

THE HOMOLOGY COVERING OF A RIEMANN SURFACE

Dedicated to Tadashi Kuroda on his sixtieth birthday

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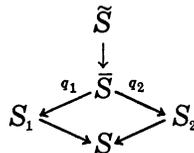
The Riemann surface \hat{S} is an Abelian cover of S if there is a regular covering $p: \hat{S} \rightarrow S$ where the group of deck transformations is Abelian.

The *homology covering* $p: \tilde{S} \rightarrow S$ is the highest Abelian covering of S ; i.e., it is the covering corresponding to the commutator subgroup of $\pi_1(S)$.

THEOREM. *Let S_1 and S_2 be closed Riemann surfaces, where S_m has genus $g_m \geq 2$. Suppose S_1 and S_2 have conformally equivalent homology covering surfaces. Then S_1 and S_2 are conformally equivalent.*

PROOF. We regard S_1 and S_2 as having the same homology cover \tilde{S} . Let Γ_m be the group of deck transformations for \tilde{S} covering S_m ; i.e., $\tilde{S}/\Gamma_m = S_m$. Let Γ be the group of conformal self-maps of \tilde{S} generated by Γ_1 and Γ_2 . It is well known that \tilde{S} is a surface of infinite genus, so the full group of conformal automorphisms of \tilde{S} is discontinuous; in particular, Γ acts discontinuously. Set $S = \tilde{S}/\Gamma$. Note that while Γ_1 and Γ_2 both act freely, Γ need not.

Since S_1 and S_2 are compact, they are finite sheeted (possibly branched) coverings of S ; i.e., both Γ_1 and Γ_2 are of finite index in Γ . It then follows that $\bar{\Gamma} = \Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2 . Let $\bar{S} = \tilde{S}/(\Gamma_1 \cap \Gamma_2)$. We have the following diagram of coverings (note that \bar{S} is a smooth (unbranched) covering of both S_1 and S_2):



Since the defining subgroup of the covering $q_m: \bar{S} \rightarrow S_m$ contains the commutator subgroup, it is normal, and the group of deck transformations is Abelian; i.e., $q_m: \bar{S} \rightarrow S_m$ is a regular Abelian covering.

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Observe that a loop w on \bar{S} lifts to a loop on \tilde{S} if and only if its projection to S_m is homologically trivial. Let $q_{m*}: H_1(\bar{S}) \rightarrow H_1(S)$ be the induced map on homology. We have shown that q_{1*} and q_{2*} have equal kernels. Since q_m is a cover map, q_{m*} is surjective. Hence, considering homology with complex coefficients, q_{1*} and q_{2*} are linear surjections from the same space with the same kernel. It follows that their images have equal dimension; i.e., $g_1 = g_2$.

Let G_m be the group of deck transformations on \bar{S} for the covering $q_m: \bar{S} \rightarrow S_m$.

By looking at automorphic forms on the universal cover, we see that there is a linear injection $q_m^*: \Omega(S_m) \rightarrow \Omega(\bar{S})$ mapping Abelian differentials (the dual space to H_1) on S_m into Abelian differentials on \bar{S} . One can regard the image $q_m^*(\Omega(S_m))$ as being the Abelian differentials on \bar{S} which are invariant under G_m . Of course, the image $q_m^*(\Omega(S_m))$ is orthogonal to the kernel, $\text{Ker}(q_{m*})$. The (complex) dimension of the annihilator of $\text{Ker}(q_{m*})$ is g_m , the dimension of the image of q_{m*} . The image $q_m^*(\Omega(S_m))$ also has (complex) dimension g_m , hence $q_m^*(\Omega(S_m))$ is the annihilator of $\text{Ker}(q_{m*})$. We have shown that the Abelian differentials on \bar{S} which are invariant under G_1 are precisely those which are invariant under G_2 .

Let G be the group of conformal self-maps of \bar{S} generated by both G_1 and G_2 . Then $S = \bar{S}/G$. Since every Abelian differential on \bar{S} , which is invariant under G_m , is invariant under all of G , $\dim(\Omega(S)) \geq g_m$ (even though the projection from \bar{S} to S may be branched, the invariant Abelian differentials still project to (regular) Abelian differentials). Thus S has genus $\geq g_m$, so the coverings $S_m \rightarrow S$ are both isomorphisms; i.e., $S = \bar{S} = S_1 = S_2$.

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