Tôhoku Math. Journ.
38 (1986), 539-549.

REMARKS ON HOLOMORPHIC FAMILIES OF RIEMANN SURFACES

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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(Received October 21, 1985)

Introduction. Let W be a domain in $\boldsymbol{C}$ or a Riemann surface and let S be a Riemann surface of type (g, n) with $3g-3+n>0$, where g is the genus of S and n is the number of punctures of S. In this paper, we shall consider holomorphic families of S over W (see Section 1, Definition 1) or a locally holomorphic mapping of W to the Teichmüller space of S, and study their boundary behavior.

In Section 1, we shall state known results and set up our notations.

In Section 2, we shall investigate a holomorphic family of S over the punctured disk $0<|z|<1$ and the behavior as $z \to 0$. Imayoshi [7] obtained a similar result. We shall show a uniqueness theorem of holomorphic families of S (Theorem 2).

In Section 3, we shall discuss holomorphic families over a general domain or a Riemann surface, and consider the problems as in Section 2.

In Section 4, we construct two examples of holomorphic families which might be of interest.

Thanks are due to Professors Y. Imayoshi and M. Taniguchi for useful and stimulating conversation. The author expresses his thanks also to the referee for valuable suggestions.

1. Preliminaries. We first recall some known results about Teichmüller spaces. Let S be a Riemann surface of type (g, n) as above and let G be a torsion free Fuchsian group acting on the upper half plane U such that U/G is conformally equivalent to S. A marked Riemann surface is a triple (S, f, S'), where S' is a Riemann surface of type (g, n) and f is a quasiconformal mapping of S onto S'. We say two marked Riemann surfaces (S, f, S') and (S, g, S'') to be equivalent if there exists a conformal mapping h of S' onto S'' such that $g^{-1}\circ h\circ f: S \to S$ is homotopic to the identity. We denote by $[S, f, S']$, or $[f, S']$ for short, the equivalence class of (S, f, S'). The set of all these equivalence classes is called the *Teichmüller space* $T(S)$ of S. It is well-known that $T(S)$ is canonically identified with the Teichmüller space $T(G)$ of a Fuchsian group G, which can be regarded as a bounded domain in C^{3g-3+n} by Bers' embedding (cf. Ahlfors [1], Bers [2]). Thus, we make no distinction between $T(S)$ and $T(G)$ in this paper. Hence both $T(S)$ and $T(G)$ have a natural complex structure.

The Teichmüller space $T(S)$ is a metric space with the *Teichmüller distance* $t_S(\cdot, \cdot)$ which is defined by

$$t_S([f, S'], [g, S''])$$
$$= \inf\{\tfrac{1}{2}\log K(h): h \text{ is a quasiconformal mapping of } S' \text{ onto } S'' \text{ homotopic to } g\circ f^{-1}\},$$

where $K(h)$ is the maximal dilatation of h. Furthermore, the Teichmüller distance is the *Kobayashi distance* in $T(S)$.

For every ϕ in $T(G)\cup\partial T(G)$ there exists a group isomorphism Θ_ϕ of G into $SL'(2, C)$. If ϕ is in $T(G)$, then $\Theta_\phi(G)$ is a quasi-Fuchsian group. If ϕ is in $\partial T(G)$, then $\Theta_\phi(G)$ is a Kleinian group called *b-group*. A b-group $\Theta_\phi(G)$ is called a *cusp* if there is a hyperbolic transformation g in G such that $\Theta_\phi(g)$ is parabolic. It is called a *totally degenerate group* if $\Omega(\Theta_\phi(G))$, the region of discontinuity of $\Theta_\phi(G)$, is connected and simply connected. It is called a *regular* b-group if the Poincaré area of $\Omega(\Theta_\phi(G))/\Theta_\phi(G)$ is twice of that of U/G. It is known that a b-group which is not a cusp is a totally degenerate group, and a regular b-group is a cusp.

Now, we shall define holomorphic families of Riemann surfaces.

