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LIAPUNOV-RAZUMIKHIN FUNCTIONS AND THE ASYMPTOTIC PROPERTIES OF THE AUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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In this paper, an "invariance principle" is introduced and developed for autonomous functional differential equations with infinite delay. Then some conditions are established to ensure that, along the solutions, the Liapunov function $V(x)$ tends to a constant.

1. Introduction. It is an interesting problem to study the asymptotic constancy of the solutions of the functional differential equations such as

$$x'(t) = -h(x(t)) + h(x(t-r))$$

which have the property that each constant function is a solution. Haddock and Terjeki [3] have done some work in this direction. In this paper, we will extend several results of [3] to the functional differential equations with infinite delay. It is easy to see that our results hold for the phase space in Hale and Kato [2]. We will restrict ourselves within the phase space C_γ, the fading memory space, for the sake of simplicity.

Consider the equation

$$x'(t) = F(x_t)\,, \qquad x_t(s) = x(t+s)\,, \quad s \leqq 0\,, \tag{1.1}$$

where $F: C_\gamma \to R^n$ is a completely continuous mapping and C_γ, $\gamma \geqq 0$, is the linear space of continuous functions ϕ: $(-\infty, 0] \to R^n$ with the property that $\lim_{s\to-\infty} e^{\gamma s}\phi(s)$ exists. The norm in this space is defined as

$$\|\phi\| = \sup_{s\leqq 0} e^{\gamma s}|\phi(s)|\ .$$

DEFINITION 1. A function $V: R^n \to R^+$ is a Liapunov function associated with the functions $p(\cdot)$ and $q(\cdot)$, if

(a) V is continuously differentiable, and

(b) for each $\phi \in C_\gamma$,

$$\sup_{s\leqq 0} p(s)V(\phi(s)) < \infty\ , \tag{1.2}$$

(c) for all t, $s \in R^-$ we have

$$p(s+t) \leqq p(s)q(t)$$

and p and $q\colon R^- \to R^+$ are continuously differentiable functions such that $p(0) = q(0) = 1$, $p'(t) \geqq 0$ for all $t \in R$, $q'(0) \geqq 0$ and $p(t)$ and $q(t)$ tends to zero as $t \to -\infty$.

Sometimes we also impose the following assumption on the function $V(\cdot)$: There are continuous and increasing functions $a(\cdot), b(\cdot)\colon R^+ \to R^+$, $a(0) = b(0) = 0$, such that for all $\phi \in C_\gamma$

$$a(|\phi(0)|) \leqq \sup_{s \leqq 0} p(s)V(\phi(s)) \leqq b(\|\phi\|) . \tag{1.3}$$

As can be verified, the space C_γ satisfies the hypotheses in Kappel [4]. Thus the following results hold (cf. [4]).

LEMMA 1.1. *Let $x(\phi)(\cdot)$ be a solution of* (1.1) *on* $[0, \infty)$ *with bounded trajectory* $\{x_t(\phi)\colon t \geqq 0\}$ *and* $\{t_n\}$ *be a sequence with $t_n \to \infty$. Then there exist a subsequence $\{s_n\}$ of $\{t_n\}$ and a continuous bounded function $\psi \in C_\gamma$ such that* $\lim_{n\to\infty} x_{s_n}(\phi)(\tau) = \psi(\tau)$ *uniformly on every compact interval in* $(-\infty, 0]$, *and* $\lim_{n\to\infty} x_{s_n}(\phi) = \psi$ *in the norm C_γ.*

LEMMA 1.2. *Let $x(\phi)(t)$ be a solution of* (1.1) *on* $[0, \infty)$ *with bounded trajectory* $\{x_t(\phi)\colon t \geqq 0\}$. *A function $\psi \in C_\gamma$ is in $\Omega(\phi)$ (i.e., there is a sequence $\{s_n\}$ with $s_n \to \infty$ such that $x_{s_n}(\phi) \to \psi$) if and only if ψ is continuous and bounded on* $(-\infty, 0]$ *and there is a sequence $\{t_n\}$ such that*

$$\lim_{n\to\infty} x_{t_n}(\phi)(\tau) = \psi(\tau)$$

uniformly on every compact interval in $(-\infty, 0]$.

