

QUATERNIONIC SUBMANIFOLDS IN QUATERNIONIC SYMMETRIC SPACES

HIROYUKI TASAKI

(Received September 24, 1985)

Contents

1. Introduction.....	513
2. Quaternionic symmetric spaces.....	514
3. Reduction of the problem.....	517
4. Preliminaries for classification of simple subalgebras of index 1.....	521
5. $\mathfrak{sl}(n, \mathbf{C})$	525
6. $\mathfrak{o}(n, \mathbf{C})$	526
7. Homology classes represented by quaternionic submanifolds.....	532
8. $G_2/SO(4)$	536
References.....	537

1. Introduction. The purposes of this paper are to classify quaternionic submanifolds in a quaternionic symmetric space and to investigate the homology classes represented by quaternionic submanifolds in a compact quaternionic symmetric spaces. The following results motivate the subject of this paper. Quaternionic submanifolds in a quaternionic Kähler manifold are minimal stable submanifolds and compact ones are homologically volume minimizing, as were proved by the author in [13].

Here we shall give the definitions of quaternionic Kähler manifolds and quaternionic submanifolds and state some properties of quaternionic submanifolds. A $4n$ -dimensional connected Riemannian manifold M is called a *quaternionic Kähler manifold*, if M has the following property: There is a point x in M such that, through an identification of $T_x(M)$ with \mathbf{H}^n , the linear holonomy group of M at x is contained in $Sp(n)Sp(1)$. In this situation, take a piecewise smooth curve τ from x to y for any point y in M and put

$$S_y = P_\tau Sp(1) P_\tau^{-1},$$

where P_τ is the parallel translation along the curve τ . S_y is independent of the choice of τ , because $Sp(1)$ is a normal subgroup of $Sp(n)Sp(1)$. We call $S = \{S_y\}_{y \in M}$ a *quaternionic structure* on M . A connected submanifold N of M is called a *quaternionic submanifold* in M , if $T_y(N)$ is invariant under the action of S_y for each y in N . Alekseevskii [1] proved that a

quaternionic submanifold in a quaternionic Kähler manifold is totally geodesic.

Kraines [8] introduced a parallel 4-form Ω on a quaternionic Kähler manifold. We call Ω the *fundamental 4-form* on a quaternionic Kähler manifold. Let $\Omega^m(\mathbf{H}^m) = c_m$, which is independent of n . On a quaternionic Kähler manifold M with the fundamental 4-form Ω the inequality

$$\frac{1}{c_m} \Omega^m \Big|_{\xi} \leq \text{vol}_{\xi}$$

holds for any oriented tangent $4m$ -plane ξ on M . The equality holds if and only if ξ is invariant under the action of the quaternionic structure on M and has a suitable orientation. This result is proved in [13], but the coefficient of Ω^m in the inequality in [13] is incorrect. So Ω^m/c_m is a calibration on M in the sense of Harvey-Lawson [4]. Consequently, a quaternionic submanifold is a minimal stable submanifold and compact one is homologically volume minimizing.

A quaternionic symmetric space is a kind of quaternionic Kähler manifold. The definition of quaternionic symmetric space is given in Section 2. Let M' be a quaternionic symmetric space and $I_0(M')$ be the identity component of the group of all isometries on M' . Let \mathfrak{g}' be the Lie algebra of $I_0(M')$ and \mathfrak{g}'^c be its complexification. In Section 3 we shall reduce the classification of $I_0(M')$ -conjugacy classes of complete quaternionic submanifolds in M' to that of $\text{Int}(\mathfrak{g}'^c)$ -conjugacy classes of complex simple subalgebras of index 1 in \mathfrak{g}'^c . Sections 4, 5 and 6 are devoted to classifying all $\text{Int}(\mathfrak{g}'^c)$ -conjugacy classes of complex simple subalgebras of index 1 in classical complex simple Lie algebras \mathfrak{g}'^c .

In Section 7 we shall consider the injectivity of the map χ from the set $\mathcal{S}(M')$ of all $I_0(M')$ -conjugacy classes of complete quaternionic submanifolds in a compact quaternionic symmetric space M' to $H_*(M'; \mathbf{R})$. The map χ assigns the $I_0(M')$ -conjugacy class represented by a quaternionic submanifold M in M' to the homology class represented by M . We shall show that χ is injective when M' is

$$\begin{aligned} G_{2,n}^c &= SU(n+2)/S(U(2) \times U(n)), & G_{4,3}^R &= SO(7)/SO(4) \times SO(3) \text{ or} \\ G_{4,4}^R &= SO(8)/SO(4) \times SO(4). \end{aligned}$$

The compact quaternionic symmetric space $G_2/SO(4)$ of exceptional type is investigated in Section 8. We shall show that there is a unique $I_0(G_2/SO(4))$ -conjugacy class of quaternionic submanifolds in $G_2/SO(4)$, which is represented by $P^2(\mathbf{C}) = G_{2,1}^c$.

2. Quaternionic symmetric spaces. We first give the definition of

quaternionic symmetric spaces and next show that a quaternionic submanifold in a quaternionic symmetric space is also a quaternionic symmetric space. By Wolf [14] all compact quaternionic symmetric spaces are constructed from compact simple Lie algebras whose ranks are greater than 1. All noncompact quaternionic symmetric spaces are obtained as the noncompact duals of compact ones. We shall review this construction in this section.

DEFINITION 2.1. A Riemannian manifold M is called a *quaternionic symmetric space* if M satisfies the following conditions:

- (i) M is a quaternionic Kähler manifold with quaternionic structure S .
- (ii) M is a symmetric space.
- (iii) S_x is contained in the linear holonomy group $H_x(M)$ for some point x in M .

REMARKS. (i) If M is a quaternionic symmetric space, then S_y is contained in $H_y(M)$ for any point y in M .

(ii) There exists a Riemannian manifold which is both quaternionic Kähler and symmetric but is not quaternionic symmetric. $S^2 \times S^2$ is such an example.

LEMMA 2.2. *Let M' be a quaternionic symmetric space and M be a complete quaternionic submanifold in M' . Then M is also a quaternionic symmetric space with respect to the induced Riemannian metric.*

PROOF. Let S be the quaternionic structure on M' . By Alekseevskii [1], M is totally geodesic in M' . Therefore M is also symmetric and quaternionic Kähler with the quaternionic structure $S|_M = \{S_y|_{T_y(M)}\}_{y \in M}$. Hence to prove Lemma 2.2 we may only show that M satisfies the condition (iii) in Definition 2.1. Fix a point x in M . We can extend each element h in $H_x(M')$ to an isometry \bar{h} of M' which satisfies $\bar{h}(x) = x$ and $(\bar{h}_*)_x = h$. For each q in S_x , the isometry \bar{q} leaves M invariant. Therefore $S_x|_{T_x(M)}$ is contained in $H_x(M)$, that is, M satisfies (iii).

Now we shall construct compact quaternionic symmetric spaces from compact simple Lie algebras whose ranks are greater than 1. Let \mathfrak{g} be a compact simple Lie algebra whose rank is greater than 1 furnished with an $\text{Int}(\mathfrak{g})$ -invariant inner product \langle, \rangle . Take a maximal Abelian subalgebra \mathfrak{t} in \mathfrak{g} . Then the complexification \mathfrak{t}^c of \mathfrak{t} is a Cartan subalgebra of the complexification \mathfrak{g}^c of \mathfrak{g} . For each element α in \mathfrak{t} , put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^c; [H, X] = \sqrt{-1}\langle \alpha, H \rangle X \text{ for each } H \in \mathfrak{t}\}.$$

An element α in $\mathfrak{t} - \{0\}$ is called a *root* if $\mathfrak{g}_\alpha \neq \{0\}$. Let \mathcal{A} denote the set of all roots. Fix a lexicographic ordering on \mathfrak{t} and denote by \mathcal{A}_+ the set of all positive roots in \mathcal{A} and by δ the highest root in \mathcal{A}_+ . Set

$$\mathfrak{g}_1 = \mathbf{R}\delta + \mathfrak{g} \cap (\mathfrak{g}_\delta + \mathfrak{g}_{-\delta}).$$

Then \mathfrak{g}_1 is a compact 3-dimensional simple subalgebra of \mathfrak{g} . It is known that the analytic subgroup G_1 of $\text{Int}(\mathfrak{g})$ corresponding to \mathfrak{g}_1 is simply connected and isomorphic to $Sp(1)$ (cf. Wolf [14, the proof of Theorem 5.4]). Let \mathfrak{z} be the centralizer of \mathfrak{g}_1 in \mathfrak{g} . The subalgebra $\mathfrak{k} = \mathfrak{g}_1 + \mathfrak{z}$ is the normalizer of \mathfrak{g}_1 in \mathfrak{g} and \mathfrak{g}_1 and \mathfrak{z} are ideals in \mathfrak{k} . Define an automorphism s of \mathfrak{g} and a linear subspace \mathfrak{p} of \mathfrak{g} by

$$s = \exp\left(\frac{2\pi}{\langle \delta, \delta \rangle} \delta\right) \in \text{Int}(\mathfrak{g})$$

and

$$\mathfrak{p} = \mathfrak{g} \cap \sum_{\alpha} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}),$$

where α runs through the elements of $\mathcal{A}_+ - \{\delta\}$ satisfying $\langle \alpha, \delta \rangle \neq 0$. The automorphism s of \mathfrak{g} is involutive, because

$$s|_{\mathfrak{t}} = \text{id}_{\mathfrak{t}}, \quad s|_{\mathfrak{p}} = -\text{id}_{\mathfrak{p}} \quad \text{and} \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{p}.$$

Hence (\mathfrak{g}, s) is a compact orthogonal symmetric Lie algebra. Denote by K the analytic subgroup of $\text{Int}(\mathfrak{g})$ corresponding to \mathfrak{k} . From the construction of K it follows that K is connected and of maximal rank in $\text{Int}(\mathfrak{g})$, so the compact symmetric space $\text{Int}(\mathfrak{g})/K$ is simply connected. Since $\mathfrak{k} = \mathfrak{g}_1 + \mathfrak{z}$ is an ideal decomposition, G_1 defines a quaternionic structure on $\text{Int}(\mathfrak{g})/K$ through the parallel translations. The symmetric space is a quaternionic Kähler manifold with the quaternionic structure defined by G_1 . The relation $G_1 \subset K$ implies that $\text{Int}(\mathfrak{g})/K$ is a quaternionic symmetric space. The quaternionic symmetric space $\text{Int}(\mathfrak{g})/K$ does not depend on the choice of a maximal Abelian subalgebra \mathfrak{t} or an ordering on \mathfrak{t} .

THEOREM 2.3. *For each compact simple Lie algebra \mathfrak{g} whose rank is greater than 1, $\text{Int}(\mathfrak{g})/K$ constructed above is a compact simply connected quaternionic symmetric space. Conversely, each compact quaternionic symmetric space is of the form $\text{Int}(\mathfrak{g})/K$ for some compact simple Lie algebra \mathfrak{g} whose rank is greater than 1. Furthermore, the noncompact quaternionic symmetric spaces are just the noncompact duals of the spaces $\text{Int}(\mathfrak{g})/K$ described above.*

PROOF. We may only prove that each compact quaternionic symmetric space is of the form $\text{Int}(\mathfrak{g})/K$ described above. Let M be a compact quaternionic symmetric space. By Wolf [14, Lemma 3.1], M is irreducible.

So the Ricci curvature tensor of M is positive definite. By Alekseevskii [1, Assertion 6], M is simply connected, so M is a compact simply connected quaternionic symmetric space. By Theorem 5.4 in Wolf [14], M is the space $\text{Int}(\mathfrak{g})/K$ for some compact simple Lie algebra \mathfrak{g} whose rank is greater than 1.

For the rest of the present paper we shall use the notations \langle , \rangle , t, Δ, \dots described above for a compact simple Lie algebra \mathfrak{g} and $\langle , \rangle', t', \Delta', \dots$ for \mathfrak{g}' .

REMARK. By Theorem 2.3, we can obtain the list of compact quaternionic symmetric spaces as in the Table.

