

SUBMERSIONS OF CR SUBMANIFOLDS

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1. Introduction. Let V be a complex Banach manifold with complex structure J . Let N be a real submanifold of V and TN its tangent bundle. We set

$$(1.1) \quad T^h N = TN \cap J(TN).$$

If $T^h N$ is a C^∞ complex subbundle of $TV|_N$, then N is called a CR submanifold of V . Assume further that V is a Kähler manifold. Let $T^v N$ be the orthogonal complement of $T^h N$ in TN ; it is a real subbundle of TN . Thus, we have an orthogonal direct sum:

$$(1.2) \quad TV|_N = T^h N \oplus T^v N \oplus T^\perp N,$$

where $T^\perp N$ is the normal bundle to N . The complex structure J leaves $T^h N$ and $T^v N \oplus T^\perp N$ invariant. We assume the following:

- (a) J interchanges $T^v N$ and $T^\perp N$;
- (b) there is a submersion $\pi: N \rightarrow W$ of N onto an almost Hermitian manifold W such that (i) $T^v N$ is the kernel of π_* and (ii) $\pi_*: T^h N \rightarrow T_{\pi(p)} W$ is a complex isometry for every $p \in N$.¹⁾

In this setting, we prove:

(1.3) **THEOREM.** *Under the assumptions stated above, W is Kähler. If H^V and H^W denote the holomorphic sectional curvature of V and W , then, for any horizontal unit vector $x \in T^h N$ of N , we have*

$$H^V(x) = H^W(\pi_* x) - 4|B(x, x)|^2,$$

where B denotes the second fundamental form of N in V .

There is nothing difficult about the proof. It is simply a combination of the equation of Gauss for the submanifold $N \subset V$ and the corresponding equation for the submersion $\pi: N \rightarrow W$. The former is classical and appears, for example, in [4] in the context of CR submanifolds. The latter is due to O'Neill [3]. The reason for singling out this theorem is its application to moduli spaces of stable holomorphic vector bundles over compact

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¹⁾ We may call $\pi: N \rightarrow W$ a Hermitian submersion.

Riemann surfaces. Let E be a fixed C^∞ complex vector bundle over a compact Riemann surface M . We fix a Kähler metric on M and a Hermitian structure h in E . Let V be the space of h -connections in E , completed with a suitable Sobolev norm. Let N be the space of irreducible Einstein h -connections, also completed in the same way. There is a natural flat Kähler metric on V , and N is a CR submanifold of V . The group G of gauge transformations of (E, h) acts on the Kähler manifold V and leaves N invariant. Moreover, G acts freely and properly on N , and N is a principal G -bundle over $W = N/G$. The G -invariant Kähler structure of V induces an almost Hermitian metric on W . In a natural way, V and W can be identified with the space of holomorphic structures in E and the moduli space of stable holomorphic structures in E , respectively. From our theorem it follows that this moduli space W has non-negative holomorphic sectional curvature. This fact has been verified computationally by Itoh [1]. Our theorem explains his calculation more geometrically. It is because of this application that we prove the theorem for infinite dimensional manifolds.

As in this application, we may replace Assumption (b) by a suitable Banach Lie group G acting on V . Such a formulation would place the theorem much closer to the set-up of the symplectic reduction theorem of Marsden-Weinstein.

In the hope that the present set-up may occur in other contexts, I separated this theorem from the problem of moduli of stable vector bundles. Perhaps the theorem is of moderate interest by itself.

The proof is divided into three parts. The immersion $N \subset V$ is studied in §2, the submersion $N \rightarrow W$ in §3, and they are combined in §4.

I would like to express my gratitude to the referee for pointing out the error in my original formulation of the theorem.

2. Submanifold N in V . In this section we recall the equation of Gauss; for its proof we refer the reader to [2.II; p. 23]. Let V be a Riemannian manifold, a Banach manifold with a Riemannian metric. Let N be a closed submanifold of V . (We do not assume here that N is a CR submanifold).