LEMMA 1.3. *Let $x(\phi)(t)$ be a solution of* (1.1) *on* $[0, \infty)$ *with bounded trajectory. Then the set $\Omega(\phi)$ of positive limit points is nonempty, bounded compact and moreover it is an invariant subset of C_γ, i.e., for each $\psi \in \Omega(\phi)$, there is a function $y\colon (-\infty, \infty) \to R$ such that*

(i) $y_t \in \Omega(\phi)$, $t \in (-\infty, \infty)$;

(ii) $y_0 = \psi$;

(iii) *y is continuously differentiable on $(-\infty, \infty)$ and*

$$y'(t) = F(y_t) , \quad t \in (-\infty, \infty) .$$

Furthermore, $x_t(\phi) \to \Omega(\phi)$ as $t \to \infty$.

2. The invariance principle. We consider the set

$$E_V(G) := \{\phi \in G\colon \sup_{s\leqq 0} p(s)V(x_t(\phi)(s)) = \sup_{s \leqq 0} p(s)V(\phi(s)),\ t \geqq 0\}$$

for a $G \subseteqq C_\gamma$, where $V(\cdot)$ is a Liapunov function associated with p and q

and $x_t(\phi)(\cdot)$ denotes the solution of (1.1) with the initial value $(0, \phi)$. As can be seen, the maximal invariant subset $M_V(G)$ in $E_V(G)$ is

$$M_V(G) = \{\phi \in G\colon \sup_{s\leqq 0} p(s)V(x_t(\phi)(s)) = \sup_{s\leqq 0} p(s)V(\phi(s)),\ t \in (-\infty, \infty)\}\ .$$

The following lemmas describe the useful properties of the function $V(\cdot)$.

LEMMA 2.1. *Suppose that $x(t)$ is a continuous function on $(-\infty, +\infty)$ satisfying $x_0 \in C_\gamma$. Then $g(t) := \sup_{s\leqq 0} p(s)V(x_t(s))$ is lower semi-continuous. Moreover, if for a sequence $\{t_k\}$ with $t_k \to \infty$ as $k \to \infty$ the sequence $\{x_{t_k}(s)\}$ converges to a $\phi(s)$ uniformly on every compact set of $(-\infty, 0]$ and $g(t_k)$ converges to a c, and if $V(x(t)) \leqq M$ for a constant M and all $t \geqq 0$, then we have*

$$\sup_{s\leqq 0} p(s)V(\phi(s)) = c\ .$$

PROOF. For $t_0 \geqq 0$ and $\varepsilon > 0$, one can find an $s_0 \in (-\infty, 0)$ such that

$$p(s_0)V(x(t_0 + s_0)) > g(t_0) - \varepsilon\ .$$

Then for any sequence $\{t_n\}$ with $t_n \to t_0$ as $n \to \infty$ we have

$$p(s_0)V(x(t_n + s_0)) \geqq g(t_0) - \varepsilon\ ,$$

when n is sufficiently large. This means that $g(t_n) \geqq g(t_0) - \varepsilon$, when n is sufficiently large. So $\liminf_{t\to t_0} g(t) \geqq g(t_0)$.