DEFINITION 1. Let W be a domain in C or a Riemann surface and let ψ be a locally holomorphic (multivalent) mapping of W to $T(S)$. We assume that ψ has an analytic continuation along every curve in W. For an arbitrary branch $\psi(w; w_0) = [f_{w,w_0}, S_{w,w_0}]$ of ψ at $w_0 \in W$, we set $\psi^\gamma(w; w_0) = [f^\gamma_{w,w_0}, S^\gamma_{w,w_0}]$ for any curve γ in W from w_0 to a point w in W, where $\psi^\gamma(w; w_0)$ is the analytic continuation of $\psi(w; w_0)$ along γ. We call the triple (W, ψ, S) a *holomorphic family of* S *over* W if S_{w_0,w_0} and $S^c_{w_0,w_0}$ are always conformally equivalent for every w_0 in W and for every closed curve c in W based at w_0. Let g be a quasiconformal self mapping of S. Then a mapping g induces an automorphism $\chi(g)$ of $T(S)$ by

$$\chi(g)([f, S']) = [f\circ g^{-1}, S'] .$$

$\chi(g)$ is called a *modular transformation* of $T(S)$ and the group $\mathrm{Mod}(S)$ of all modular transformations of $T(S)$ is called the *modular group* of $T(S)$. It is known that $\chi(g)$ is a holomorphic isometry with respect to t_S and $\chi(g)$ depends only on the homotopy class $[g]$ of g. So, we set $\chi(g) = \chi[g]$.

Here, we shall introduce the *Thurston-Bers classification* of modular transformations. Let $\chi[g]$ be in $\mathrm{Mod}(S) - \{\mathrm{id.}\}$. We set

$$a(\chi[g]) = \inf\{t_S(\phi, \chi[g](\phi)) \colon \phi \in T(S)\} .$$

We call it elliptic if it has a fixed point in $T(S)$, parabolic if $a(\chi[g]) = 0$ and there is no fixed point, hyperbolic if $a(\chi[g]) > 0$ and there is a point ϕ in $T(S)$ with $a(\chi[g]) = t_S(\phi, \chi[g](\phi))$, and pseudo-hyperbolic if $a(\chi[g]) > 0$ and $a(\chi[g]) < t_S(\phi, \chi[g](\phi))$ for all ϕ in $T(S)$. These definitions are equivalent to those defined in terms of homotopy class $[g]$ (cf. Bers [3]).

2. Holomorphic families of Riemann surfaces over the punctured disk. In this section, we shall consider a holomorphic family (D^*, ψ, S) of a Riemann surface S over the punctured disk $D^* = \{z \in \boldsymbol{C}; 0 < |z| < 1\}$.

Fix a point w_0 in D^* and a branch $\psi(\cdot; w_0)$ at w_0. For a circle $c = \{w_0 e^{i\theta}; 0 \leqq \theta \leqq 2\pi\}$, there exists $\chi_c \in \mathrm{Mod}(S)$ such that $\psi^c(w_0; w_0) = \chi_c(\psi(w_0; w_0))$, because S_{w_0, w_0} and $S^c_{w_0, w_0}$ are conformally equivalent to each other. Furthermore, χ_c depends only on the homotopy class c in D^* and does not depend on the choice of w_0 and $\psi(\cdot, w_0)$. It is known that if χ_c is the identity, then ψ is a single valued holomorphic mapping of D^* to $T(S)$, and has an extension to a holomorphic mapping of the unit disk D to $T(S)$. If χ_c is of finite order, then a similar result also holds (cf. Imayoshi [7]).

Here, we shall consider only the case that χ_c is of infinite order.

THEOREM 1. *Let (D^*, ψ, S) be a holomorphic family of S over the punctured disk D^*. If χ_c is of infinite order, then $\Theta_\phi(G)$ is a regular b-group for every accumulation point ϕ of $\{\psi^{c^n}(w_0; w_0)\}_{n=1}^{\pm\infty}$.*

PROOF. Let $\pi \colon U \to D^*$ be the universal covering of D^* with the covering transformation group $\langle A \rangle = \{A^n; n \in \boldsymbol{Z}\}$, where A is a parabolic Möbius transformation keeping U fixed. Then, there is a holomorphic mapping $\Psi \colon U \to T(S)$ so that $\Psi(A(z)) = \chi_c(\Psi(z))$ $(z \in U)$ and the following diagram is commutative:

$$\begin{tikzcd} U \arrow[r, "\Psi"] \arrow[d, "\pi"'] & T(S) \\ D^* \arrow[ur, "\psi"'] & \end{tikzcd}$$