TABLE

	\mathfrak{g}^C	Rank	Dimension
$G_{2,1}^C$	$\mathfrak{sl}(3, C)$	1	4
$G_{2,n}^C$ ($n \geq 2$)	$\mathfrak{sl}(n+2, C)$	2	$4n$
$G_{4,3}^R$	$\mathfrak{o}(7, C)$	3	12
$G_{4,n}^R$ ($n \geq 4$)	$\mathfrak{o}(n+4, C)$	4	$4n$
$P^n(\mathbf{H})$ ($n \geq 1$)	$\mathfrak{sp}(n+1, C)$	1	$4n$
$(\mathfrak{e}_6, \mathfrak{su}(6) + \mathfrak{sp}(1))$	\mathfrak{e}_6^C	4	40
$(\mathfrak{e}_7, \mathfrak{o}(12) + \mathfrak{sp}(1))$	\mathfrak{e}_7^C	4	64
$(\mathfrak{e}_8, \mathfrak{e}_7 + \mathfrak{sp}(1))$	\mathfrak{e}_8^C	4	112
$(\mathfrak{f}_4, \mathfrak{sp}(3) + \mathfrak{sp}(1))$	\mathfrak{f}_4^C	4	28
$G_2/SO(4)$	\mathfrak{g}_2^C	2	8

3. Reduction of the problem. Let \mathfrak{g}' be a compact simple Lie algebra whose rank is greater than 1. Then $M' = \text{Int}(\mathfrak{g}')/K'$ constructed in Section 2 is a compact quaternionic symmetric space. In this section we shall reduce the classification of $I_0(M')$ -conjugacy classes of complete quaternionic submanifolds in M' to that of $\text{Int}(\mathfrak{g}')$ -conjugacy classes of simple subalgebras of index 1 in \mathfrak{g}' whose ranks are greater than 1 (Theorem 3.1). Next for a compact semisimple Lie algebra \mathfrak{g}' we shall reduce the classification of $\text{Int}(\mathfrak{g}')$ -conjugacy classes of semisimple subalgebras in \mathfrak{g}' to that of $\text{Int}(\mathfrak{g}'^C)$ -conjugacy classes of complex semisimple subalgebras in \mathfrak{g}'^C (Proposition 3.3).

From now on, for any compact simple Lie algebra \mathfrak{g} , we shall consider the invariant inner product \langle , \rangle on \mathfrak{g} normalized by the condition $\langle \delta, \delta \rangle = 2$, which is independent of the choice of a maximal Abelian subalgebra t . Let \mathfrak{g} be a simple subalgebra in a compact simple Lie algebra \mathfrak{g}' and $\iota: \mathfrak{g} \rightarrow \mathfrak{g}'$ be the inclusion map. The *index* j , of the simple subalgebra \mathfrak{g} in \mathfrak{g}' is defined by the equation:

$$j_i \langle X, Y \rangle = \langle \iota(X)\iota, (Y) \rangle'$$

for all X and Y in \mathfrak{g} , after Dynkin. By Theorem 2.2 in Dynkin [3], the index j_i is a positive integer.

For a simple subalgebra \mathfrak{g} in a compact simple Lie algebra \mathfrak{g}' , denote by $c(\mathfrak{g})$ the $\text{Int}(\mathfrak{g}')$ -conjugacy class represented by \mathfrak{g} , and by $\mathcal{E}_1(\mathfrak{g}')$ the set of all $\text{Int}(\mathfrak{g}')$ -conjugacy classes of simple subalgebras of index 1 in \mathfrak{g}' whose ranks are greater than 1. Similarly, for a complete quaternionic submanifold M in the compact quaternionic symmetric space $M' = \text{Int}(\mathfrak{g}')/K'$, denote by $c(M)$ the $I_0(M')$ -conjugacy class represented by M , and by $\mathcal{E}(M')$ the set of all $I_0(M')$ -conjugacy classes of complete quaternionic submanifolds in M' . Note that $I_0(M')$ leaves the quaternionic structure on M' invariant.

We shall construct a map φ from $\mathcal{E}_1(\mathfrak{g}')$ to $\mathcal{E}(M')$. Take and fix a maximal Abelian subalgebra \mathfrak{t}' in \mathfrak{g}' . For each element a in $\mathcal{E}_1(\mathfrak{g}')$ we can take a representative \mathfrak{g} of a such that $\mathfrak{g} \cap \mathfrak{t}'$ is a maximal Abelian subalgebra in \mathfrak{g} . Since the index j_i of \mathfrak{g} in \mathfrak{g}' is 1, the image of the highest root δ in Δ is a long root in Δ' , by Theorem 2.4 in Dynkin [3]. So we can suppose $\iota(\delta) = \delta'$ and $\iota(\mathfrak{g}_i) = \mathfrak{g}'_i$. The relations

$$\iota(\mathfrak{k}) \subset \mathfrak{k}' \quad \text{and} \quad \iota(\mathfrak{p}) \subset \mathfrak{p}'$$

are obtained, because $s' = \iota(s)$. The relations imply that $\text{Int}(\mathfrak{g})/K$ is a totally geodesic submanifold in $M' = \text{Int}(\mathfrak{g}')/K'$. Since $[\mathfrak{g}'_i, \iota(\mathfrak{p})] = \iota([\mathfrak{g}_i, \mathfrak{p}]) = \iota(\mathfrak{p})$, the submanifold $\text{Int}(\mathfrak{g})/K$ is a quaternionic submanifold in M' . We define φ by

$$\varphi(a) = c(\text{Int}(\mathfrak{g})/K) \in \mathcal{E}(M') .$$

THEOREM 3.1. *Let \mathfrak{g}' be a compact simple Lie algebra whose rank is greater than 1 and $M' = \text{Int}(\mathfrak{g}')/K'$ be the corresponding compact quaternionic symmetric space. Then the map φ from $\mathcal{E}_1(\mathfrak{g}')$ to $\mathcal{E}(M')$ is well-defined and bijective.*

PROOF. We first show that φ is well-defined. Let $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ be simple subalgebras in \mathfrak{g}' which satisfy the following conditions:

- (i) The indices of $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ in \mathfrak{g}' are equal to 1;
 - (ii) The subalgebras $\mathfrak{t}^{(1)} = \mathfrak{g}^{(1)} \cap \mathfrak{t}'$ and $\mathfrak{t}^{(2)} = \mathfrak{g}^{(2)} \cap \mathfrak{t}'$ are maximal Abelian subalgebras in $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$, respectively;
 - (iii) $\iota^{(1)}(\delta^{(1)}) = \iota^{(2)}(\delta^{(2)}) = \delta'$, where $\iota^{(t)}: \mathfrak{g}^{(t)} \rightarrow \mathfrak{g}'$ is the inclusion and $\delta^{(t)}$ is the highest root in $\Delta^{(t)}$ for $t = 1, 2$;
 - (iv) There is an element g in $\text{Int}(\mathfrak{g}')$ such that $g(\mathfrak{g}^{(1)}) = \mathfrak{g}^{(2)}$.
- The condition (iv) is equivalent to $c(\mathfrak{g}^{(1)}) = c(\mathfrak{g}^{(2)})$. We can take an element g in (iv) such that $g(\mathfrak{g}_i^{(1)}) = \mathfrak{g}_i^{(2)}$. Since $\mathfrak{g}_i^{(1)} = \mathfrak{g}_i^{(2)} = \mathfrak{g}'_i$, we have $g(\mathfrak{g}'_i) = \mathfrak{g}'_i$.

To prove that the submanifolds $\text{Int}(\mathfrak{g}^{(1)})/K^{(1)}$ and $\text{Int}(\mathfrak{g}^{(2)})/K^{(2)}$ are $I_0(M')$ -conjugate, we must show that $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ are K' -conjugate in \mathfrak{g}' . So the following lemma implies that φ is well-defined.

LEMMA 3.2. *The normalizer of \mathfrak{g}'_1 in $\text{Int}(\mathfrak{g}')$ coincides with K' .*

PROOF. Let N' denote the normalizer of \mathfrak{g}'_1 in $\text{Int}(\mathfrak{g}')$. Then the identity component of N' coincides with K' . We may only prove that $N' \subset K'$.

For each n in N' , $n|_{\mathfrak{g}'_1}: \mathfrak{g}'_1 \rightarrow \mathfrak{g}'_1$ is an automorphism of \mathfrak{g}'_1 , hence there exists an element g_1 in G'_1 such that $g_1 n|_{\mathfrak{g}'_1} = \text{id}_{\mathfrak{g}'_1}$. This equation implies that

$$(3.1) \quad (g_1 n) g (g_1 n)^{-1} g^{-1} = e$$

for all g in G'_1 .

Let $\pi: \tilde{G} \rightarrow \text{Int}(\mathfrak{g}')$ be the universal covering group of $\text{Int}(\mathfrak{g}')$ and \tilde{G}_1 be the analytic subgroup of \tilde{G} corresponding to \mathfrak{g}'_1 . Take elements \tilde{g}_1 in \tilde{G}_1 and \tilde{n} in \tilde{G} such that $\pi(\tilde{g}_1) = g_1$ and $\pi(\tilde{n}) = n$. By (3.1),

$$\pi((\tilde{g}_1 \tilde{n}) \tilde{g} (\tilde{g}_1 \tilde{n})^{-1} \tilde{g}^{-1}) = e$$

for all \tilde{g} in \tilde{G}_1 . Since the kernel of π is discrete,

$$(3.2) \quad (\tilde{g}_1 \tilde{n}) \tilde{g} (\tilde{g}_1 \tilde{n})^{-1} \tilde{g}^{-1} = e$$

for all \tilde{g} in \tilde{G}_1 .

Let \tilde{s} be the element in \tilde{G} defined by

$$\tilde{s} = \exp_{\tilde{a}} \left(\frac{2\pi}{\langle \delta', \delta' \rangle'} \delta' \right),$$

which is contained in \tilde{G}_1 and involutive. Put

$$\tilde{K} = \{x \in \tilde{G}; \tilde{s} x \tilde{s}^{-1} = x\}.$$

Then the Lie algebra of \tilde{K} is \mathfrak{k}' and \tilde{K} is connected (see, for example, Theorem 8.2 of Chapter VII in Helgason [5]). By (3.2), $\tilde{g}_1 \tilde{n}$ is contained in \tilde{K} , so \tilde{n} is an element of \tilde{K} . Therefore

$$n = \pi(\tilde{n}) \in \pi(\tilde{K}) = K',$$

which completes the proof of Lemma 3.2.

Since the isotropy subgroup of $I_0(M')$ at the origin is K' , the map φ is injective. We have to prove that φ is surjective.

For each element b in $\mathcal{E}(M')$ we can take a representative M of b such that M contains the origin of $M' = \text{Int}(\mathfrak{g}')/K'$. By Lemma 2.2, M is a compact quaternionic symmetric space. The Lie algebra \mathfrak{g} of the

group of all isometries of M is naturally imbedded in \mathfrak{g}' in such a way that $\mathfrak{k} \subset \mathfrak{k}'$ and $\mathfrak{p} \subset \mathfrak{p}'$. Here we show that $\iota(\mathfrak{g}_1) = \mathfrak{g}'_1$. Since the quaternionic structure of M is the restriction of that of M' to M , there is a Lie algebra homomorphism ρ from \mathfrak{g}_1 to \mathfrak{g}'_1 satisfying $\iota([U, X]) = [\rho(U), \iota(X)]$ for $U \in \mathfrak{g}_1$ and $X \in \mathfrak{p}$. For $U \in \mathfrak{g}_1$ and $X, Y \in \mathfrak{p}$ we have

$$\begin{aligned} \iota([U, [X, Y]]) &= -[\iota(X), \iota([Y, U])] - [\iota(Y), \iota([U, X])] \\ &= -[\iota(X), [\iota(Y), \rho(U)]] - [\iota(Y), [\rho(U), \iota(X)]] \\ &= [\rho(U), \iota([X, Y])] . \end{aligned}$$

Noting that $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$, we obtain

$$(3.3) \quad \iota([U, X]) = [\rho(U), \iota(X)]$$

for $U \in \mathfrak{g}_1$ and $X \in \mathfrak{g}$. Hence $\iota(\mathfrak{g}_1) = \iota([\mathfrak{g}_1, \mathfrak{g}_1]) = [\rho(\mathfrak{g}_1), \iota(\mathfrak{g}_1)] \subset \mathfrak{g}'_1$, because of (3.3) and $\mathfrak{k}' = \mathfrak{g}'_1 + \mathfrak{z}'$. Therefore we have $\iota(\mathfrak{g}_1) = \mathfrak{g}'_1$. This implies that the index of \mathfrak{g} in \mathfrak{g}' is 1. Therefore $\varphi(c(\mathfrak{g})) = c(M) = b$. Thus Theorem 3.1 has been proved.