Now we prove the last part of Lemma 2.1. If $c = 0$, we have

$$0 \leqq \limsup_{k\to\infty} p(s_0)V(x_{t_k}(s_0)) \leqq \limsup_{k\to\infty} \sup_{s\leqq 0} p(s)V(x_{t_k}(s)) = 0\ ,$$

for any fixed $s_0 \leqq 0$. This means $\lim_{k\to\infty} V(x_{t_k}(s_0)) = V(\phi(s_0)) = 0$. And then $V(\phi(s)) = 0$ for all $s \leqq 0$. If $c > 0$, we can find a positive constant r such that $q(t)\cdot\sup_{s\leqq 0} p(s)V(x(s)) \leqq c/2$ for all $t \leqq -r$ and $M\cdot p(-r) \leqq c/2$. Then for $t_k \geqq r$, we have

$$\sup_{s\leqq -t_k} p(s)V(x_{t_k}(s)) = \sup_{t\leqq 0} p(t - t_k)V(x(t)) \leqq q(-t_k)\sup_{t\leqq 0} p(t)V(x(t)) \leqq c/2$$

and

$$\sup_{-t_k\leqq s\leqq -r} p(s)V(x_{t_k}(s)) \leqq p(-r)M \leqq c/2\ .$$

These inequalities and the fact that $g(t_k) \to c > 0$ as $k \to \infty$ mean that

$$\begin{aligned}
&\sup_{s\leqq 0} p(s)V(x_{t_k}(s)) \\
&\qquad = \max\{\sup_{s\leqq -t_k} p(s)V(x_{t_k}(s)),\ \sup_{-t_k\leqq s\leqq -r} p(s)V(x_{t_k}(s)),\ \sup_{-r\leqq s\leqq 0} p(s)V(x_{t_k}(s))\} \\
&\qquad = \sup_{-r\leqq s\leqq 0} p(s)V(x_{t_k}(s))\ ,
\end{aligned}$$

for sufficiently large k and then

$$c = \lim_{k\to\infty} \sup_{s\leqq 0} p(s)V(x_{t_k}(s)) = \lim_{k\to\infty} \sup_{-r\leqq s\leqq 0} p(s)V(x_{t_k}(s)) = \sup_{-r\leqq s\leqq 0} p(s)V(\phi(s)) \\ \leqq \sup_{s\leqq 0} p(s)V(\phi(s)) .$$

On the other hand, for any $L > 0$ and $\varepsilon > 0$, there is an N such that

$$\sup_{-L\leqq s\leqq 0} p(s)V(\phi(s)) \leqq \varepsilon + \sup_{-L\leqq s\leqq 0} p(s)V(x_{t_k}(s)) \leqq \varepsilon + \sup_{s\leqq 0} p(s)V(x_{t_k}(s)) ,$$

when $t_k \geqq N$. Let $k\to\infty$ we get $\sup_{-L\leqq s\leqq 0} p(s)V(\phi(s)) \leqq \varepsilon + c$, and this means $\sup_{s\leqq 0} p(s)V(\phi(s)) \leqq c$.

REMARK. In Lemma 2.1 $g(t)$ is continuous on the right: Indeed, for any $t \geqq t_0$ we have $\Delta t := t_0 - t \leqq 0$ and then

$$g(t) = \sup_{s\leqq 0} p(s)V(x(t+s)) = \sup_{s\leqq 0} p(s)V(x(t_0 + s - \Delta t)) \\ = \sup_{u+\Delta t\leqq 0} p(u+\Delta t)V(x(t_0+u)) \leqq q(\Delta t)\cdot \sup_{u\leqq -\Delta t} p(u)V(x(t_0+u)) .$$

From this we get $\limsup_{t\to t_0+} g(t) \leqq g(t_0)$. This inequality and the lower semicontinuity of $g(t)$ imply the continuity on the right. If the condition (c) in Definition 1 is true for $t\in R^+$, we can give the proof of the continuity on the left of $g(t)$ in the same way. But that condition may be too strong for applications.

