Tôhoku Math. Journ.
38 (1986), 561-562.

THE HOMOLOGY COVERING OF A RIEMANN SURFACE

Dedicated to Tadashi Kuroda on his sixtieth birthday

BERNARD MASKIT*

(Received November 8, 1985)

The Riemann surface $\hat{S}$ is an Abelian cover of S if there is a regular covering $p: \hat{S} \to S$ where the group of deck transformations is Abelian.

The *homology covering* $p: \tilde{S} \to S$ is the highest Abelian covering of S; i.e., it is the covering corresponding to the commutator subgroup of $\pi_1(S)$.

THEOREM. *Let S_1 and S_2 be closed Riemann surfaces, where S_m has genus $g_m \geqq 2$. Suppose S_1 and S_2 have conformally equivalent homology covering surfaces. Then S_1 and S_2 are conformally equivalent.*

PROOF. We regard S_1 and S_2 as having the same homology cover $\tilde{S}$. Let Γ_m be the group of deck transformations for $\tilde{S}$ covering S_m; i.e., $\tilde{S}/\Gamma_m = S_m$. Let Γ be the group of conformal self-maps of $\tilde{S}$ generated by Γ_1 and Γ_2. It is well known that $\tilde{S}$ is a surface of infinite genus, so the full group of conformal automorphisms of $\tilde{S}$ is discontinuous; in particular, Γ acts discontinuously. Set $S = \tilde{S}/\Gamma$. Note that while Γ_1 and Γ_2 both act freely, Γ need not.

Since S_1 and S_2 are compact, they are finite sheeted (possibly branched) coverings of S; i.e., both Γ_1 and Γ_2 are of finite index in Γ. It then follows that $\bar{\Gamma} = \Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2. Let $\bar{S} = \tilde{S}/(\Gamma_1 \cap \Gamma_2)$. We have the following diagram of coverings (note that $\bar{S}$ is a smooth (unbranched) covering of both S_1 and S_2):

$$\begin{array}{ccccc} & & \tilde{S} & & \\ & & \downarrow & & \\ & \overset{q_1}{\swarrow} & \bar{S} & \overset{q_2}{\searrow} & \\ S_1 & & & & S_2 \\ & \searrow & & \swarrow & \\ & & S & & \end{array}$$

Since the defining subgroup of the covering $q_m: \bar{S} \to S_m$ contains the commutator subgroup, it is normal, and the group of deck transformations is Abelian; i.e., $q_m: \bar{S} \to S_m$ is a regular Abelian covering.

* Research supported in part by NSF grant #MCS 8401280.

Observe that a loop w on $\bar{S}$ lifts to a loop on $\tilde{S}$ if and only if its projection to S_m is homologically trivial. Let $q_{m*}\colon H_1(\bar{S}) \to H_1(S)$ be the induced map on homology. We have shown that q_{1*} and q_{2*} have equal kernels. Since q_m is a cover map, q_{m*} is surjective. Hence, considering homology with complex coefficients, q_{1*} and q_{2*} are linear surjections from the same space with the same kernel. It follows that their images have equal dimension; i.e., $g_1 = g_2$.

Let G_m be the group of deck transformations on $\bar{S}$ for the covering $q_m\colon \bar{S} \to S_m$.

By looking at automorphic forms on the universal cover, we see that there is a linear injection $q_m^*\colon \Omega(S_m) \to \Omega(\bar{S})$ mapping Abelian differentials (the dual space to H_1) on S_m into Abelian differentials on $\bar{S}$. One can regard the image $q_m^*(\Omega(S_m))$ as being the Abelian differentials on $\bar{S}$ which are invariant under G_m. Of course, the image $q_m^*(\Omega(S_m))$ is orthogonal to the kernel, $\mathrm{Ker}(q_{m*})$. The (complex) dimension of the annihilator of $\mathrm{Ker}(q_{m*})$ is g_m, the dimension of the image of q_{m*}. The image $q_m^*(\Omega(S_m))$ also has (complex) dimension g_m, hence $q_m^*(\Omega(S_m))$ is the annihilator of $\mathrm{Ker}(q_{m*})$. We have shown that the Abelian differentials on $\bar{S}$ which are invariant under G_1 are precisely those which are invariant under G_2.

Let G be the group of conformal self-maps of $\bar{S}$ generated by both G_1 and G_2. Then $S = \bar{S}/G$. Since every Abelian differential on $\bar{S}$, which is invariant under G_m, is invariant under all of G, $\dim(\Omega(S)) \geqq g_m$ (even though the projection from $\bar{S}$ to S may be branched, the invariant Abelian differentials still project to (regular) Abelian differentials). Thus S has genus $\geqq g_m$, so the coverings $S_m \to S$ are both isomorphisms; i.e., $S = \bar{S} = S_1 = S_2$.

STATE UNIVERSITY OF NEW YORK AT STONY BROOK
STONY BROOK, NY 11794
U.S.A.