Since the Teichmüller distance is the Kobayashi distance on $T(S)$, we have, by the well-known holomorphic decreasing property,

$$\rho_U(z, A(z)) \geqq t_S(\Psi(z), \Psi(A(z))) = t_S(\Psi(z), \chi_c(\Psi(z))) \geqq a(\chi_c) ,$$

where $\rho_U(\cdot, \cdot)$ is the Poincaré (Kobayashi) distance on U. And we have

$$0 = \inf_{z \in U} \rho_U(z, A(z)) \geqq a(\chi_c)\ , \quad \text{i.e., } a(\chi_c) = 0$$

because of the parabolicity of A. Since χ_c is of infinite order, χ_c is not elliptic (Bers [3]) but parabolic. Therefore, by [12, Theorem 3.1] every accumulation point of $\{\Psi^{c^n}(w_0; w_0)\}_{n=0}^{\pm\infty} = \{\chi_c^n(\Psi(z_0))\}_{n=0}^{\pm\infty}$ $(\pi(z_0) = w_0)$ is a regular b-group. Thus, the statement of Theorem 1 is proved.

For every open arc $\gamma = \{\gamma(t); 0 \leqq t < 1\}$ in D^* with $\gamma(0) = w_0$ and $\lim_{t\to 1} \gamma(t) = 0$, we set $\gamma_s = \{\gamma(t); 0 \leqq t \leqq s\}$ $(s < 1)$ and $f(s) = \psi^{\gamma_s}(\gamma(s); w_0) \in T(S)$.

THEOREM 2. *Let (D^*, ψ, S) be a holomorphic family of S over the punctured disk D^*. If χ_c is of infinite order, then the cluster set of $f(s)$ as $s \to 1$ consists of cusps on $\partial T(S)$. Especially, if* $\dim T(S) = 1$, *then* $\lim_{s\to 1} f(s)$ *exists and it is a regular b-group.*

PROOF. Since χ_c is a parabolic modular transformation, there is a hyperbolic Möbius transformation g in G such that $(\text{trace}\,\Theta(\chi_c^n(z); g)^2 \to 4$ as $n \to \pm\infty$ for any $\phi \in T(G)$, where $\Theta(\phi, \cdot)$ is the group isomorphism $\Theta_\phi(\cdot)$ of G for $\phi \in T(G) \cup \partial T(G)$ referred to in Section 1 (cf. [12, the proof of Theorem 3.1]). Set $\tilde{c} = \pi^{-1}(c) \subset U$. Then $\tilde{c}$ is a horocycle tangent to ∂U at the fixed point z_A of A. Noting that $\chi_c^n(\Psi(z_0)) = \Psi(A^n(z_0))$ is in $\Psi(\tilde{c})$ for $z_0 \in \pi^{-1}(w_0)$ and $\rho_U(A^n(z_0), A^{n+1}(z_0)) = \rho_U(z_0, A(z_0)) < +\infty$, we conclude that $(\text{trace}\,\Theta(\Psi(z); g) \to 4$ as $z \to z_A$ along $\tilde{c}$ (see [11, Proof of Theorem 5 (b)-(ii)]). Since $(\text{trace}\,\Theta(\Psi(z); g))^2$ is a bounded holomorphic function on U, we have from the Poisson integral on $\partial\tilde{c}$

$$(\text{trace}\,\Theta(\Psi(z); g))^2 \to 4$$

as $z \to z_A$ from the inside of $\tilde{c}$. For $\gamma = \{\gamma(t) \in D^*; 0 \leqq t < 1,\ \gamma(0) = 0, \lim_{t\to 1} \gamma(t) = 0\}$, $\tilde{\gamma} = \pi^{-1}(\gamma) = \{\tilde{\gamma}(t); 0 \leqq t < 1\}$ is an open arc terminating at z_A from the inside of $\tilde{c}$. Hence $4 = \lim_{s\to 1} (\text{trace}\,\Theta(\Psi(\tilde{\gamma}(s)); g))^2 = \lim_{s\to 1} (\text{trace}\,\Theta(\Psi^{\gamma_s}(\gamma(s); w_0); g))^2 = \lim_{s\to 1} (\text{trace}\,\Theta(f(s); g))^2$ and we have the first assertion of Theorem 2.