LEMMA 2.2. *Suppose that $x(\cdot)$ is a solution of the equation* (1.1) *on* $[0, \infty)$. *Then for $t \geqq 0$ we have*

$$D^+\{g(t)\} \leqq \begin{cases} -q'(0)\sup_{s\leqq 0} p(s)V(x_t(s)) & or \\ (p'(0) - q'(0))V(x(t)) + V'_{(1.1)}(x_t) , \end{cases}$$

where $g(t) = \sup_{s\leqq 0} p(s)V(x_t(s))$, $D^+\{g(t)\} = \limsup_{h\to 0+} \{g(t+h) - g(t)\}/h$ and $V'_{(1.1)}(x_t) = \limsup_{h\to 0+} \{V(x(t+h)) - V(x(t))\}/h$. The latter case holds when $V(x(t)) = \sup_{s\leqq 0} p(s)V(x_t(s))$.

PROOF. For any fixed $t \geqq 0$,

$$\{\sup_{s\leqq 0} p(s)V(x_{t+h}(s)) - \sup_{s\leqq 0} p(s)V(x_t(s))\}/h \\ = \{\sup_{s\leqq h} p(s)V(x(t+s))p(s-h)/p(s) - \sup_{s\leqq 0} p(s)V(x(t+s))\}/h \\ \leqq \{q(-h)\sup_{s\leqq h} p(s)V(x(t+s)) - \sup_{s\leqq 0} p(s)V(x(t+s))\}/h .$$

If for an $h = h_0 > 0$ we have $\sup_{s\leqq h} p(s)V(x(t+s)) = \sup_{s\leqq 0} p(s)V(x(t+s))$, it must be true for each $h \in (0, h_0)$. And then

$$\{\sup_{s\leqq 0} p(s)V(x_{t+h}(s)) - \sup_{s\leqq 0} p(s)V(x_t(s))\}/h \leqq (q(-h) - 1)\sup_{s\leqq 0} p(s)V(x_t(s))/h .$$

Let $h \to 0+$ in this inequality, and we get

$$D^+\{\sup_{s \leqq 0} p(s)V(x_t(s))\} \leqq -q'(0)\sup_{s \leqq 0} p(s)V(x_t(s)) .$$

If such an h_0 does not exist, we have

$$\begin{aligned}\sup_{s \leqq 0} & p(s)V(x(t+s)) \\ & < \sup_{s \leqq h} p(s)V(x(t+s)) = \max\{\sup_{s \leqq 0} p(s)V(x(t+s)), \sup_{0 \leqq s \leqq h} p(s)V(x(t+s))\} \\ & = \sup_{0 \leqq s \leqq h} p(s)V(x(t+s)) = p(\xi)V(x(t+\xi)) ,\end{aligned}$$

for a $\xi = \xi(h) \in (0, h]$. Let $h \to 0+$, and we get $V(x(t)) = \sup_{s \leqq 0} p(s)V(x_t(s))$. And then

$$\begin{aligned}\{\sup_{s \leqq 0} & p(s)V(x_{t+h}(s)) - \sup_{s \leqq 0} p(s)V(x_t(s))\}/h \\ & \leqq \{q(-h)p(\xi)V(x(t+\xi)) - V(x(t))\}/h \\ & = \{p(\xi)V(x(t+\xi)) - V(x(t))\}/\xi \cdot q(-h) \cdot \xi/h + (q(-h) - 1)/h \cdot V(x(t)) \\ & \leqq \{p(\xi)V(x(t+\xi)) - V(x(t))\}/\xi \cdot q(-h) + (q(-h) - 1)/h \cdot V(x(t)) .\end{aligned}$$

Let $h \to 0+$, and we get

$$D^+\{g(t)\} \leqq (p'(0) - q'(0))V(x(t)) + V'_{(1.1)}(x_t) .$$

The following lemma is obvious.

LEMMA 2.3. *Suppose that $x(\cdot)$ is a solution of the equation* (1.1) *on* $[0, \infty)$, *that $V(\cdot)$ is a Liapunov function and that*

$$\lim_{t \to \infty} \sup_{s \leqq 0} p(s)V(x(t+s)) = c , \quad c < \infty .$$

Then there is a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n \to \infty} V(x(t_n)) = c .$$

The following theorem plays a fundamental role in this paper. It can be seen as an extension of Kappel's result in [4].