Let \mathfrak{g}' be a compact semisimple Lie algebra. For a semisimple subalgebra \mathfrak{g} in \mathfrak{g}' , denote by $c(\mathfrak{g})$ the $\text{Int}(\mathfrak{g}')$ -conjugacy class represented by \mathfrak{g} , and by $\mathcal{E}(\mathfrak{g}')$ the set of all $\text{Int}(\mathfrak{g}')$ -conjugacy classes of semisimple subalgebras in \mathfrak{g}' . When \mathfrak{g}' is simple, $\mathcal{E}_1(\mathfrak{g}')$ is a subset of $\mathcal{E}(\mathfrak{g}')$. The complexification \mathfrak{g}'^c of \mathfrak{g}' is a complex semisimple Lie algebra. For a complex semisimple subalgebra \mathfrak{h} in \mathfrak{g}'^c , denote by $c(\mathfrak{h})$ the $\text{Int}(\mathfrak{g}'^c)$ -conjugacy class represented by \mathfrak{h} , and by $\mathcal{E}(\mathfrak{g}'^c)$ the set of all $\text{Int}(\mathfrak{g}'^c)$ -conjugacy classes of complex semisimple subalgebras in \mathfrak{g}'^c . Now we define a map ψ from $\mathcal{E}(\mathfrak{g}')$ to $\mathcal{E}(\mathfrak{g}'^c)$ by

$$\psi(c(\mathfrak{g})) = c(\mathfrak{g}^c)$$

for each semisimple subalgebra \mathfrak{g} in \mathfrak{g}' .

We shall give a brief proof of the following proposition.

PROPOSITION 3.3. *Let \mathfrak{g}' be a compact semisimple Lie algebra. Then the map ψ from $\mathcal{E}(\mathfrak{g}')$ to $\mathcal{E}(\mathfrak{g}'^c)$ is well-defined and bijective.*

PROOF. Clearly ψ is well-defined, because $\text{Int}(\mathfrak{g}')$ is naturally imbedded in $\text{Int}(\mathfrak{g}'^c)$.

First the surjectivity of ψ follows from the existence of compact real forms in a complex semisimple Lie algebra and the conjugacy of maximal compact subalgebras in a real semisimple Lie algebras.

Next the injectivity of ψ follows from Lemma 1 in Ihara [7].

We can define the index of a complex simple subalgebra in a complex simple Lie algebra, using their compact real forms. Denote by $\mathcal{E}_1(\mathfrak{g}'^c)$

the set of all $\text{Int}(\mathfrak{g}^c)$ -conjugacy classes of complex simple subalgebras of index 1 in \mathfrak{g}^c whose ranks are greater than 1. For a compact simple Lie algebra \mathfrak{g}' , the map ψ preserves the indices. Hence the classification of all elements in $\mathcal{E}(M')$ is reduced to that of all elements in $\mathcal{E}_1(\mathfrak{g}^c)$, by Theorem 3.1 and Proposition 3.3.

We end this section by stating some properties about complex simple subalgebras of index 1. Let \mathfrak{g}' be a compact simple Lie algebra. For each c in $\mathcal{E}_1(\mathfrak{g}^c)$, take a representative \mathfrak{g}^c in c such that $\mathfrak{g} = \mathfrak{g}^c \cap \mathfrak{g}'$ is a compact real form of \mathfrak{g}^c . Take maximal Abelian subalgebras \mathfrak{t} and \mathfrak{t}' in \mathfrak{g} and \mathfrak{g}' , respectively, in such a way that $\mathfrak{t} \subset \mathfrak{t}'$. Denote by $\iota: \mathfrak{g}^c \rightarrow \mathfrak{g}'^c$ the inclusion. The following lemma holds, by Theorem 2.4 in Dynkin [3].

LEMMA 3.4. *In the above situation, if α is a long root in Δ , then $\iota(\alpha)$ is also a long root in Δ' . Furthermore, we have $\iota(\mathfrak{g}_\alpha) = \mathfrak{g}'_{\iota(\alpha)}$.*

COROLLARY 3.5. *If Δ is of type A , D or E , we have $\iota(\mathfrak{g}_\alpha) = \mathfrak{g}'_{\iota(\alpha)}$ for all α in Δ .*

PROOF. If Δ is of type A , D or E , then all roots in Δ are long roots. Hence Lemma 3.4 implies Corollary 3.5.

A subalgebra \mathfrak{g}^c in \mathfrak{g}^c is said to be regular, if $\iota(\mathfrak{g}_\alpha) = \mathfrak{g}'_{\iota(\alpha)}$ for each α in Δ .

4. Preliminaries for classification of simple subalgebras of index 1.

In this section we shall review a general method of classifying subalgebras in a classical complex Lie algebra developed by Mal'cev in [9] and describe concretely each of classical complex simple Lie algebras.

Let \mathfrak{g}^c be one of the classical complex simple Lie algebras $\mathfrak{o}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$. Before considering $\text{Int}(\mathfrak{g}^c)$ -conjugacy classes of complex simple subalgebras in \mathfrak{g}^c , we consider $\text{Int}(\mathfrak{g}^c)$ -conjugacy classes of pairs (\mathfrak{h}, ι) of complex simple Lie algebras \mathfrak{h} and homomorphisms $\iota: \mathfrak{h} \rightarrow \mathfrak{g}^c$. Since $\mathfrak{o}(n, \mathbb{C}) \subset \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C}) \subset \mathfrak{sl}(2n, \mathbb{C})$, we can regard $\iota: \mathfrak{h} \rightarrow \mathfrak{g}^c$ as a complex linear representation of \mathfrak{h} . The following theorem is obtained from Theorem 1 in Mal'cev [9].

THEOREM 4.1. *Let ι_1 and ι_2 be homomorphisms from a complex simple Lie algebra \mathfrak{h} to \mathfrak{g}^c . Then ι_1 and ι_2 are equivalent as complex linear representations if and only if the following conditions are satisfied:*

- (i) ι_1 and ι_2 are $O(n, \mathbb{C})$ -conjugate in the case $\mathfrak{g}^c = \mathfrak{o}(n, \mathbb{C})$.
- (ii) ι_1 and ι_2 are $SL(n, \mathbb{C})$ -conjugate in the case $\mathfrak{g}^c = \mathfrak{sl}(n, \mathbb{C})$.
- (iii) ι_1 and ι_2 are $Sp(n, \mathbb{C})$ -conjugate in the case $\mathfrak{g}^c = \mathfrak{sp}(n, \mathbb{C})$.

COROLLARY 4.2. *Let \mathfrak{g}^c be one of $\mathfrak{o}(2n + 1, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$.*

Let ι_1 and ι_2 be homomorphisms from a complex simple Lie algebra \mathfrak{h} to \mathfrak{g}^c . The homomorphisms ι_1 and ι_2 are $\text{Int}(\mathfrak{g}^c)$ -conjugate if and only if ι_1 and ι_2 are equivalent as complex linear representations.

We shall consider the case $\mathfrak{g}^c = \mathfrak{o}(2n, \mathbb{C})$. Take an element σ in $O(2n, \mathbb{C})$ such that $\det(\sigma) = -1$. For a complex simple Lie algebra \mathfrak{h} and a homomorphism ι from \mathfrak{h} to \mathfrak{g}^c , define ι^σ by

$$\iota^\sigma(X) = \sigma\iota(X)\sigma^{-1}$$

for all X in \mathfrak{h} . The following theorem is obtained from Theorems 2 and 3 in Mal'cev [9].

THEOREM 4.3. *Let \mathfrak{g}^c be $\mathfrak{o}(2n, \mathbb{C})$ and σ be an element in $O(2n, \mathbb{C})$ such that $\det(\sigma) = -1$. For a complex simple Lie algebra \mathfrak{h} and a homomorphism $\iota: \mathfrak{h} \rightarrow \mathfrak{g}^c$, ι and ι^σ are $\text{Int}(\mathfrak{g}^c)$ -conjugate if and only if ι is orthogonally reducible.*

By Corollary 4.2 and Theorem 4.3, we can construct all $\text{Int}(\mathfrak{g}^c)$ -conjugacy classes of pairs (\mathfrak{h}, ι) of complex simple Lie algebras \mathfrak{h} and homomorphisms $\iota: \mathfrak{h} \rightarrow \mathfrak{g}^c$ from all equivalence classes of complex linear representations for each classical complex simple Lie algebra \mathfrak{g}^c .

Now we shall consider $\text{Int}(\mathfrak{g}^c)$ -conjugacy classes of complex simple subalgebras in a classical complex simple Lie algebra \mathfrak{g}^c . After classifying $\text{Int}(\mathfrak{g}^c)$ -conjugacy classes of pairs (\mathfrak{h}, ι) , we must decide whether $\iota_1(\mathfrak{h})$ and $\iota_2(\mathfrak{h})$ are $\text{Int}(\mathfrak{g}^c)$ -conjugate subalgebras or not, for distinct $\text{Int}(\mathfrak{g}^c)$ -conjugacy classes of (\mathfrak{h}, ι_1) and (\mathfrak{h}, ι_2) . If $\iota_1(\mathfrak{h})$ and $\iota_2(\mathfrak{h})$ are $\text{Int}(\mathfrak{g}^c)$ -conjugate, we can take a representative ι_2 such that $\iota_1(\mathfrak{h}) = \iota_2(\mathfrak{h})$. So there is an element τ in $\text{Aut}(\mathfrak{h})$ such that $\iota_2 = \iota_1 \circ \tau$. Since $\text{Int}(\mathfrak{h})$ is naturally imbedded in $\text{Int}(\mathfrak{g}^c)$, we may suppose that τ is an outer automorphism of \mathfrak{h} . If \mathfrak{h} does not have any outer automorphisms, this does not happen. If \mathfrak{h} has outer automorphisms, we must decide which of $\text{Int}(\mathfrak{g}^c)$ -conjugacy classes of (\mathfrak{h}, ι) coincides with the $\text{Int}(\mathfrak{g}^c)$ -conjugacy class of $(\mathfrak{h}, \iota_1 \circ \tau)$ for each (\mathfrak{h}, ι_1) and $\tau \bmod \text{Int}(\mathfrak{h})$ in $\text{Aut}(\mathfrak{h})/\text{Int}(\mathfrak{h})$. By identifying $\text{Int}(\mathfrak{g}^c)$ -conjugacy classes of (\mathfrak{h}, ι) and (\mathfrak{h}, ι_1) such that (\mathfrak{h}, ι) and $(\mathfrak{h}, \iota_1 \circ \tau)$ are $\text{Int}(\mathfrak{g}^c)$ -conjugate for some τ in $\text{Aut}(\mathfrak{h})$, we can classify $\text{Int}(\mathfrak{g}^c)$ -conjugacy classes of complex simple subalgebras in \mathfrak{g}^c .

We shall apply the method mentioned above to each classical complex simple Lie algebra in Sections 5 and 6.

Let \mathfrak{g}^c be a complex simple Lie algebra with compact real form \mathfrak{g} and $\rho: \mathfrak{g}^c \rightarrow \mathfrak{gl}(V)$ be an irreducible complex linear representation. Take a maximal Abelian subalgebra \mathfrak{t} in \mathfrak{g} and put

$$V_\lambda = \{v \in V; \rho(H)v = \sqrt{-1}\langle \lambda, H \rangle v \text{ for each } H \text{ in } \mathfrak{t}\}$$

for each element λ in \mathfrak{t} . An element λ in \mathfrak{t} is called a *weight* of ρ if $V_\lambda \neq \{0\}$. Let A_ρ denote the set of all weights of ρ . Put $m_\lambda = \dim_{\mathbf{C}} V_\lambda$ for λ in A_ρ .

Here we concretely describe each of classical complex simple Lie algebras. Let $M_n(\mathbf{F})$ be the set of all $n \times n$ \mathbf{F} -matrices and $E_{i,j} = (\delta_{ai}\delta_{bj})_{1 \leq a,b \leq n} \in M_n(\mathbf{F})$.

(i) $\mathfrak{g}^{\mathbf{C}} = \mathfrak{sl}(n+1, \mathbf{C})$: A_n -type

$$\mathfrak{sl}(n+1, \mathbf{C}) = \{X \in M_{n+1}(\mathbf{C}); \text{tr}(X) = 0\}.$$

A compact real form \mathfrak{g} of $\mathfrak{sl}(n+1, \mathbf{C})$ is given by

$$\mathfrak{g} = \mathfrak{su}(n+1) = \{X \in M_{n+1}(\mathbf{C}); {}^t\bar{X} + X = 0, \text{tr}(X) = 0\}.$$

Define an invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} by

$$\langle X, Y \rangle = -\text{tr}(XY)$$

for X and Y in \mathfrak{g} . Put

$$\mathfrak{t} = \left\{ \sqrt{-1}(t_1 E_{1,1} + \cdots + t_{n+1} E_{n+1,n+1}); t_i \in \mathbf{R}, \sum_{i=1}^{n+1} t_i = 0 \right\}.$$

Then \mathfrak{t} is a maximal Abelian subalgebra in \mathfrak{g} . Set

$$e_i = \sqrt{-1} E_{i,i}$$

for $1 \leq i \leq n+1$. The root system Δ of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{t}^{\mathbf{C}}$ is $\{e_i - e_j; 1 \leq i \neq j \leq n+1\}$. Since $\langle e_i - e_j, e_i - e_j \rangle = 2$, the inner product $\langle \cdot, \cdot \rangle$ defined above is the normalized invariant inner product on \mathfrak{g} .

$$\omega_i = e_1 + \cdots + e_i - \frac{i}{n+1} \sum_{j=1}^{n+1} e_j \quad (1 \leq i \leq n)$$

is a fundamental weight system of Δ .