If $\dim T(S) = 1$, then a cusp is a regular b-group [9, Theorem 5]. The set $\{\phi \in T(S) \cup \partial T(S) \subset C; (\text{trace}\,\Theta(\phi; g))^2 = 4\}$ is discrete for every hyperbolic $g \in G$. Hence $\lim_{s\to 1} f(s) \in \{\phi \in T(S) \cup \partial T(S); (\text{trace}\,\Theta(\phi; g))^2 = 4$, for some hyperbolic $g \in G\}$ exists.

REMARK. When $1 < \dim T(S) < +\infty$, the author does not know whether the cluster set of $f(s)$ as $s \to 1$ consists of only regular b-groups. Imayoshi [7] showed that if an arc γ is contained in an angular domain $\{z \in D^*; \theta_1 < \arg z < \theta_2 \leqq \theta_1 + 2\pi\}$, then $f(s)$ converges to a regular b-group as $s \to 1$. Hence we can show that if

$$(*) \qquad \sup_{0<\theta\leqq 2\pi} \{|\gamma(t_\theta)|\,|\gamma(t'_\theta)|^{-1};\ 0 \leqq t_\theta < t'_\theta < 1,\ \arg\gamma(t_\theta) = \arg\gamma(t'_\theta) = \theta \text{ and } \arg\gamma(t) \neq \theta \text{ for every } t \text{ in } (t_\theta, t'_\theta)\} = M < +\infty\ ,$$

then $f(s)$ converges to a regular b-group as $s \to 1$. Indeed, the condition (*) implies that

$$|\gamma(t_{n,\theta})| \leqq M^n |\gamma(t_{0,\theta})|\ ,$$

where $\gamma(t_{n,\theta})$ is a point on γ satisfying

$$\int_{t_{0,\theta}}^{t_{n,\theta}} d\arg\gamma(t) = 2n\pi\ , \quad n \in \boldsymbol{N}\ , \quad 0 \leqq \theta < 2\pi\ .$$

Thus, when we take $\pi(z) = e^{2\pi i z}$, $z_A = \infty$ and $\gamma(z) = z + 1$, there exists $z_{n,\theta}$ such that $\pi(z_{n,\theta}) = \gamma(t_{n,\theta})$ and

$$\arg z_{n,\theta} \geqq 2\pi |z_{n,\theta}|/\log M\ .$$

Hence, we conclude that $\rho_U(\tilde{\gamma}, \tilde{\gamma}_0) = M' < \infty$, where $\tilde{\gamma} = \pi^{-1}(\gamma)$ and $\tilde{\gamma}_0 = \pi^{-1}(\{rw_0/|w_0|;\ 0 < r < 1\})$ and $\rho_U(\tilde{\gamma}, \tilde{\gamma}_0)$ is the Poincaré distance between $\tilde{\gamma}$ and $\tilde{\gamma}_0$. Thus, there exists a point $z_0(z)$ on $\tilde{\gamma}_0$ for each z on $\tilde{\gamma}$ such that $\rho_U(z, z_0(z)) \leqq M'$. Since the Teichmüller distance is the Kobayashi distance, we have $t_S(\Psi(z), \Psi(z_0(z))) \leqq \rho_U(z, z_0(z)) \leqq M'$. Hence, from [4, Lemma 1] we have the assertion.

Next, we shall show the following uniqueness theorem of holomorphic families over D^*.