THEOREM 2.1. *Suppose that there are a Liapunov function $V(\cdot)$ and a positive invariant closed subset $G \subseteqq C_\gamma$ such that for each $\phi \in G$ with $V(\phi(0)) = \sup_{s \leqq 0} p(s)V(\phi(s))$, we have $V'_{(1.1)}(\phi) \leqq -(p'(0) - q'(0))V(\phi(0))$. Then we have $\Omega(\phi) \subseteqq M_V(G) \subseteqq E_V(G)$ for such $\phi \in G$ that the solution $x(\phi)(\cdot)$ is defined and bounded on $[0, \infty)$. That is, $x_t(\phi) \to M_V(G)$ as $t \to \infty$.*

REMARK. It is quite obvious that the inequality (1.2) for V may hold for a $\phi \in G$.

PROOF. Because of the boundedness of the set $\{x_t(\phi): t \geqq 0\}$, the positive limit set $\Omega(\phi)$ is nonempty, compact and invariant. Moreover $x_t(\phi)\to\Omega(\phi)$ as $t\to\infty$ by Lemma 1.3. Clearly, $\Omega(\phi)\subset G$.

From Lemma 2.2, we have $D^+\{g(t)\}\leqq 0$ for all $t\geqq 0$, where $g(t)=\sup_{s\leqq 0}p(s)V(x_t(s))$. That is, the function $g(t)$ is nonincreasing in t and bounded below, and hence $\lim_{t\to\infty}g(t)$ exists. We denote the limit by c. Thus for all $\psi\in\Omega(\phi)$, we have $\sup_{s\leqq 0}p(s)V(\psi(s))=c$ by Lemma 2.1. From the invariance of $\Omega(\phi)$ we have $x_t(\psi)\in\Omega(\phi)$ and $\sup_{s\leqq 0}p(s)V(x_t(\psi)(s))=c$ for all t. That is, $\psi\in E_V(G)$, $\Omega(\phi)\subseteqq E_V(G)$ and $\Omega(\phi)\subseteqq M_V(G)\subseteqq E_V(G)$.

DEFINITION 2. The zero solution of (1.1) is stable with respect to G, if for a given $\varepsilon>0$ there is a $\delta>0$ such that $\phi\in G$ and $\|\phi\|<\delta$ imply $\|x_t(\phi)\|<\varepsilon$ for all $t\geqq 0$.

Similarly, one can give the definition of the asymptotic stability of the zero solution of (1.1).

COROLLARY. *Suppose that* $F(0)=0$ *and there are a Liapunov function* $V(\cdot)$ *with the property* (1.3) *and a positive invariant subset* $G\subseteqq C_r$ *such that for some positive constant* a:

(i) $V(0)=0$, $V(x)>0$ *if* $x\neq 0$ *and* $|x|<a$;

(ii) $V'_{(1.1)}(\phi)<-(p'(0)-q'(0))V(\phi(0))$ *for each* $\phi\in G$ *such that* $\|\phi\|\neq 0$ *and* $\|\phi\|<a$ *and* $V(\phi(0))=\sup_{s\leqq 0}p(s)V(\phi(s))$.

Then the zero solution is asymptotically stable with respect to G.

PROOF. From Lemma 2.2, the function $g(t)=\sup_{s\leqq 0}p(s)V(x_t(\phi)(s))$ is decreasing in t whenever $\|\phi\|>0$ is small enough. By the condition (1.3) we know that, for a given $\varepsilon>0$ there is a $\delta>0$ such that $t\geqq 0$, $\phi\in G$ and $\|\phi\|<\delta$ imply that $\|x_t(\phi)\|<\varepsilon$. Then the stability of the zero solution respect to G follows.