(ii) $\mathfrak{g}^{\mathbf{C}} = \mathfrak{o}(2n+1, \mathbf{C})$: B_n -type

$$\mathfrak{o}(2n+1, \mathbf{C}) = \{X \in M_{2n+1}(\mathbf{C}); {}^tX + X = 0\}.$$

A compact real form \mathfrak{g} of $\mathfrak{o}(2n+1, \mathbf{C})$ is given by

$$\mathfrak{g} = \mathfrak{o}(2n+1, \mathbf{R}) = \{X \in M_{2n+1}(\mathbf{R}); {}^tX + X = 0\}.$$

Define an invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} by

$$\langle X, Y \rangle = -\text{tr}(XY)/2$$

for X and Y in \mathfrak{g} . Put

$$\mathfrak{t} = \{t_1(E_{1,2} - E_{2,1}) + \cdots + t_n(E_{2n-1,2n} - E_{2n,2n-1}); t_i \in \mathbf{R}\}.$$

Then \mathfrak{t} is a maximal Abelian subalgebra in \mathfrak{g} . Set

$$e_i = E_{2i-1, 2i} - E_{2i, 2i-1}$$

for $1 \leq i \leq n$. The root system Δ of \mathfrak{g}^c with respect to \mathfrak{t}^c is $\{\pm e_i, \pm e_j; 1 \leq i < j \leq n\} \cup \{\pm e_i; 1 \leq i \leq n\}$. Since $\langle e_i + e_j, e_i + e_j \rangle = \langle e_i - e_j, e_i - e_j \rangle = 2$, the inner product \langle , \rangle defined above is the normalized invariant inner product on \mathfrak{g} .

$$\begin{aligned} \omega_i &= e_1 + \cdots + e_i \quad (1 \leq i \leq n - 1), \\ \omega_n &= (e_1 + \cdots + e_n)/2 \end{aligned}$$

is a fundamental weight system of Δ .

(iii) $\mathfrak{g}^c = \mathfrak{sp}(n, \mathbf{C})$: C_n -type

$$\mathfrak{sp}(n, \mathbf{C}) = \left\{ \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & -{}^t Z_1 \end{bmatrix}; Z_i \in M_n(\mathbf{C}), {}^t Z_2 = Z_2, {}^t Z_3 = Z_3 \right\}.$$

A compact real form \mathfrak{g} of $\mathfrak{sp}(n, \mathbf{C})$ is given by

$$\mathfrak{g} = \mathfrak{sp}(n) = \left\{ \begin{bmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{bmatrix}; Z_i \in M_n(\mathbf{C}), {}^t \bar{Z}_1 + Z_1 = 0, {}^t Z_2 = Z_2 \right\}.$$

Define an invariant inner product \langle , \rangle on \mathfrak{g} by

$$\langle X, Y \rangle = -\text{tr}(XY)$$

for X and Y in \mathfrak{g} . Put

$$\mathfrak{t} = \{\sqrt{-1}(t_1(E_{1,1} - E_{n+1, n+1}) + \cdots + t_n(E_{n,n} - E_{2n, 2n})); t_i \in \mathbf{R}\}.$$

Then \mathfrak{t} is a maximal Abelian subalgebra in \mathfrak{g} . Set

$$e_i = \sqrt{-1}(E_{i,i} - E_{n+i, n+i})/\sqrt{2}$$

for $1 \leq i \leq n$. The root system Δ of \mathfrak{g}^c with respect to \mathfrak{t}^c is $\{\pm\sqrt{2}e_i\} \cup \{(\pm e_i \pm e_j)/\sqrt{2}; 1 \leq i < j \leq n\}$. Since $\langle \sqrt{2}e_i, \sqrt{2}e_i \rangle = 2$, the inner product \langle , \rangle defined above is the normalized invariant inner product on \mathfrak{g} .

$$\omega_i = (e_1 + \cdots + e_i)/\sqrt{2} \quad (1 \leq i \leq n)$$

is a fundamental weight system of Δ .

(iv) $\mathfrak{g}^c = \mathfrak{o}(2n, \mathbf{C})$: D_n -type

$$\mathfrak{o}(2n, \mathbf{C}) = \{X \in M_{2n}(\mathbf{C}); {}^t X + X = 0\}.$$

A compact real form \mathfrak{g} of $\mathfrak{o}(2n, \mathbf{C})$ is given by

$$\mathfrak{g} = \mathfrak{o}(2n, \mathbf{R}) = \{X \in M_{2n}(\mathbf{R}); {}^t X + X = 0\}.$$

Define an invariant inner product \langle , \rangle on \mathfrak{g} by

$$\langle X, Y \rangle = -\text{tr}(XY)/2$$

for X and Y in \mathfrak{g} . Put

$$t = \{t_1(E_{1,2} - E_{2,1}) + \cdots + t_n(E_{2n-1,2n} - E_{2n,2n-1}); t_i \in \mathbf{R}\}.$$

Then t is a maximal Abelian subalgebra in \mathfrak{g} . Set

$$e_i = E_{2i-1,2i} - E_{2i,2i-1}$$

for $1 \leq i \leq n$. The root system Δ of \mathfrak{g}^c with respect to t^c is $\{\pm e_i \pm e_j; 1 \leq i < j \leq n\}$. Since $\langle e_i + e_j, e_i + e_j \rangle = \langle e_i - e_j, e_i - e_j \rangle = 2$, the inner product \langle , \rangle defined above is the normalized invariant inner product on \mathfrak{g} .

$$\begin{aligned} \omega_i &= e_1 + \cdots + e_i \quad (1 \leq i \leq n - 2), \\ \omega_{n-1} &= (e_1 + \cdots + e_{n-1} - e_n)/2, \\ \omega_n &= (e_1 + \cdots + e_{n-1} + e_n)/2 \end{aligned}$$

is a fundamental weight system of Δ .

REMARK. The natural inclusions $\mathfrak{sl}(n, \mathbf{C}) \rightarrow \mathfrak{sl}(n + 1, \mathbf{C})$, $\mathfrak{o}(n, \mathbf{C}) \rightarrow \mathfrak{o}(n + 1, \mathbf{C})$, $\mathfrak{sp}(n, \mathbf{C}) \rightarrow \mathfrak{sp}(n + 1, \mathbf{C})$, $\mathfrak{sp}(n, \mathbf{C}) \rightarrow \mathfrak{sl}(2n, \mathbf{C})$ are of index 1, but the inclusion $\mathfrak{o}(n, \mathbf{C}) \rightarrow \mathfrak{sl}(n, \mathbf{C})$ is of index 2.

5. $\mathfrak{sl}(n, \mathbf{C})$. In this section we shall classify all elements in $\mathcal{E}_1(\mathfrak{g}^c)$ for $\mathfrak{g}^c = \mathfrak{sl}(n, \mathbf{C})$. The corresponding compact quaternionic symmetric space is the complex Grassmann manifold $G_{2, n-2}^c = SU(n)/S(U(n-2) \times U(2))$. Since $G_{2,1}^c = P^2(\mathbf{C})$ has no proper quaternionic submanifolds, we may suppose that $n \geq 4$. The rank of the symmetric space $M' = G_{2, n-2}^c$ is 2. Let \mathfrak{g}^c be a complex simple subalgebra of index 1 in \mathfrak{g}^c whose rank is greater than 1. The compact quaternionic symmetric space M corresponding to \mathfrak{g}^c is a quaternionic submanifold, in particular, a totally geodesic submanifold in M' . Hence the rank of M is less than or equal to 2. According to Table in Section 2, $\mathfrak{g}^c = \mathfrak{sl}(k, \mathbf{C})$, $\mathfrak{sp}(k, \mathbf{C})$ or \mathfrak{g}_2^c .

Let

$$\iota = \iota_1 \oplus \cdots \oplus \iota_d$$

be an irreducible decomposition of $\iota: \mathfrak{g}^c \rightarrow \mathfrak{sl}(n, \mathbf{C})$. Then

$$j_i = j_{i_1} + \cdots + j_{i_d},$$

by Theorem 2.3 in Dynkin [3]. Since $j_i = 1$, one of j_{i_1}, \dots, j_{i_d} is 1 and the others are all 0. So we suppose that ι is an irreducible complex linear representation.

Take a long root α in Δ , a weight λ of ι , and a nonzero vector v in V_λ . By the definition,

$$\iota(\alpha)v = \sqrt{-1}\langle \lambda, \alpha \rangle v.$$

Therefore

$$\langle \iota(\alpha), \iota(\alpha) \rangle' = -\text{tr}(\iota(\alpha)\iota(\alpha)) = \sum_{\lambda \in A_i} m_\lambda \langle \lambda, \alpha \rangle^2,$$

that is,

$$j_i = \sum_{\lambda \in A_i} m_\lambda \langle \lambda, \alpha \rangle^2 / 2.$$

Since $\langle \lambda, \alpha \rangle = 2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$ is an integer, we obtain the following lemma.

LEMMA 5.1. *Let \mathfrak{g}^c be a complex simple subalgebra in $\mathfrak{sl}(n, \mathbb{C})$ and assume that \mathfrak{g}^c is irreducible in $\mathfrak{sl}(n, \mathbb{C})$. The index of \mathfrak{g}^c is equal to 1 if and only if, for a long root α in A , there is a unique weight λ_0 in A , such that $\langle \lambda, \alpha \rangle = 0$ for all λ in $A_i - \{\lambda_0, -\lambda_0\}$, $\langle \lambda_0, \alpha \rangle = 1$, and $m_{\lambda_0} = 1$.*

We apply this criterion to $\mathfrak{g}^c = \mathfrak{sl}(k, \mathbb{C})$, $\mathfrak{sp}(k, \mathbb{C})$, and \mathfrak{g}_2^c .

In the case $\mathfrak{g}^c = \mathfrak{sl}(k, \mathbb{C})$, the highest weight of $\iota: \mathfrak{sl}(k, \mathbb{C}) \rightarrow \mathfrak{sl}(n, \mathbb{C})$ of index 1 is ω_1 or ω_{k-1} , by the concrete description of ω_i in Section 4 and Lemma 5.1. Hence $k = n$ and $\iota: \mathfrak{sl}(n, \mathbb{C}) \rightarrow \mathfrak{sl}(n, \mathbb{C})$ is an automorphisms.

In the case $\mathfrak{g}^c = \mathfrak{sp}(k, \mathbb{C})$, the highest weight of $\iota: \mathfrak{sp}(k, \mathbb{C}) \rightarrow \mathfrak{sl}(n, \mathbb{C})$ of index 1 is ω_1 . The irreducible complex linear representation with highest weight ω_1 is nothing but the natural inclusion $\iota: \mathfrak{sp}(k, \mathbb{C}) \rightarrow \mathfrak{sl}(2k, \mathbb{C})$.

In the case $\mathfrak{g}^c = \mathfrak{g}_2^c$, all irreducible complex linear representations of \mathfrak{g}_2^c are orthogonal. See Mal'cev [9]. So we may suppose that the image of an irreducible complex linear representation $\iota: \mathfrak{g}_2^c \rightarrow \mathfrak{sl}(n, \mathbb{C})$ is contained in $\mathfrak{o}(n, \mathbb{C})$. Since the index of the inclusion $\mathfrak{o}(n, \mathbb{C}) \rightarrow \mathfrak{sl}(n, \mathbb{C})$ is 2, we obtain $j_i \geq 2$. Therefore there is no inclusion $\iota: \mathfrak{g}_2^c \rightarrow \mathfrak{sl}(n, \mathbb{C})$ of index 1.

The following theorem summarizes the above argument.

THEOREM 5.2. *For $n \geq 4$,*

$$\mathcal{E}_1(\mathfrak{sl}(n, \mathbb{C})) = \{c(\mathfrak{sl}(k, \mathbb{C})); 3 \leq k \leq n - 1\} \cup \{c(\mathfrak{sp}(k, \mathbb{C})); 2 \leq k \leq [n/2]\},$$

where $\mathfrak{sp}(k, \mathbb{C}) \subset \mathfrak{sl}(2k, \mathbb{C}) \subset \mathfrak{sl}(n, \mathbb{C})$ are the natural inclusions.