We consider a totally geodesic disk $\boldsymbol{D}$ in $T(S)$, namely, there is a holomorphic embedding F of U into $T(G)$ such that $\boldsymbol{D} = F(U)$ and F is isometric with respect to $\rho_U(\cdot, \cdot)$ and $t_S(\cdot, \cdot)$. Suppose that there exists $\chi \in \mathrm{Mod}(S)$ such that χ is parabolic and $\chi(\boldsymbol{D}) = \boldsymbol{D}$. Then $A = F^{-1}\circ\chi\circ F$ is a Möbius transformation keeping U fixed and is parabolic. Indeed, if A is not parabolic, then A is elliptic or hyperbolic. Since χ has no fixed point, neither does A and A must be hyperbolic. Therefore, there exists a non-Euclidean line L on U such that $A(L) = L$. $F(L)$ is a geodesic line with respect to $t_S(\cdot, \cdot)$ and $\chi(F(L)) = F(L)$. This implies that χ is a hyperbolic modular transformation by [3, Corollary 1 to Theorem 5], and yields a contradiction. Thus A is parabolic. Marden-Masur [9] gives a certain parabolic modular transformation keeping a geodesic disk fixed.

Since $D^* \simeq U/\langle A\rangle$, we can easily show that there is an $f\colon D^* \to \boldsymbol{D}/\langle\chi\rangle$ so that the following diagram is commutative:

$$(**) \qquad \begin{array}{ccc} U & \xrightarrow{\quad F \quad} & \boldsymbol{D} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\pi}} \\ D^* & \xrightarrow{\quad f \quad} & \boldsymbol{D}/\langle \chi \rangle \end{array}$$

where $\tilde{\pi}$: $\boldsymbol{D} \to \boldsymbol{D}/\langle \chi \rangle$ is the natural projection. Then, (D^*, ψ, S) is a holomorphic family of S over D^*, where $\psi = f \circ \tilde{\pi}^{-1}$. We call this holomorphic family *canonical* for $\boldsymbol{D}$ and χ. The argument in Marden-Masur [9, §3] implies the following theorem which shows that the canonical holomorphic family is unique in a certain sense.

THEOREM 3. *Let (D^*, ψ_j, S) be canonical holomorphic families for $\boldsymbol{D}$ and χ in* Mod(S) *of S over D^* $(j = 1, 2)$. If $\psi_1(w_0; w_0) = \psi_2(w_0; w_0)$ for a point w_0 in D^*, then $\psi_1 = \psi_2$ on D^*.*

3. In this section, we shall consider more general W.

THEOREM 4. *Let E be a compact subset of the unit disk D with capacity positive and let $(D - E, \psi, S)$ be a holomorphic family of S over $D - E$. Then there exist a point e in E and an open arc $\gamma = \{\gamma(t); 0 \leqq t < 1\}$ in $D - E$ with* $\lim_{t\to 1} \gamma(t) = e$ *such that* $\lim_{s\to 1} f(s)$ *exists and is not a cusp, where $f(s)$ is the same as that in Theorem 1. Hence if* $\lim_{s\to 1} f(s)$ *belongs to $\partial T(S)$, then it is a totally degenerate group.*

PROOF. Let π: $D \to D - E$ be the universal covering surface of $D - E$ and let u_E be the harmonic measure of E with respect to $\partial(D - E)$. Note that π is a bounded holomorphic function and $u_E \circ \pi$ (>0) is the harmonic measure of $\partial D - \pi^{-1}(\partial D)$ with respect to ∂D, where $\pi^{-1}(\partial D)$ is the subset of ∂D corresponding to the border curve ∂D of $D - E$ via (the continuous extension of) π. Then, we can take a measurable subset α of ∂D with positive measure in such a way that the radial limits $\lim_{r\to 1} \pi(re^{i\theta})$ and $\lim_{r\to 1} u_E \circ \pi(re^{i\theta})$ exist for every $e^{i\theta}$ in α and $\lim_{r\to 1} u_E \circ \pi(re^{i\theta}) = 1$. Hence, the open arc $\gamma_\theta = \{\pi(re^{i\theta}); 0 < r < 1\}$ in $D - E$ is terminating at a point on E for every $e^{i\theta}$ in α. Applying [11, Theorem 5 (a)-ii)] to $\Psi = \psi \circ \pi$: $D \to T(S)$, we verify that there exists an $e^{i\theta}$ in α such that γ_θ is our desired open arc in $D - E$. The last statement is an immediate consequence from Maskit [10, Theorem 4].

Next, we shall show a uniqueness theorem for holomorphic families of S over a parabolic Riemann surface.