From Theorem 2.1, for $\phi\in G$ with small norm, $\Omega(\phi)$ is nonempty and invariant, and $x_t(\phi)\to\Omega(\phi)\subseteqq E_V(G)$ as $t\to\infty$. Thus for each $\psi\in\Omega(\phi)$, the function $\sup_{s\leqq 0}p(s)V(x_t(\psi)(s))$ is a constant when $t\in(-\infty,+\infty)$.

We have to show that $\Omega(\phi)=\{0\}$. If $\psi\in\Omega(\phi)$ and $\psi\neq 0$, then $c:=\sup_{s\leqq 0}p(s)V(x_t(\psi)(s))>0$ for all t. Then for all $s\leqq 0$ and all t we have

$$V(x_t(\psi)(s))=V(x_{t+s}(\psi)(0))\leqq\sup_{u\leqq 0}p(u)V(x_{t+s}(\psi)(u))=c$$

and we can find a $r>0$ such that $p(s)\leqq 1/2$ for all $s\leqq -r$. And then

$$\begin{aligned}c&=\sup_{s\leqq 0}p(s)V(x_t(\psi)(s))=\max\{\sup_{s\leqq -r}p(s)V(x_t(\psi)(s)),\ \sup_{-r\leqq s\leqq 0}p(s)V(x_t(\psi)(s))\}\\&\leqq\max\{\sup_{s\leqq -r}p(s)c,\ \sup_{-r\leqq s\leqq 0}p(s)V(x_t(\psi)(s))\}\,.\end{aligned}$$

Thus the function $V(x_t(\psi)(0))$ is not always less than c for all $t\geqq 0$ and can not be greater than c for any $t \geqq 0$. Thus we can find a $t_0 > 0$ such that

$$V(x_{t_0}(\psi)(0)) = c = \sup_{s\leqq 0} p(s)V(x_{t_0}(\psi)(s)) .$$

Then $V'_{(1.1)}(x_{t_0}(\psi)) < 0$ by (ii). This equality also implies that the function $V(x(\psi)(t))$ has a maximal value at t_0, and $V'_{(1.1)}(x_{t_0}(\psi)) = 0$, which is a contradiction.

EXAMPLE. Consider the system

$$x'(t) = F(ax(t) + \int_{-\infty}^{0} g(s)x(t+s)ds) , \tag{2.1}$$

where $F: R^n \to R^n$ is continuous, g is integrable and for a $\gamma > 0$,

$$\int_{-\infty}^{0} |g(t)|e^{-\gamma s}ds < a ,$$

and $x\cdot F(u) < 0$ for any $x, u \in R^n$ such that $x\cdot u > 0$ (here "$\cdot$" stands for the inner product in the Euclidean space). Let $V(x) = x\cdot x/2$ and $p(s) = q(s) = e^{2\gamma s}$ for $s \in R$. Then Condition (1.3) is satisfied. Since

$$V'_{(2.1)}(\phi) = \phi(0)\cdot F(a\phi(0) + \int_{-\infty}^{0} g(s)\phi(s)ds)$$

and for $\phi \neq 0$ and $V(\phi(0)) = \sup_{s\leqq 0} e^{2\gamma s}V(\phi(s))$

$$\left|\phi(0)\int_{-\infty}^{0} g(s)\phi(s)ds\right| \leqq |\phi(0)|\int_{-\infty}^{0} |g(s)|\,|\phi(s)|ds \leqq |\phi(0)|^2\int_{-\infty}^{0} |g(s)|e^{-\gamma s}ds ,$$

we have

$$\phi(0)\cdot(a\phi(0) + \int_{-\infty}^{0} g(s)\phi(s)ds) = a|\phi(0)|^2 + \phi(0)\cdot\int_{-\infty}^{0} g(s)\phi(s)ds > 0 ,$$

and this means $V'_{(2.1)}(\phi) < 0$. Then the asymptotic stability of the zero solution follows.

3. Asymptotic constancy of V along solutions. In this section we shall find some sufficient conditions for the asymptotic constancy of $V(\cdot)$ along solutions as $t \to \infty$. In some cases, this implies that the solution $x(t)$ tends to a constant as $t \to \infty$.