Now we obtain the classification of quaternionic submanifolds in $G_{2,n}^c$.

THEOREM 5.3. *Let $P^k(\mathbf{H}) \subset G_{2,2k}^c$ be the inclusion induced by the natural inclusion $\mathfrak{sp}(k + 1, \mathbb{C}) \subset \mathfrak{sl}(2k + 2, \mathbb{C})$. For $n \geq 2$,*

$$\mathcal{E}(G_{2,n}^c) = \{c(G_{2,k}^c); 1 \leq k \leq n - 1\} \cup \{c(P^k(\mathbf{H})); 1 \leq k \leq [n/2]\}.$$

REMARK. Chen-Nagano [2] found these submanifolds as totally geodesic submanifolds in $G_{2,n}^c$.

6. $\mathfrak{o}(n, \mathbb{C})$. Let $\mathfrak{g}'^c = \mathfrak{o}(n, \mathbb{C})$ and \mathfrak{g}^c be a complex simple subalgebra of index 1 in \mathfrak{g}'^c whose rank is greater than 1. We denote by $\iota: \mathfrak{g}^c \rightarrow \mathfrak{g}'^c$ the inclusion. Let

$$\iota = \iota_1 \oplus \cdots \oplus \iota_d$$

be an orthogonally irreducible decomposition of ι . Then

$$j_i = j_{i_1} + \cdots + j_{i_d}$$

by [3, Theorem 2.3]. Since $j_i = 1$, one of j_{i_1}, \dots, j_{i_d} is 1 and the others are all 0. So we suppose that ι is orthogonally irreducible. We need the following theorem due to Mal'cev.

THEOREM 6.1 ([9, Theorem 4]). *Let $\iota: \mathfrak{g}^c \rightarrow \mathfrak{o}(n, \mathbb{C})$ be orthogonally irreducible. Then ι is also irreducible in $\mathfrak{sl}(n, \mathbb{C})$ or there is an irreducible complex representation ρ of \mathfrak{g}^c such that ι is equivalent to $\rho \oplus \rho^*$, where ρ^* is the contragredient of ρ .*

In the case that ι is also irreducible in $\mathfrak{sl}(n, \mathbb{C})$, from the definition of the normalized invariant inner product on $\mathfrak{o}(n, \mathbb{C})$ it follows that

$$\langle \iota(\alpha), \iota(\alpha) \rangle' = -\text{tr}(\iota(\alpha)\iota(\alpha))/2 = \sum_{\lambda \in \Delta_i} m_\lambda \langle \lambda, \alpha \rangle^2 / 2$$

for a root α in Δ . Hence

$$j_i = \sum_{\lambda \in \Delta_i} m_\lambda \langle \lambda, \alpha \rangle^2 / 4$$

for a long root α in Δ . On the other hand, for a long root α in Δ there are distinct integers i and j such that the matrix $\iota(\alpha)$ is equivalent to the matrix $\pm e_i \pm e_j$ described in (ii) $\mathfrak{g}^c = \mathfrak{o}(2n + 1, \mathbb{C})$ and (iv) $\mathfrak{g}^c = \mathfrak{o}(2n, \mathbb{C})$ of Section 2, by Lemma 3.4. The matrix $\pm e_i \pm e_j$ has eigenvalues $\sqrt{-1}$ with multiplicity 2, $-\sqrt{-1}$ with multiplicity 2, and 0 with multiplicity $n - 4$. We have

$$\langle \lambda, \alpha \rangle = \pm 1, 0$$

for λ in Δ_i . From these we obtain the following lemma.

LEMMA 6.2. *Let \mathfrak{g}^c be a complex simple subalgebra in $\mathfrak{o}(n, \mathbb{C})$ and assume that \mathfrak{g}^c is irreducible in $\mathfrak{sl}(n, \mathbb{C})$. The index of \mathfrak{g}^c in $\mathfrak{o}(n, \mathbb{C})$ is equal to 1 if and only if for a long root α in Δ and all weights λ in Δ_i ,*

$$\langle \lambda, \alpha \rangle = \pm 1, 0 \quad \text{and} \quad \sum_{\lambda \in \Delta_i, \langle \lambda, \alpha \rangle = 1} m_\lambda = 2.$$

In the case that there is an irreducible complex representation ρ of \mathfrak{g}^c such that ι is equivalent to $\rho \oplus \rho^*$ we consider the weights of ρ . Let V be the representation space of ρ . From the definition of the normalized invariant inner product on $\mathfrak{o}(n, \mathbb{C})$ it follows that

$$\langle \iota(\alpha), \iota(\alpha) \rangle' = -\text{tr}(\iota(\alpha)\iota(\alpha))/2 = \sum_{\lambda \in \Delta_\rho} m_\lambda \langle \lambda, \alpha \rangle^2$$

for a root α in Δ , because $\mathbb{C}^n = V \oplus V^*$. Hence

$$j_\iota = \sum_{\lambda \in A_\rho} m_\lambda \langle \lambda, \alpha \rangle^2 / 2$$

for a long root α in A . Since $\langle \lambda, \alpha \rangle$ is an integer, we obtain the following lemma.

LEMMA 6.3. *Let \mathfrak{g}^c be a complex simple subalgebra in $\mathfrak{o}(n, \mathbb{C})$ and assume that ι is orthogonally irreducible and that there is an irreducible complex representation ρ of \mathfrak{g}^c such that ι is equivalent to $\rho \oplus \rho^*$. The index of \mathfrak{g}^c in $\mathfrak{o}(n, \mathbb{C})$ is equal to 1 if and only if for a long root α in A there is a unique weight λ_0 in A_ρ such that $\langle \lambda, \alpha \rangle = 0$ for all λ in $A_\rho - \{\lambda_0, -\lambda_0\}$, $\langle \lambda_0, \alpha \rangle = 1$, and $m_{\lambda_0} = 1$.*

We apply the criteria described in Lemma 6.2 and Lemma 6.3 to each complex simple Lie algebra \mathfrak{g}^c .

The irreducible complex linear representations ρ of \mathfrak{g}^c described in Lemma 6.3 have been listed in Section 5, that is, the identity map $\rho_1: \mathfrak{sl}(n, \mathbb{C}) \rightarrow \mathfrak{sl}(n, \mathbb{C})$, its contragredient ρ_1^* , and the natural inclusion $\rho_2: \mathfrak{sp}(n, \mathbb{C}) \rightarrow \mathfrak{sl}(2n, \mathbb{C})$.

The contragredient ρ_1^* of ρ_1 is given by $\rho_1^*(X) = -{}^tX$ for X in $\mathfrak{sl}(n, \mathbb{C})$. Since

$$\begin{aligned} & \begin{bmatrix} I & I \\ \sqrt{-1}I & -\sqrt{-1}I \end{bmatrix} \begin{bmatrix} X & \\ & -{}^tX \end{bmatrix} \begin{bmatrix} I & I \\ \sqrt{-1}I & -\sqrt{-1}I \end{bmatrix}^{-1} \\ &= \frac{1}{2} \begin{bmatrix} X - {}^tX & -\sqrt{-1}(X + {}^tX) \\ \sqrt{-1}(X + {}^tX) & X - {}^tX \end{bmatrix}, \end{aligned}$$

the representation $\rho_1 \oplus \rho_1^*$ is equivalent to

$$\begin{aligned} & \iota: \mathfrak{sl}(n, \mathbb{C}) \rightarrow \mathfrak{sl}(2n, \mathbb{C}) \\ & : X \mapsto \frac{1}{2} \begin{bmatrix} X - {}^tX & -\sqrt{-1}(X + {}^tX) \\ \sqrt{-1}(X + {}^tX) & X - {}^tX \end{bmatrix}. \end{aligned}$$

The image of ι is contained in $\mathfrak{o}(2n, \mathbb{C})$, so $\iota: \mathfrak{sl}(n, \mathbb{C}) \rightarrow \mathfrak{o}(2n, \mathbb{C})$. For an element σ in $O(2n, \mathbb{C})$ with $\det \sigma = -1$, the homomorphisms ι and ι^σ from $\mathfrak{sl}(n, \mathbb{C})$ to $\mathfrak{o}(2n, \mathbb{C})$ are not $\text{Int}(\mathfrak{o}(2n, \mathbb{C}))$ -conjugate by Theorem 4.3, because ι is orthogonally irreducible. The order of the group $\text{Aut}(\mathfrak{sl}(n, \mathbb{C}))/\text{Int}(\mathfrak{sl}(n, \mathbb{C}))$ is 2 and the unique nontrivial element in it is $\rho_1^* \bmod \text{Int}(\mathfrak{sl}(n, \mathbb{C}))$. For all X in $\mathfrak{sl}(n, \mathbb{C})$,

$$\begin{aligned} \iota \circ \rho_1^*(X) &= \frac{1}{2} \begin{bmatrix} X - {}^tX & \sqrt{-1}(X + {}^tX) \\ -\sqrt{-1}(X + {}^tX) & X - {}^tX \end{bmatrix} \\ &= \begin{bmatrix} I & \\ & -I \end{bmatrix} \iota(X) \begin{bmatrix} I & \\ & -I \end{bmatrix}^{-1}. \end{aligned}$$

If n is even, ι and $\iota \circ \rho_1^*$ are $\text{Int}(\mathfrak{o}(2n, \mathbb{C}))$ -conjugate. Hence the subalgebras $\iota(\mathfrak{sl}(n, \mathbb{C}))$ and $\iota^\circ(\mathfrak{sl}(n, \mathbb{C}))$ are not $\text{Int}(\mathfrak{o}(2n, \mathbb{C}))$ -conjugate. If n is odd, taking $\sigma = \begin{bmatrix} I & \\ & -I \end{bmatrix}$, we obtain $\iota \circ \rho_1^* = \iota^\circ$, hence the subalgebras $\iota(\mathfrak{sl}(n, \mathbb{C}))$ and $\iota^\circ(\mathfrak{sl}(n, \mathbb{C}))$ are $\text{Int}(\mathfrak{o}(2n, \mathbb{C}))$ -conjugate. Next we consider $\mathfrak{sl}(n, \mathbb{C}) \subset \mathfrak{o}(2n, \mathbb{C}) \subset \mathfrak{o}(m, \mathbb{C})$, where $2n < m$. Denote by $\iota_1: \mathfrak{o}(2n, \mathbb{C}) \rightarrow \mathfrak{o}(m, \mathbb{C})$ the natural inclusion. By Theorem 4.3, $c(\iota_1 \circ \iota(\mathfrak{sl}(n, \mathbb{C})))$ is a unique $\text{Int}(\mathfrak{o}(m, \mathbb{C}))$ -conjugacy class of simple subalgebras of index 1 in $\mathfrak{o}(m, \mathbb{C})$ which are obtained from the equivalence class of $(\mathfrak{sl}(n, \mathbb{C}), \iota_1 \circ \iota)$.

The representation $\rho_2 \oplus \rho_2^*$ of $\mathfrak{sp}(n, \mathbb{C})$ is equivalent to

$$\iota: \mathfrak{sp}(n, \mathbb{C}) \rightarrow \mathfrak{o}(4n, \mathbb{C})$$

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & -{}^t Z_1 \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} Z_1 - {}^t Z_1 & Z_2 - {}^t Z_3 & -\sqrt{-1}(Z_1 + {}^t Z_1) & -\sqrt{-1}(Z_2 + {}^t Z_3) \\ Z_3 - {}^t Z_2 & Z_1 - {}^t Z_1 & -\sqrt{-1}(Z_3 + {}^t Z_2) & \sqrt{-1}(Z_1 + {}^t Z_1) \\ \sqrt{-1}(Z_1 + {}^t Z_1) & \sqrt{-1}(Z_2 + {}^t Z_3) & Z_1 - {}^t Z_1 & Z_2 - {}^t Z_3 \\ \sqrt{-1}(Z_3 + {}^t Z_2) & -\sqrt{-1}(Z_1 + {}^t Z_1) & Z_3 - {}^t Z_2 & Z_1 - {}^t Z_1 \end{bmatrix}.$$

For an element σ in $O(4n, \mathbb{C})$ with $\det \sigma = -1$, the homomorphisms ι and ι° from $\mathfrak{sp}(n, \mathbb{C})$ to $\mathfrak{o}(4n, \mathbb{C})$ are not $\text{Int}(\mathfrak{o}(4n, \mathbb{C}))$ -conjugate by Theorem 4.3, because ι is orthogonally irreducible. The group $\text{Aut}(\mathfrak{sp}(n, \mathbb{C}))/\text{Int} \mathfrak{sp}(n, \mathbb{C})$ is trivial, hence $\iota(\mathfrak{sp}(n, \mathbb{C}))$ and $\iota^\circ(\mathfrak{sp}(n, \mathbb{C}))$ are not $\text{Int}(\mathfrak{o}(4n, \mathbb{C}))$ -conjugate. Next we consider $\mathfrak{sp}(n, \mathbb{C}) \subset \mathfrak{o}(4n, \mathbb{C}) \subset \mathfrak{o}(m, \mathbb{C})$, where $4n < m$. Denote by $\iota_1: \mathfrak{o}(4n, \mathbb{C}) \rightarrow \mathfrak{o}(m, \mathbb{C})$ the natural inclusion. By Theorem 4.3, $c(\iota_1 \circ \iota(\mathfrak{sp}(n, \mathbb{C})))$ is a unique $\text{Int}(\mathfrak{o}(m, \mathbb{C}))$ -conjugacy class of simple subalgebras of index 1 in $\mathfrak{o}(m, \mathbb{C})$ which are obtained from the equivalence class of $(\mathfrak{sp}(n, \mathbb{C}), \iota_1 \circ \iota)$.