THEOREM 5. *Suppose that W is a Riemann surface in O_G, namely,*

W has no Green's functions. Let (W, ψ_j, S) be holomorphic families of S over W $(j = 1, 2)$. Then $\psi_1 = \psi_2$ if $\psi_1(w_0; w_0) = \psi_2(w_0; w_0)$ for a fixed point w_0 in W, and if for each closed curve c in W there is a χ_c in $\mathrm{Mod}(S)$ *such that $\psi_j^c(w_0; w_0) = \chi_c(\psi_j(w_0; w_0))$ $(j = 1, 2)$.*

PROOF. First, we suppose that the universal covering surface of W is (conformally equivalent to) the unit disk D, and let $\pi: D \to W$ be the natural projection. Then, there are holomorphic mappings Ψ_j $(j = 1, 2)$: $D \to T(S)$ so that the following diagram is commutative:

$$\begin{tikzcd} D \arrow[r, "\Psi_j"] \arrow[d, "\pi"'] & T(S) \\ W \arrow[ur, "\psi_j"'] & \end{tikzcd}$$

Since W belongs to O_G, the covering transformation group G of W is a Fuchsian group of *divergence type* (cf. Tsuji [13]). Hence

$$(1) \qquad \sum_{m=0}^{\infty} (1 - |a_m|) = +\infty ,$$

where $\{a_m\}_{m=0}^{\infty} = \pi^{-1}(w_0)$.

For each closed curve c based at w_0 there exists g_c in G such that $\Psi_j(g_c(a_0)) = \Psi_j(a_m) = \psi_j(\pi(a_m)) = \psi_j^c(w_0; w_0) = \chi_c(\psi_j(w_0; w_0))$ $(j = 1, 2)$ for some m. Therefore, we have

$$(2) \qquad \Psi_1(a_m) = \Psi_2(a_m) , \qquad m = 0, 1, 2, \cdots .$$

As we identify $T(S)$ with a bounded domain in C^{3g-3+n} by Bers' embedding, we can write $\Psi_j(z) = (\Psi_j^1(z), \cdots, \Psi_j^{3g-3+n}(z))$ $(j = 1, 2$ and $z \in D)$, where Ψ_j^k $(j = 1, 2,\ k = 1, \cdots, 3g - 3 + n)$ are bounded holomorphic functions on D. From the equation (2), we have

$$(3) \qquad \Psi_j^k(a_m) = \Psi_j^k(a_m) , \quad k = 1, \cdots, 3g - 3 + n \quad \text{and} \quad j = 1, 2 .$$

On the other hand, the equation (1) implies that $\{a_m\}_{m=0}^{\infty}$ does not satisfy the Blaschke condition (cf. Garnett [6]). Thus, bounded holomorphic functions $\Psi_1^k - \Psi_2^k$ on D must vanish identically $(k = 1, \cdots, 3g - 3 + n)$. So, $\Psi_1 = \Psi_2$ and we proved the theorem when the universal covering surface of W is the unit disk.

Next, if the universal covering of W is not the unit disk, then it is the entire plane $\boldsymbol{C}$ or the Riemann sphere $\hat{\boldsymbol{C}}$. In either case, there are holomorphic mappings Ψ_j $(j = 1, 2)$ so that the following diagram is commutative;

$$\begin{tikzcd} C \text{ or } \widehat{C} \arrow[r, "\Psi_j"] \arrow[d, "\pi"'] & T(S) \\ W \arrow[ur, "\psi_j"'] & \end{tikzcd}$$

Since $T(S)$ is a bounded domain in C^{3g-3+n}, Ψ_j must be a constant. Since $\psi_1(a_0) = \psi_2(a_0)$, $\Psi_1 = \Psi_2$ and $\psi_1 = \psi_2$. The proof of Theorem 5 is completed.

REMARK. A result similar to Theorem 5 was proved by Borel-Narashimhan [5], but our proof is different from theirs. Recently, Imayoshi-Shiga [8] have obtained an extension of Theorem 5.

4. We shall construct two examples of holomorphic families of Riemann surfaces. The first is an example for which Theorem 4 can be applied, and the second is an example for which the exceptional set E is of null capacity but the same result as in Theorem 3 does not hold.