Let $V(\cdot)$ be a Liapunov function. Denote $K_V := \{\phi \in C_\gamma: V(\phi(s)) = V(\phi(0))$, for all $s \leqq 0\}$.

LEMMA 3.1. *Suppose that $V(\cdot)$ is a Liapunov function and G is an invariant and closed subset of C_γ such that $\lim_{t\to\infty}\sup_{s\leqq 0} p(s)V(x_t(\phi)(s))$ exists for such $\phi \in G$ that the solution $x(\phi)(\cdot)$ has the bounded trajectory. If $\phi \in G$, if the solution $x(\phi)(\cdot)$ has bounded trajectory and if $\Omega(\phi) \subseteqq K_V$,* *then we have* $\lim_{t\to\infty} V(x(\phi)(t)) = \lim_{t\to\infty} \sup_{s\leqq 0} p(s)V(x_t(\phi)(s)))$.

PROOF. By Lemma 2.3, there is a sequence $\{t'_n\}$ with $t'_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} V(x(\phi)(t'_n)) = \lim_{t\to\infty} \sup_{s\leqq 0} p(s)V(x_t(\phi)(s)) =: c\ .$$

If the lemma is false, we can find a sequence $\{t_n\}$ with $t_n \to \infty$ such that $V(x(\phi)(t_n)) \to a < c$, as $n \to \infty$. Because of the compactness of the sequence $\{x_{t_n}(\phi)\}$ (cf. Lemma 1.1), we can assume that $x_{t_n}(\phi) \to \psi \in \Omega(\phi) \subseteqq K_V$. Then for $s \in (-\infty, 0)$ we have $V(\psi(s)) = V(\psi(0)) = a$, and then $\sup_{s\leqq 0} p(s)V(\psi(s)) = a$. On the other hand, the fact that $x_{t_n}(\phi) \to \psi$ implies $\lim_{n\to\infty} x_{t_n}(\phi)(s) = \psi(s)$ uniformly on every compact subset of $(-\infty, 0]$ and this result means

$$\sup_{s\leqq 0} p(s)V(\psi(s)) = \lim_{t\to\infty} \sup_{s\leqq 0} p(s)V(x_t(\phi)(s)) = c\ ,$$

by Lemma 2.1, which is a contradiction.

THEOREM 3.1. *Suppose that $V(\cdot)$ is a Liapunov function and G is an invariant and closed subset of C_γ such that*

(i) *for each $\phi \in G$ such that $V(\phi(0)) = \sup_{s\leqq 0} p(s)V(\phi(s))$ we have*

$$V'_{(1.1)}(\phi) \leqq -(p'(0) - q'(0))V(\phi(0))\ ;$$

(ii) *for each $\phi \in G$ such that $V(\phi(0)) = \sup_{s\leqq 0} p(s)V(\phi(s))$ and $V'_{(1.1)}(\phi) = 0$ we have $\phi \in K_V$.*

Then $\Omega(\phi) \subseteqq K_V$ for each $\phi \in G$ such that the solution $x(\phi)(\cdot)$ has the bounded trajectory, and then $V(x_t(\phi)(0))$ tends to a constant as $t \to \infty$.

PROOF. By Lemma 3.1, we only have to prove $\Omega(\phi) \subseteqq K_V$ whenever $\phi \in G$ and $x(\phi)(\cdot)$ is bounded. Following the same reasoning as in the proof of Corollary to Theorem 2.1, we know that for given $\psi \in \Omega(\phi)$ there is a sufficiently large t_0 such that $V(x_{t_0}(\psi)(0)) = \sup_{s\leqq 0} p(s)V(x_{t_0}(\psi)(s))$ and $V'_{(1.1)}(x_{t_0}(\psi)) = 0$. Then we have $x_{t_0}(\psi) \in K_V$, that is, $V(x(\psi)(t_0 + s)) = V(x(\psi)(t_0))$ for all $s \leqq 0$, by (ii). And then the fact that $x(\psi)(s) = \psi(s)$ for all $s \leqq 0$ implies $\psi \in K_V$.