We classify elements in $\mathcal{E}_1(\mathfrak{o}(n, \mathbb{C}))$ whose orthogonally irreducible parts satisfy the condition in Lemma 6.2. Let \mathfrak{g}^c be a complex simple subalgebra in $\mathfrak{o}(n, \mathbb{C})$ and assume that \mathfrak{g}^c is irreducible in $\mathfrak{sl}(n, \mathbb{C})$. We apply the criterion described in Lemma 6.2 to each complex simple Lie algebra \mathfrak{g}^c .

In the case $\mathfrak{g}^c = \mathfrak{sl}(k, \mathbb{C})$, the highest weight of $\iota: \mathfrak{sl}(k, \mathbb{C}) \rightarrow \mathfrak{o}(n, \mathbb{C})$ of index 1 is only ω_2 of $\mathfrak{sl}(4, \mathbb{C})$, by the concrete description of ω_i in Section 4 and Lemma 6.2. Hence $k = 4$ and $n = 6$. The complex irreducible representation of $\mathfrak{sl}(4, \mathbb{C})$ with highest weight ω_2 gives an isomorphism of $\mathfrak{sl}(4, \mathbb{C})$ to $\mathfrak{o}(6, \mathbb{C})$.

In the case $\mathfrak{g}^c = \mathfrak{o}(2k + 1, \mathbb{C})$, the highest weight of $\iota: \mathfrak{o}(2k + 1, \mathbb{C}) \rightarrow \mathfrak{o}(n, \mathbb{C})$ of index 1 is ω_1 of $\mathfrak{o}(2k + 1, \mathbb{C})$ or ω_3 of $\mathfrak{o}(7, \mathbb{C})$. The complex irreducible representation of $\mathfrak{o}(2k + 1, \mathbb{C})$ with highest weight ω_1 is nothing but the identity of $\mathfrak{o}(2k + 1, \mathbb{C})$. That of $\mathfrak{o}(7, \mathbb{C})$ with ω_3 is the spin

representation $\text{spin}: \mathfrak{o}(7, \mathbb{C}) \rightarrow \mathfrak{o}(8, \mathbb{C})$. For an element σ in $O(8, \mathbb{C})$ with $\det \sigma = -1$, the homomorphisms spin and spin^σ from $\mathfrak{o}(7, \mathbb{C})$ to $\mathfrak{o}(8, \mathbb{C})$ are not $\text{Int}(\mathfrak{o}(8, \mathbb{C}))$ -conjugate by Theorem 4.3. The group $\text{Aut}(\mathfrak{o}(7, \mathbb{C}))/\text{Int}(\mathfrak{o}(7, \mathbb{C}))$ is trivial, hence $\text{spin}(\mathfrak{o}(7, \mathbb{C}))$ and $\text{spin}^\sigma(\mathfrak{o}(7, \mathbb{C}))$ are not $\text{Int}(\mathfrak{o}(8, \mathbb{C}))$ -conjugate. For $n > 8$, subalgebras $\text{spin}(\mathfrak{o}(7, \mathbb{C}))$ and $\text{spin}^\sigma(\mathfrak{o}(7, \mathbb{C}))$ in $\mathfrak{o}(n, \mathbb{C})$ are $\text{Int}(\mathfrak{o}(n, \mathbb{C}))$ -conjugate, by Theorem 4.3.

In the case $\mathfrak{g}^c = \mathfrak{sp}(k, \mathbb{C})$, the highest weight of $\iota: \mathfrak{sp}(k, \mathbb{C}) \rightarrow \mathfrak{o}(n, \mathbb{C})$ of index 1 is only ω_2 of $\mathfrak{sp}(2, \mathbb{C})$. The complex irreducible representation of $\mathfrak{sp}(2, \mathbb{C})$ with highest weight ω_2 gives an isomorphism of $\mathfrak{sp}(2, \mathbb{C})$ to $\mathfrak{o}(5, \mathbb{C})$.

In the case $\mathfrak{g}^c = \mathfrak{o}(2k, \mathbb{C})$, the highest weight of $\iota: \mathfrak{o}(2k, \mathbb{C}) \rightarrow \mathfrak{o}(n, \mathbb{C})$ of index 1 is ω_1 of $\mathfrak{o}(2k, \mathbb{C})$ or ω_3, ω_4 of $\mathfrak{o}(8, \mathbb{C})$. The complex irreducible representation of $\mathfrak{o}(2k, \mathbb{C})$ of highest weight ω_1 is the identity of $\mathfrak{o}(2k, \mathbb{C})$. That of $\mathfrak{o}(8, \mathbb{C})$ with highest weight ω_3 or ω_4 gives an outer automorphism of $\mathfrak{o}(8, \mathbb{C})$.

In the case that \mathfrak{g}^c is of type E . It is impossible that the Dynkin diagram of \mathfrak{g}^c is a subdiagram of the extended Dynkin diagram of $\mathfrak{o}(n, \mathbb{C})$, hence $\mathfrak{o}(n, \mathbb{C})$ does not admit a regular subalgebra which is isomorphic to \mathfrak{g}^c by Theorems 5.1, 5.2 and 5.3 in Dynkin [3]. Therefore there is no inclusion $\mathfrak{g}^c \rightarrow \mathfrak{o}(n, \mathbb{C})$ of index 1 because of Corollary 3.5.

In the case $\mathfrak{g}^c = \mathfrak{f}_4^c$, we shall first describe the root system of the exceptional Lie algebra \mathfrak{f}_4^c . Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of \mathbb{R}^4 . Put

$$\Delta = \{\pm e_i; 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j; 1 \leq i < j \leq 4\} \cup \{(\pm e_1 \pm e_2 \pm e_3 \pm e_4)/2\}.$$

Then Δ is the root system of \mathfrak{f}_4^c . Since $\langle e_i + e_j, e_i + e_j \rangle = \langle e_i - e_j, e_i - e_j \rangle = 2$, the inner product $\langle \ , \ \rangle$ is the normalized invariant inner product.

$$\begin{aligned} \omega_1 &= e_1 + e_2, & \omega_2 &= 2e_1 + e_2 + e_3, \\ \omega_3 &= (3e_1 + e_2 + e_3 + e_4)/2, & \omega_4 &= e_1 \end{aligned}$$

is a fundamental weight system of Δ . The weight ω_4 is a unique dominant weight λ such that $\langle \lambda, \alpha \rangle = \pm 1$ or 0 for all long roots α in Δ . Since all $(\pm e_1 \pm e_2 \pm e_3 \pm e_4)/2$ are weights of the complex irreducible representation $\iota: \mathfrak{f}_4^c \rightarrow \mathfrak{o}(26, \mathbb{C})$ with highest weight ω_4 , the index of ι is not 1 by Lemma 6.2. Hence there is no inclusion $\mathfrak{f}_4^c \rightarrow \mathfrak{o}(n, \mathbb{C})$ of index 1.

In the case $\mathfrak{g}^c = \mathfrak{g}_2^c$, we shall first describe the root system of the exceptional Lie algebra \mathfrak{g}_2^c . Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of \mathbb{R}^3 . Put

$$\begin{aligned} \Delta &= \{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_3 - e_1), \pm(e_1 + e_2 - 2e_3)/3, \\ &\quad \pm(e_2 + e_3 - 2e_1)/3, \pm(e_3 + e_1 - 2e_2)/3\}. \end{aligned}$$

Then Δ is the root system of \mathfrak{g}_2^c . Since $\langle e_i - e_j, e_i - e_j \rangle = 2$, the inner product \langle , \rangle is the normalized invariant inner product.

$$\omega_1 = (e_1 + e_2 - 2e_3)/3, \quad \omega_2 = e_1 - e_3$$

is a fundamental weight system of Δ . The weight ω_1 is a unique dominant weight λ such that $\langle \lambda, \alpha \rangle = \pm 1$ or 0 for all long roots α in Δ . The complex irreducible representation $\iota: \mathfrak{g}_2^c \rightarrow \mathfrak{o}(7, C)$ with highest weight ω_1 is of index 1 by Lemma 6.2. This represents a unique $\text{Int}(\mathfrak{o}(7, C))$ -conjugacy class of homomorphisms of index 1 from \mathfrak{g}_2^c to $\mathfrak{o}(7, C)$ by Corollary 4.2. Hence the subalgebra $\iota(\mathfrak{g}_2^c)$ represents a unique $\text{Int}(\mathfrak{o}(7, C))$ -conjugacy class of simple subalgebras of index 1 in $\mathfrak{o}(7, C)$ which are isomorphic to \mathfrak{g}_2^c . Similar results hold in $\mathfrak{o}(n, C)$ for $n \geq 8$.

The following theorem summarizes the above argument.

THEOREM 6.4. *Take an element σ in $O(m, C)$ with $\det \sigma = -1$ and let $\mathfrak{sl}(n, C) \subset \mathfrak{o}(2n, C)$, $\mathfrak{sp}(n, C) \subset \mathfrak{o}(4n, C)$, $\text{spin}(\mathfrak{o}(7, C)) \subset \mathfrak{o}(8, C)$ and $\mathfrak{g}_2^c \subset \mathfrak{o}(7, C)$ be the inclusions described above. Then $\mathcal{E}_1(\mathfrak{o}(m, C))$ is given as follows.*

$$\mathcal{E}_1(\mathfrak{o}(7, C)) = \{c(\mathfrak{o}(5, C)), c(\mathfrak{o}(6, C)), c(\mathfrak{sl}(3, C)), c(\mathfrak{g}_2^c)\}.$$

$$\begin{aligned} \mathcal{E}_1(\mathfrak{o}(8, C)) = & \{c(\mathfrak{o}(k, C)); 5 \leq k \leq 7\} \cup \{c(\mathfrak{sl}(4, C)), c(\sigma \mathfrak{sl}(4, C)\sigma^{-1}), c(\mathfrak{sl}(3, C))\} \\ & \cup \{c(\mathfrak{sp}(2, C)), c(\sigma \mathfrak{sp}(2, C)\sigma^{-1})\} \\ & \cup \{c(\mathfrak{g}_2^c), c(\text{spin}(\mathfrak{o}(7, C))), c(\sigma \text{spin}(\mathfrak{o}(7, C))\sigma^{-1})\}. \end{aligned}$$

$$\begin{aligned} \mathcal{E}_1(\mathfrak{o}(4n, C)) = & \{c(\mathfrak{o}(k, C)); 5 \leq k \leq 4n - 1\} \cup \{c(\mathfrak{sl}(k, C)); 3 \leq k \leq 2n\} \\ & \cup \{c(\sigma \mathfrak{sl}(2n, C)\sigma^{-1})\} \cup \{c(\mathfrak{sp}(k, C)); 2 \leq k \leq n\} \\ & \cup \{c(\sigma \mathfrak{sp}(n, C)\sigma^{-1})\} \cup \{c(\mathfrak{g}_2^c), c(\text{spin}(\mathfrak{o}(7, C)))\} \end{aligned}$$

for $n \geq 3$.

$$\begin{aligned} \mathcal{E}_1(\mathfrak{o}(4n + r, C)) = & \{c(\mathfrak{o}(k, C)); 5 \leq k \leq 4n + r - 1\} \\ & \cup \{c(\mathfrak{sl}(k, C)); 3 \leq k \leq 2n + [r/2]\} \cup \{c(\mathfrak{sp}(k, C)); 2 \leq k \leq n\} \\ & \cup \{c(\mathfrak{g}_2^c), c(\text{spin}(\mathfrak{o}(7, C)))\} \end{aligned}$$

for $n \geq 2$ and $r = 1, 2$ or 3 .