Let S be a Riemann surface of type $(0, 4)$. Then it is well-known that $T(S) = T(0, 4) \simeq U$ and $M(S) = M(0, 4) = T(S)/\mathrm{Mod}(S) \simeq \hat{C} - \{0, 1, \infty\}$. Hence, the modular group can be identified with a finitely generated Fuchsian group Γ with $U/\Gamma \simeq \hat{C} - \{0, 1, \infty\}$. We denote by $a_1, \cdots, a_k$ $(\in \hat{C} - \{0, 1, \infty\})$ the images of elliptic fixed points of Γ via the natural projection $\pi_1 : U\ (= T(0, 4)) \to U/\Gamma\ (= M(0, 4))$, and set $R' = \hat{C} - \{0, 1, \infty, a_1, \cdots, a_k\}$. In this situation, we have the following examples:

EXAMPLE 1. Take a closed disk K in R', and consider a torsion free Fuchsian group G_K acting on U so that $U/G_K \simeq R' - K$. Obviously, G_K is finitely generated and the second kind. Let π_2 be the natural projection of $\Omega(G_K)$ onto $\Omega(G_K)/G_K$, where $\Omega(G_K)$ is the region of discontinuity of G_K, and let B be an open disk with $E = B \cap \Lambda(G_K) \neq \emptyset$ and $\partial B \subset \Omega(G_K)$, where $\Lambda(G_K)$ is the limit set of G_K. Since $\Omega(G_K)/G_K$ is a double of U/G_K, the Riemann surface $\Omega(G_K)/G_K$ is of type $(0, 6 + 2k)$. Hence we consider $\Omega(G_K)/G_K$ as a subset of $M(0, 4) \simeq C - \{0, 1, \infty\}$. Then, a mapping $\pi_2|_{B-E}$ is considered as a holomorphic mapping of $B - E$ to $M(0, 4)$. Hence, a triple $(B - E, p^{-1}\circ\pi_2, S)$ is a holomorphic family of S over $B - E$, where p: $T(S) \to M(0, 4)$ (actually p: $U \to \hat{C} - \{0, 1, \infty\}$) is the natural projection of $T(S)$ onto $M(0, 4)$. It is known that the capacity of E is positive (cf. Tsuji [13]). Therefore, we obtain an example to which Theorem 3 is applicable.

Furthermore, it is easily seen that there exist a sequence $\{z_n\}_{n=1}^{\infty}$ in

$B - E$ converging to a point in E and g in G_K such that g induces a hyperbolic modular transformation χ, namely $\pi_2 \circ g(z) = p \circ \chi(p^{-1} \circ \pi_2(z))$ $(z \in B - E)$, and $z_n = g^n(z_1)$. For example, take small circles c_0, c_1 in R' such that

$$c_j(t) = j + \varepsilon e^{2\pi i t}\,, \quad 0 \leqq t < 1\,, \quad j = 0, 1\,.$$

And take a smooth Jordan curve β in R' from ε to $1 - \varepsilon$. Then both γ in Γ and g in G_K corresponding to $c_0 \beta c_1^{-1} \beta^{-1}$ are hyperbolic. As we regard $\hat{C} - \{0, 1, \infty\}$ as $M(0, 4)$, χ in Mod(S) corresponding to $c_0 \beta c_1^{-1} \beta^{-1}$ (consequently, corresponding to the conjugacy class of γ) must be hyperbolic. Thus, the desired χ is obtained. Then $p^{-1} \circ \pi_2(z_n) = \chi(p^{-1} \circ \pi_2(z_{n-1})) = \cdots = \chi^{n-1}(p^{-1} \circ \pi_2(z_1))$. Hence, from [4, Theorem 1] every accumulation point of $\{p^{-1} \circ \pi_2(z_n)\}_{n=1}^{\infty}$ is a non-cusped totally degenerate group in $\partial T(S)$.

