \geqq 4$, and $A(n, p_2) = A(n, p_3) = (n/6)(36n^3 - 348n^2 + 249n - 72) > 0$ for $n \geqq 9$, we obtain $A(n, p) > 0$ for $5 \leqq p \leqq 2n - 2$, $n \geqq 9$. However, for $n = 4, 5, 6, 7$ and 8, $A(n, p_2) = A(n, p_3) < 0$, Consider, for example, the case $n = 4$. Then, $p_2 = 4 - \sqrt{10} = 0.837 \cdots$, $p_3 = 4 + \sqrt{10} = 7.162 \cdots$, $A(4, 0) > 0$, $A(4, 1) < 0$, $A(4, 2) > 0$, so the graph of $A(4, p)$ is as in the Figure. It follows that $A(4, p) > 0$ for $p \neq 1, 7$. Similarly, $A(n, p) \geqq 0$ for $n = 5, 6, 7, 8$ and $p \neq 1, 2n - 1$. In the same way, $P_1(n, 2) \geqq 0$, $P_1(n, 3) > 0$ and $P_1(n, 4) > 0$.

LEMMA 8. *For all n and p, $0 \leqq p \leqq 2n$, except $n = 2$, $p = 2$,*

$$\left(\frac{1}{2} - \frac{1}{2n}\right)P_2 + P_3 > 0 .$$

The proof is similar to that of Lemma 7.

We now complete the proof of Theorem 1. For $n = 2$, $p = 2$, the theorem was proved in [4]. For all other n and p, with the possible exception of $p = 1$ and $p = 2n - 1$, $n = 1, \cdots, 7$, the theorem is a consequence of Lemmas 7 and 8.

The proof of Theorem 2 employs Lemmas 3 and 6, and is similar to that of Theorem 1.

REMARKS. (a) For $p = 0$ and 1, Theorem 1 was proved by Chen and Vanhecke [3].

(b) For $p = 1$ or 3 and $n = 2$, Theorem 1 may be proved if one replaces the cohomologically Einstein condition by the stronger condition $b_2(M) = 1$. Indeed, in this case, $b_i(M) = b_i(CP_2)$, $i = 0, \cdots, 4$, so the Euler-Poincaré characteristic $\chi(M) = \chi(CP_2)$. By the Gauss-Bonnet formula

$$\int_M (|R|^2 - 4|S|^2 + \rho^2)dV = \int_{CP_2} (|R'|^2 - 4|S'|^2 + \rho'^2)dV' . \tag{4.1}$$

From Lemma 2.

$$\int_M \left(|S|^2 - \frac{\rho^2}{2}\right)dV = \int_{CP_2} \left(|S'|^2 - \frac{\rho'^2}{2}\right)dV' . \tag{4.2}$$

Moreover, (2.3) implies

$$\int_M \left(-\frac{11}{20}|R|^2 + \frac{43}{10}|S|^2 - \rho^2\right)dV = \int_{CP_2}\left(-\frac{11}{20}|R'|^2 + \frac{43}{10}|S'|^2 - \rho'^2\right)dV' . \tag{4.3}$$

The relations (4.1)–(4.3) give rise to

$$\int_M |R|^2 dV = \int_{CP_2} |R'|^2 dV' , \quad \int_M |S|^2 dV = \int_{CP_2} |S'|^2 dV' , \quad \int_M \rho^2 dV = \int_{CP_2} \rho'^2 dV' ,$$

from which

$$\int_M \left(|S|^2 - \frac{\rho^2}{2}\right) dV = \int_{CP_2} \left(|S'|^2 - \frac{\rho'^2}{2}\right) dV' = 0 ,$$

and this implies that g in an Einstein metric with $\rho = \rho'$. It follows from Lemma 1 that (M, g) is isometric with $(\boldsymbol{CP}_2, g_0)$. Thus, *if* (M, g) *is a compact Kaehler manifold with* $b_2(M) = 1$, *and if* $\mathrm{Spec}^1(M, g) = \mathrm{Spec}^1(\boldsymbol{CP}_2, g_0)$ (*or* $\mathrm{Spec}^3(M, g) = \mathrm{Spec}^3(\boldsymbol{CP}_2, g_0)$), *then* (M, g) *is holomorphically isometric with* $(\boldsymbol{CP}_2, g_0)$.

(c) The case $p = 1$ and $n = 1$ in Theorem 2 was proved by Tanno [9].

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