Now we obtain the classification of quaternionic submanifolds in $G_{4,n}^R$.

THEOREM 6.5. *Take an element σ in $O(m + 4)$ with $\det \sigma = -1$ and assume that σ leaves the quaternionic structure on $G_{4,m}^R$ invariant. Let $G_{2,n}^c \subset G_{4,2n}^R$, $P^n(\mathbf{H}) \subset G_{4,4n}^R$, $G_{4,3}^R(\text{spin}) \subset G_{4,4}^R$ and $G_2/SO(4) \subset G_{4,3}^R$ be the inclusions induced by $\mathfrak{sl}(n + 2, C) \subset \mathfrak{o}(2n + 4, C)$, $\mathfrak{sp}(n + 1, C) \subset \mathfrak{o}(4n + 4, C)$, $\text{spin}(\mathfrak{o}(7, C)) \subset \mathfrak{o}(8, C)$ and $\mathfrak{g}_2^c \subset \mathfrak{o}(7, C)$, respectively. Then $\mathcal{E}(G_{4,m}^R)$ is given as follows.*

$$\mathcal{E}(G_{4,3}^R) = \{c(G_{4,1}^R), c(G_{4,2}^R), c(G_{2,1}^c), c(G_2/SO(4))\}.$$

$$\mathcal{E}(G_{4,4}^R) = \{c(G_{4,k}^R); 1 \leq k \leq 3\} \cup \{c(G_{2,2}^C), c(\sigma G_{2,2}^C), c(G_{2,1}^C)\} \cup \{c(P^1(\mathbf{H}), c(\sigma P^1(\mathbf{H}))\} \\ \cup \{c(G_2/SO(4)), c(G_{4,3}^R(\text{spin})), c(\sigma G_{4,3}^C(\text{spin}))\} .$$

$$\mathcal{E}(G_{4,4n}^R) = \{c(G_{4,k}^R); 1 \leq k \leq 4n - 1\} \cup \{c(G_{2,k}^C); 1 \leq k \leq 2n\} \cup \{c(\sigma G_{2,2n}^C)\} \\ \cup \{c(P^k(\mathbf{H})); 1 \leq k \leq n\} \cup \{c(\sigma P^n(\mathbf{H}))\} \cup \{c(G_2/SO(4)), c(G_{4,3}^R(\text{spin}))\}$$

for $n \geq 2$.

$$\mathcal{E}(G_{4,4n+r}^R) = \{c(G_{4,k}^R); 1 \leq k \leq 4n + r - 1\} \cup \{c(G_{2,k}^C); 1 \leq k \leq 2n + [r/2]\} \\ \cup \{c(P^k(\mathbf{H})); 1 \leq k \leq n\} \cup \{c(G_2/SO(4)), c(G_{4,3}^R(\text{spin}))\}$$

for $n \geq 1$ and $r = 1, 2$ or 3 .

REMARKS. Chen-Nagano [2] found many of these submanifolds as totally geodesic submanifolds in $G_{4,n}^R$.

The compact quaternionic symmetric space M' satisfying $g'^c = \mathfrak{sp}(n + 1, \mathbf{C})$ is $P^n(\mathbf{H}) = Sp(n + 1)/Sp(1) \times Sp(n)$. In $P^n(\mathbf{H})$, all complete quaternionic submanifolds are $P^m(\mathbf{H})$ ($1 \leq m \leq n - 1$). Furthermore, all compact minimal stable submanifolds in $P^n(\mathbf{H})$ are $P^m(\mathbf{H})$ ($1 \leq m \leq n - 1$), which was proved by Howard-Wei [6] and Ohnita [10]. So we omit this case.

7. Homology classes represented by quaternionic submanifolds. In this section we shall investigate the homology classes represented by compact quaternionic submanifolds in compact quaternionic symmetric spaces $G_{2,n}^C$, $G_{4,3}^R$ and $G_{4,4}^R$, using the results obtained in Sections 5 and 6.

Let M' be a compact quaternionic symmetric space. Define a map $\chi: \mathcal{E}(M') \rightarrow H_*(M'; \mathbf{R})$ by

$$\chi(c(M)) = [M] ,$$

where $[M]$ is the homology class represented by M . If complete quaternionic submanifolds M_1 and M_2 are $I_0(M')$ -conjugate, M_1 and M_2 are isotopic, in particular, homologous. A complete quaternionic submanifold in a compact quaternionic symmetric space is a compact oriented submanifold. Therefore χ is well-defined.

The aim of this section is to show that χ is injective when M' is one of $G_{2,n}^C$, $G_{4,3}^R$ and $G_{4,4}^R$. Consequently, *in these spaces a complete quaternionic submanifold is a unique volume minimizing submanifold up to $I_0(M')$ -conjugacy in its homology class.* Indeed, a quaternionic submanifold M in M' is a volume minimizing submanifold in its homology class. Let Ω be the fundamental 4-form on M' . If a compact oriented submanifold M_1 in M' is also volume minimizing in the homology class $[M]$, then

$$\int_{M_1} \Omega^k/c_k = \int_M \Omega^k/c_k = \text{vol}(M) = \text{vol}(M_1) = \int_{M_1} \text{vol}_{M_1} ,$$

where $\dim(M) = \dim(M_1) = 4k$. Since $\text{vol}_{M_1} = \Omega^k/c_k$ on M_1 , M_1 is also a quaternionic submanifold. If the map χ for M' is injective, then M and M_1 are $I_0(M')$ -conjugate. Therefore a quaternionic submanifold in M' is a unique volume minimizing submanifold in its homology class up to $I_0(M')$ -conjugacy.

REMARK. A uniqueness theorem of certain homologically volume minimizing submanifolds in a compact simple Lie group was obtained by Ohnita and the author in [11].

LEMMA 7.1. *Let M_1 and M_2 be compact quaternionic submanifolds in a quaternionic Kähler manifold M' . If M_1 and M_2 are homologous in M' , then $\text{vol}(M_1) = \text{vol}(M_2)$.*

PROOF. Let Ω be the fundamental 4-form on M' . Put $\dim(M_1) = \dim(M_2) = 4k$. Then we obtain

$$\text{vol}(M_1) = \int_{M_1} \Omega^k/c_k = \int_{M_2} \Omega^k/c_k = \text{vol}(M_2) .$$

THEOREM 7.2. *The following map is injective.*

$$\chi: \mathcal{E}(G_{2,n}^c) \rightarrow H_*(G_{2,n}^c; \mathbf{R}) .$$

PROOF. By Theorem 5.3, we may only prove $[G_{2,k}^c] \neq [P^k(\mathbf{H})]$ in $H_*(G_{2,n}^c; \mathbf{R})$ for $1 \leq k \leq [n/2]$.

Suppose $[G_{2,k}^c] = [P^k(\mathbf{H})]$. Since $G_{2,k}^c$ is a Hermitian symmetric space, we can consider the fundamental 2-form ω on the Kähler manifold $G_{2,n}^c$. The submanifold $G_{2,k}^c$ is a complex submanifold in $G_{2,n}^c$. On the other hand, $P^k(\mathbf{H})$ is a totally real submanifold in the Kähler manifold $G_{2,n}^c$. Therefore

$$\text{vol}(G_{2,k}^c) = \int_{G_{2,k}^c} \omega^{2k}/(2k)! = \int_{P^k(\mathbf{H})} \omega^{2k}/(2k)! = 0 ,$$

which is a contradiction. Hence $[G_{2,n}^c] \neq [P^k(\mathbf{H})]$.

THEOREM 7.3. *The following maps are injective.*

$$\chi: \mathcal{E}(G_{4,3}^R) \rightarrow H_*(G_{4,3}^R; \mathbf{R}) ,$$

$$\chi: \mathcal{E}(G_{4,4}^R) \rightarrow H_*(G_{4,4}^R; \mathbf{R}) .$$

PROOF. By Theorem 6.5, the map $\mathcal{E}(G_{4,3}^R) \rightarrow \mathcal{E}(G_{4,4}^R)$ induced by the inclusion $G_{4,3}^R \rightarrow G_{4,4}^R$ is injective and the diagram

$$\begin{array}{ccc} \mathcal{E}(G_{4,3}^R) & \xrightarrow{\chi} & H_*(G_{4,3}^R; \mathbf{R}) \\ \downarrow & & \downarrow \\ \mathcal{E}(G_{4,4}^R) & \xrightarrow{\chi} & H_*(G_{4,4}^R; \mathbf{R}) \end{array}$$

is commutative. Hence it is sufficient to prove that the map $\chi: \mathcal{C}(G_{4,4}^R) \rightarrow H_*(G_{4,4}^R; \mathbf{R})$ is injective.

First we shall construct 4 distinct quaternionic structures on $G_{4,4}^R$. The root system Δ of $\mathfrak{o}(8, \mathbf{C})$ is given by $\Delta = \{\pm e_i \pm e_j; 1 \leq i < j \leq 4\}$. Since every roots in Δ are long roots, each of them is the highest root with respect to a suitable ordering. Let

$$\begin{aligned} \delta_1 &= e_1 - e_2, \\ \Delta_t &= \{\alpha \in \Delta; \langle \alpha, \delta_1 \rangle = 0\} \cup \{\pm \delta_1\}, \\ \Delta_p &= \{\alpha \in \Delta - \{\pm \delta_1\}; \langle \alpha, \delta_1 \rangle \neq 0\}. \end{aligned}$$

Then the decomposition

$$\begin{aligned} \mathfrak{o}(8) &= \mathfrak{k} + \mathfrak{p}, \\ \mathfrak{k} &= \mathfrak{t} + \mathfrak{o}(8) \cap \sum_{\alpha \in \Delta_t} \mathfrak{g}_\alpha, \\ \mathfrak{p} &= \mathfrak{o}(8) \cap \sum_{\alpha \in \Delta_p} \mathfrak{g}_\alpha \end{aligned}$$

determines the quaternionic symmetric space $G_{4,4}^R$. Denote by S_1 the quaternionic structure constructed from $\mathbf{R}\delta_1 + \mathfrak{o}(8) \cap (\mathfrak{g}_{\delta_1} + \mathfrak{g}_{-\delta_1})$. Set

$$\delta_2 = e_1 + e_2, \quad \delta_3 = e_3 - e_4 \quad \text{and} \quad \delta_4 = e_3 + e_4.$$

For each $2 \leq j \leq 4$,

$$\begin{aligned} \Delta_t &= \{\alpha \in \Delta; \langle \alpha, \delta_j \rangle = 0\} \cup \{\pm \delta_j\}, \\ \Delta_p &= \{\alpha \in \Delta - \{\pm \delta_j\}; \langle \alpha, \delta_j \rangle \neq 0\}, \end{aligned}$$

hence $\mathbf{R}\delta_j + \mathfrak{o}(8) \cap (\mathfrak{g}_{\delta_j} + \mathfrak{g}_{-\delta_j})$ also determines a quaternionic structure on $G_{4,4}^R$, which is denoted by S_j . We call a quaternionic submanifold with respect to the quaternionic structure S_j on $G_{4,4}^R$ an S_j -quaternionic submanifold in $G_{4,4}^R$. Let Ω_j be the fundamental 4-form with respect to the quaternionic structure S_j on $G_{4,4}^R$ for $1 \leq j \leq 4$.

LEMMA 7.4. *Let M_1 and M_2 be complete S_1 -quaternionic submanifolds in $G_{4,4}^R$. If M_1 is an S_j -quaternionic submanifold and if M_2 is not an S_j -quaternionic submanifold for the same j ($2 \leq j \leq 4$), then the homology classes represented by M_1 and M_2 are distinct.*

PROOF. Let $\dim(M_1) = \dim(M_2) = 4k$. Suppose $[M_1] = [M_2]$, then $\text{vol}(M_1) = \text{vol}(M_2)$ by Lemma 7.1. On the other hand,

$$\text{vol}(M_1) = \int_{M_1} \Omega_j^k / c_k = \int_{M_2} \Omega_j^k / c_k < \text{vol}(M_2),$$

which is a contradiction. Hence $[M_1] \neq [M_2]$.