EXAMPLE 2. We set $R = R' - I$, where I is the line on R' from 0 to ∞. Taking infinite copies $R_0, R_1, R_{-1}, R_2, R_{-2}, \cdots$ of R, we identify, along I, the upper edge on R_n with the lower edge on R_{n+1} $(n \in \boldsymbol{Z})$. Thus, an infinite sheeted unbranched covering surface $\hat{R}$ of $R' \subset M(0, 4)$ is obtained. We denote by f the projection of $\hat{R}$ onto R'. It is easily seen that $\hat{R}$ is planar and belongs to O_G, i.e., $\hat{R}$ admits no Green's functions. Actually, $\hat{R}$ is conformally equivalent to the sphere with infinite punctures converging to a point because 0 and ∞ are logarithmic branch points of the covering ($\hat{R}$ is regarded as $f^{-1}(R') = \boldsymbol{C} - f^{-1}(\{1, a_1, a_2, \cdots, a_k\})$, where $f(z) = e^z$). Take a small closed disk B in $\hat{R}$ and a Möbius transformation $F\colon \hat{C} \to \hat{C}$ which maps $\hat{C} - B$ onto the unit disk. Set $E = D - F(\hat{R} - B)$. The capacity of E is zero, since $E = F(\hat{C} - \hat{R}) = F(f^{-1}(\{1, a_1, \cdots, a_k\})) \cup F(\infty)$ is a countable set. We set $W = F(\hat{R} - B) = D - E$ and $\psi = \pi_1^{-1} \circ f \circ F^{-1}$. Since ψ is a locally holomorphic mapping of W to $T(0, 4)$, a triple (W, ψ, S) is a holomorphic family of S over W. From the construction and Theorem 1, we see that each point of $F(f^{-1}(1))$ satisfies the property required in Theorem 1. Furthermore, by the same argument as in Example 1, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in W $(= D - E)$ such that $z_n \to z_0$ as $n \to \infty$ and every accumulation point of $\{\psi(z_n)\}_{n=1}^{\infty}$ is a non-cusped totally degenerate group in $\partial T(S)$.

REMARKS. (1) For a holomorphic family (W, ψ, S) in Example 2, we can consider that a Riemann surface $S_p = \hat{C} - \{0, 1, \infty, w(p)\} = \hat{C} - \{0, 1, \infty, f \circ F^{-1}(p)\}$ $(p \in W)$ of type $(0, 4)$ is associated (holomorphically) to each point p in W, namely $\psi(p) = [f_p, S_p]$ with a suitable quasiconformal mapping $f_p\colon S \to S_p$. Then it is easily shown that $w(p)$ $(\neq 0, 1, \infty)$ is a holomorphic function on W, and $V = \cup_{p \in W} S_p$ is naturally a 2-

dimensional complex manifold. Furthermore, $V' = W \times (\hat{C} - \{0, 1, \infty\}) - \{(p, w(p)); p \in W\}$ is obviously a Stein manifold. Indeed, the set $\{(p, w(p); p \in W\} = \{(p, w) \in V'; w(p) - p = 0\}$ is a analytic hypersurface with codimension 1. Hence V is a Stein manifold.

(2) For a compact subset E of the unit disk D, we consider a holomorphic family $(D - E, \psi, S)$ of a Riemann surface S of type (g, n) $(3g - 3 + n > 0)$ over $D - E$. Let $M(g, n)$ be the moduli space of Riemann surfaces of type (g, n). Then we can define a holomorphic mapping $\alpha: D - E \to M(g, n)$ by

$$\alpha(p) = [S_p]\,, \quad \text{for every} \quad p \in D - E\,,$$

where $[S_p]$ is a point in $M(g, n)$ determined by $\psi(p)$. It is known (cf. [7]) that if E is discrete, then α is extended to a holomorphic mapping $\hat{\alpha}: D \to M(g, n)$, where $\hat{M}(g, n)$ is a reasonable compactification of $M(g, n)$. Hence, the following problem is raised;

When the logarithmic capacity of E is zero, is α extended to a holomorphic mapping $\hat{\alpha}: D \to \hat{M}(g, n)$?

But Example 2 gives a counterexample to this problem. Indeed, in this case, $\alpha(p) = w(p) = f \circ F^{-1}(p)$ for every p in $D - E$. α is extended to a holomorphic mapping $\alpha_0: D - \{F(\infty)\} \to \hat{M}(g, n) = \hat{C}$. However, α is not extended to a holomorphic mapping on D because $f(z) = e^z$ has an essential singularity at $z = \infty$.

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