EXAMPLE. Consider the scalar equation

$$x'(t) = \{-e^{r^2}x(t) + x(t-r) - x(t)x(t-r)\}\int_{-\infty}^0 e^{2\gamma s}|x_t(s) - x_t(0)|ds\ . \tag{3.1}$$

One can verify that the set $G := \{\phi \in C_\gamma : \phi(s) \geqq 0,\ s \leqq 0\}$ is a closed set and is positive invariant with respect to (3.1). Let $V(x) = x^2/2$, $p(s) = e^{-2s^2}$ for $s \leqq 0$, $p(s) = e^{2s^2}$ for $s \geqq 0$, and $q(s) = e^{-s^2}$ for $s \leqq 0$. Then $V(\cdot)$ is a Liapunov function.

$$V'_{(3.1)}(\phi) = \{-e^{r^2}\phi^2(0) + \phi(0)\phi(-r) - \phi^2(0)\phi(-r)\}\int_{-\infty}^0 e^{2\gamma s}|\phi(s) - \phi(0)|ds\ .$$

Since $-e^{r^2}\phi^2(0) + \phi(0)\phi(-r) \leqq 0$ for such $\phi \in G$ that $V(\phi(0)) = \sup_{s\leqq 0} e^{-2s^2} \times V(\phi(s))$, we have $V'_{(3.1)}(\phi) \leqq 0$ for such a ϕ. Moreover, if $V'_{(3.1)}(\phi) = 0$, we can get

$$\{-e^{r^2}\phi^2(0) + \phi(0)\phi(-r) - \phi^2(0)\phi(-r)\} \int_{-\infty}^{0} e^{2rs}|\phi(s) - \phi(0)|ds = 0 \, .$$

This implies that $\phi(s) = \phi(0)$ for all $s \leqq 0$, or $\phi \in K_V$. Since $V(\cdot)$ has the property that $V(x) \to \infty$ as $|x| \to \infty$, the boundedness of the solutions of (3.1) follows from (i) in Theorem 3.1. And then each solution of (3.1) tends to a constant by Theorem 3.1.

REMARK. We cannot directly apply Theorem 3.1 to the equation (3.1) with the function e^{2rs} in the integrant replaced by $|f(s)|e^{2rs}$, if f is continuous and bounded on $(-\infty, 0]$ but may be zero on a subset of $(-\infty, 0]$. In particular, for this equation, one can not readily verify that $\phi \in K_V$, for each $\phi \in G$ such that $V(\phi(0)) = \sup_{s\leqq 0} p(s)V(\phi(s))$ and $V'_{(3.1)}(\phi) = 0$ (using $V(x) = x^2/2$). Following the idea of Haddock and Terjeki [3, § 3], one can eliminate this difficulty.

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REFERENCES

[1] C. CORDUNEANU AND V. LAKSHMIKANTHAM, Equations with unbounded delay, A survey, Nonlinear Analysis Theory, Methods and Applications 4 (1980), 831–877.

[2] J. K. HALE AND J. KATO, Phase space for retarded equations with infinite delay, Funkcial. Ekvac. 21 (1978), 11–41.

[3] J. R. HADDOCK AND J. TERJEKI, Liapunov-Razumikhin functions and an invariance principle for functional differential equations, J. Differential Equations 48 (1983), 95–122.

[4] F. KAPPEL, The invariance of limit sets for autonomous functional differential equations, SIAM J. Appl. Math. 19 (1970), 408–419.

[5] M. PARROTT, Convergence of solutions of infinite delay differential equations with an underlying space of continuous functions, Lecture Notes in Math. 846, Springer-Verlag, Berlin, Heidelberg, New York, 1981, 280–289.

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