By Theorem 6.5, a 4-dimensional S_1 -quaternionic submanifold in $G_{4,4}^R$ coincides with one of $G_{4,1}^R$, $G_{2,1}^C$, $P^1(\mathbf{H})$ and $\sigma P^1(\mathbf{H})$ up to $I_0(G_{4,4}^R)$ -conjugacy, where σ is an element in $O(8)$ with $\det \sigma = -1$. The corresponding complex simple subalgebras of index 1 in $\mathfrak{o}(8, \mathbf{C})$ are $\mathfrak{o}(5, \mathbf{C})$, $\mathfrak{sl}(3, \mathbf{C})$, $\mathfrak{sp}(2, \mathbf{C})$ and $\sigma \mathfrak{sp}(2, \mathbf{C})\sigma^{-1}$.

The root system $\Delta(\mathfrak{sl}(3, \mathbf{C}))$ is given as follows, by Corollary 3.5.

$$\Delta(\mathfrak{sl}(3, \mathbf{C})) = \{\pm(e_i - e_j); 1 \leq i < j \leq 3\} \subset \Delta.$$

$\Delta(\mathfrak{sl}(3, \mathbf{C}))$ contains δ_1 . Hence $G_{2,1}^C$ is an S_1 -quaternionic submanifold, but it is not an S_j -quaternionic submanifold for $2 \leq j \leq 4$, by the proof of the surjectivity of φ in Theorem 3.1.

The subalgebra $\mathfrak{o}(5, \mathbf{C})$ is orthogonally reducible in $\mathfrak{o}(8, \mathbf{C})$, so

$$\Delta(\mathfrak{o}(5, \mathbf{C})) = \{\pm e_1 \pm e_2, \pm e_1, \pm e_2\}.$$

$\Delta(\mathfrak{o}(5, \mathbf{C}))$ contains δ_1 and δ_2 . Therefore $G_{4,1}^R$ is an S_1 - and S_2 -quaternionic submanifold, but it is not an S_3 - or S_4 -quaternionic submanifold.

The inclusion $\mathfrak{sp}(2, \mathbf{C}) \rightarrow \mathfrak{o}(8, \mathbf{C})$ is explicitly represented in Section 6. By a suitable permutation of coordinates, $\Delta(\mathfrak{sp}(2, \mathbf{C}))$ is given by

$$\Delta(\mathfrak{sp}(2, \mathbf{C})) = \{\pm(e_1 - e_2), \pm(e_3 - e_4), (\pm(e_1 - e_2) \pm (e_3 - e_4))/2\}.$$

$\Delta(\mathfrak{sp}(2, \mathbf{C}))$ contains δ_1 and δ_3 . Therefore $P^1(\mathbf{H})$ is an S_1 - and S_3 -quaternionic submanifold, but it is not an S_2 - or S_4 -quaternionic submanifold. Put

$$\sigma = \begin{bmatrix} 1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & 1 & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & -1 \end{bmatrix} \in O(8).$$

Then

$$\sigma e_j \sigma^{-1} = e_j, \quad \text{for } 1 \leq j \leq 3, \quad \text{and } \sigma e_4 \sigma^{-1} = -e_4.$$

Hence

$$\Delta(\sigma \mathfrak{sp}(2, \mathbf{C})\sigma^{-1}) = \{\pm(e_1 - e_2), \pm(e_3 + e_4), (\pm(e_1 - e_2) \pm (e_3 + e_4))/2\}.$$

$\Delta(\sigma \mathfrak{sp}(2, \mathbf{C})\sigma^{-1})$ contains δ_1 and δ_4 . Therefore $\sigma P^1(\mathbf{H})$ is an S_1 - and S_4 -quaternionic submanifold, but it is not an S_2 - or S_3 -quaternionic submanifold.

It follows from the above argument and Lemma 7.4 that the homology classes represented by $G_{2,1}^C$, $G_{4,1}^R$, $P^1(\mathbf{H})$ and $\sigma P^1(\mathbf{H})$ are pairwise distinct.

By Theorem 6.5, an 8-dimensional S_1 -quaternionic submanifold in $G_{4,4}^R$ coincides with one of $G_{4,2}^R$, $G_{2,2}^C$, $\sigma G_{2,2}^C$ and $G_2/SO(4)$ up to $I_0(G_{4,4}^R)$ -conjugacy.

The corresponding complex simple subalgebras of index 1 in $\mathfrak{o}(8, C)$ are $\mathfrak{o}(6, C)$, $\mathfrak{sl}(4, C)$, $\sigma \mathfrak{sl}(4, C)\sigma^{-1}$ and \mathfrak{g}_2^C . The root system of them are given as follows.

$$\begin{aligned} \Delta(\mathfrak{o}(6, C)) &= \{\pm e_i \pm e_j; 1 \leq i < j \leq 3\}, \\ \Delta(\mathfrak{sl}(4, C)) &= \{\pm(e_i - e_j); 1 \leq i < j \leq 4\}, \\ \Delta(\sigma \mathfrak{sl}(4, C)\sigma^{-1}) &= \{\pm(e_i - e_j); 1 \leq i < j \leq 3\} \cup \{\pm(e_i + e_4); 1 \leq i \leq 3\}, \\ \Delta(\mathfrak{g}_2^C) &= \{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_3 - e_1), \pm(e_1 + e_2 - 2e_3)/3, \\ &\quad \pm(e_2 + e_3 - 2e_1)/3, \pm(e_3 + e_1 - 2e_2)/3\}. \end{aligned}$$

Therefore it follows from Lemma 7.4 that the homology classes represented by $G_{4,2}^R$, $G_{2,2}^C$, $\sigma G_{2,2}^C$ and $G_2/SO(4)$ are pairwise distinct.

By Theorem 6.5, a 12-dimensional S_1 -quaternionic submanifold in $G_{4,4}^R$ coincides with one of $G_{4,3}^R$, $G_{4,3}^R(\text{spin})$ and $\sigma G_{4,3}^R(\text{spin})$ up to $I_0(G_{4,4}^R)$ -conjugacy. The corresponding complex simple subalgebras of index 1 in $\mathfrak{o}(8, C)$ are $\mathfrak{o}(7, C)$, $\text{spin}(\mathfrak{o}(7, C))$ and $\sigma \text{spin}(\mathfrak{o}(7, C))\sigma^{-1}$.

The root system $\Delta(\mathfrak{o}(7, C))$ is given by

$$\Delta(\mathfrak{o}(7, C)) = \{\pm e_i \pm e_j; 1 \leq i < j \leq 3\} \cup \{\pm e_i; 1 \leq i \leq 3\}.$$

Using the explicit realization of the spin representation $\text{spin}: \mathfrak{o}(7, C) \rightarrow \mathfrak{o}(8, C)$ in Sato-Kimura [12, (5, 31)], we can obtain the root system

$$\begin{aligned} \Delta(\text{spin}(\mathfrak{o}(7, C))) &= \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_1 + e_4), \pm(e_2 - e_3), \pm(e_2 \pm e_4), \\ &\quad \pm(e_3 + e_4), \pm(-e_1 + e_2 + e_3 + e_4)/2, \\ &\quad \pm(e_1 - e_2 + e_3 + e_4)/2, \pm(e_1 + e_2 - e_3 + e_4)/2\}. \end{aligned}$$

So

$$\begin{aligned} \Delta(\sigma \text{spin}(\mathfrak{o}(7, C))\sigma^{-1}) &= \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_1 - e_4), \pm(e_2 - e_3), \\ &\quad \pm(e_2 - e_4), \pm(e_3 - e_4), \pm(-e_1 + e_2 + e_3 - e_4)/2, \\ &\quad \pm(e_1 - e_2 + e_3 - e_4)/2, \pm(e_1 + e_2 - e_3 - e_4)/2\}. \end{aligned}$$

Therefore it follows from Lemma 7.4 that the homology classes represented by $G_{4,3}^R$, $G_{4,3}^R(\text{spin})$ and $\sigma G_{4,3}^R(\text{spin})$ are pairwise distinct, which completes the proof of Theorem 7.3.

8. $G_2/SO(4)$. Quaternionic submanifolds in $G_2/SO(4)$ are investigated in this section.

The root system $\Delta(\mathfrak{sl}(3, C))$ is a subsystem in $\Delta(\mathfrak{g}_2^C)$. So there is an inclusion $\mathfrak{sl}(3, C) \rightarrow \mathfrak{g}_2^C$ whose image is a regular subalgebra in \mathfrak{g}_2^C .

THEOREM 8.1.

$$\mathcal{E}_1(\mathfrak{g}_2^C) = \{c(\mathfrak{sl}(3, C))\},$$

where $\mathfrak{sl}(3, \mathbf{C}) \rightarrow \mathfrak{g}_2^c$ is the inclusion described above.

PROOF. The rank of \mathfrak{g}^c whose $\text{Int}(\mathfrak{g}_2^c)$ -conjugacy class $c(\mathfrak{g}^c)$ is contained in $\mathcal{E}_1(\mathfrak{g}_2^c)$ is 2. So \mathfrak{g}^c is $\mathfrak{sl}(3, \mathbf{C})$ or $\mathfrak{o}(5, \mathbf{C})$. By Lemma 3.4, long roots in $\Delta(\mathfrak{g}^c)$ are also long roots in $\Delta(\mathfrak{g}_2^c)$. It is impossible, if $\mathfrak{g}^c = \mathfrak{o}(5, \mathbf{C})$. If $\mathfrak{g}^c = \mathfrak{sl}(3, \mathbf{C})$, it is a regular subalgebra in \mathfrak{g}_2^c by Corollary 3.5 and it is unique up to $\text{Int}(\mathfrak{g}_2^c)$ -conjugacy.

From Theorem 8.1 we obtain the classification of quaternionic submanifolds in $G_2/SO(4)$.

THEOREM 8.2. *Let $G_{2,1}^c \rightarrow G_2/SO(4)$ be the inclusion induced by the inclusion $\mathfrak{sl}(3, \mathbf{C}) \rightarrow \mathfrak{g}_2^c$ described above. Then*

$$\mathcal{E}(G_2/SO(4)) = \{c(G_{2,1}^c)\}.$$

Since $\mathcal{E}(G_2/SO(4))$ has only one element, it is trivial that the map

$$\chi: \mathcal{E}(G_2/SO(4)) \rightarrow H_*(G_2/SO(4); \mathbf{R})$$

is injective.

REFERENCES

- [1] D. V. ALEKSEEVSKII, Compact quaternion spaces, *Functional Anal. Appl.* 2 (1968), 109-114.
- [2] B. Y. CHEN AND T. NAGANO, Totally geodesic submanifolds of symmetric spaces II, *Duke Math. J.* 45 (1978), 405-425.
- [3] E. B. DYNKIN, Semisimple subalgebras of semisimple Lie algebras, *Mat. Sb.* 30 (1952), 349-462. = *Amer. Math. Soc. Translations Ser. 2*, 6 (1960), 111-244.
- [4] R. HARVEY AND H. B. LAWSON, JR., Calibrated geometry, *Acta Math.* 148 (1982), 47-157.
- [5] S. HELGASON, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978.
- [6] R. HOWARD AND S. W. WEI, On the existence and nonexistence of stable submanifolds and currents in positively curved manifolds and the topology of submanifolds in Euclidean spaces, preprint.
- [7] S. IHARA, Holomorphic imbeddings of symmetric domains, *J. Math. Soc. Japan* 19 (1967), 261-302.
- [8] V. Y. KRAINES, Topology of quaternionic manifolds, *Trans. Amer. Math. Soc.* 122 (1966), 357-367.
- [9] A. I. MAL'CEV, On semisimple subgroups of Lie groups, *Izv. Akad. Nauk SSSR. Ser. Mat.* 8 (1944), 143-174. = *Amer. Math. Soc. Translations Ser. 1*, 9 *Lie groups* (1962), 172-213.
- [10] Y. OHNITA, Stable minimal submanifolds in compact rank one symmetric spaces, *Tôhoku Math. J.* 38 (1986), 199-217.
- [11] Y. OHNITA AND H. TASAKI, Uniqueness of certain 3-dimensional homologically volume minimizing submanifolds in compact simple Lie groups, *Tsukuba J. Math.* 10 (1986), 11-16.
- [12] M. SATO AND T. KIMURA, A classification of irreducible prehomogeneous vector spaces and their relative invariants, *Nagoya Math. J.* 65 (1977), 1-155.

- [13] H. TASAKI, Certain minimal or homologically volume minimizing submanifolds in compact symmetric spaces, *Tsukuba J. Math.* 9 (1985), 117-131.
- [14] J. A. WOLF, Complex homogeneous contact manifolds and quaternionic symmetric spaces, *J. Math. Mech.* 14 (1965), 1033-1047.

DEPARTMENT OF MATHEMATICS
TOKYO GAKUGEI UNIVERSITY
KOGANEI, TOKYO 184
JAPAN