Let ∇^V and ∇^N denote covariant differentiation operators of V and N . The proof in [2.I; p. 160] of the existence and uniqueness of the Levi-Civita connections (or the Riemannian connections) ∇^V and ∇^N is valid in the infinite dimensional situation.

Given tangent vector fields x, y of N , we can write

$$(2.1) \quad \nabla_x^V y = \nabla_x^N y + B(x, y),$$

where $\nabla_x^N y$ is the tangential part and $B(x, y)$ is the normal part of $\nabla_x^V y$. Then B is a symmetric bilinear mapping

$$B: TN \times TN \rightarrow T^\perp N,$$

where $T^\perp N$ is the normal bundle of N . It is called the second fundamental form of N in V .

The curvature R^V of V is given by

$$(2.2) \quad R^V(x, y) = [\nabla_x^V, \nabla_y^V] - \nabla_{[x, y]}^V.$$

It is a 2-form with values in $\text{End}(TV)$. With the same symbol R^V we denote the corresponding quadrilinear form. Thus

$$(2.3) \quad R^V(w, z, x, y) = \langle R^V(x, y)z, w \rangle,$$

where \langle, \rangle stands for the given Riemannian inner product for V . We define R^N in a similar manner. As a quadrilinear form, the curvature R^N of N is related to R^V by the following equation of Gauss.

$$(2.4) \quad R^V(w, z, x, y) = R^N(w, z, x, y) + \langle B(x, z), B(y, w) \rangle - \langle B(y, z), B(x, w) \rangle.$$

3. Submersion of N onto W . In this section we extract parts of O'Neill's paper [3] which are needed in our proof. Let N and W be Riemannian manifolds and $\pi: N \rightarrow W$ a Riemannian submersion. By a submersion π we mean that both π and its differential π_* are surjective. Using the metric of N we decompose the tangent bundle TN into a direct sum

$$(3.1) \quad TN = T^h N \oplus T^v N,$$

where $T^v N = \text{Ker } \pi_*$ is the *vertical part* and its orthogonal complement $T^h N$ is the *horizontal part*. A submersion π is said to be Riemannian if $\pi_*: T_p^h N \rightarrow T_{\pi(p)} W$ is an isometry at each point p of N .

A horizontal vector field x of N is said to be *basic* if it induces (or comes from) a vector field x_* on the base manifold W . This vector field is sometimes denoted by $\pi_* x$; thus $x_* = \pi_* x$. Clearly, $x \mapsto x_*$ gives a one-to-one correspondence between the basic vector fields of N and the vector fields of W .

The following lemma is from O'Neill [3].

(3.2) LEMMA. *Let x and y be basic vector fields of N . Then*

- (a) $\langle x, y \rangle = \langle x_*, y_* \rangle \circ \pi$;
- (b) *the horizontal part $[x, y]^h$ of $[x, y]$ is a basic vector field and corresponds to $[x_*, y_*]$, i.e., $\pi_*([x, y]^h) = [x_*, y_*]$;*
- (c) $[\xi, x]$ *is vertical for any vertical vector field ξ of N ;*
- (d) $(\nabla_x^N y)^h$ *is the basic vector field corresponding to $\nabla_{x_*}^W y_*$.*

In (d) above, ∇^W is the covariant differentiation on W . We define the corresponding operator $\tilde{\nabla}^W$ for basic vector fields of N by setting

$$(3.3) \quad \tilde{\nabla}_x^W y = (\nabla_x^N y)^h \quad \text{for basic vector fields } x, y .$$

Then $\tilde{\nabla}_x^W y$ is a basic vector field by (d) and

$$(3.4) \quad \pi_*(\tilde{\nabla}_x^W y) = \nabla_{x_*}^W y_* .$$

We define C by

$$(3.5) \quad \nabla_x^N y = \tilde{\nabla}_x^W y + C(x, y) ,$$

where $C(x, y)$ denotes the vertical part of $\nabla_x^N y$. It is easy to check that C defines a bilinear mapping $T^h N \times T^h N \rightarrow T^v N$. Moreover, C is skew-symmetric and satisfies

$$(3.6) \quad 2C(x, y) = [x, y]^v .$$

In fact, if ξ is any vertical vector field of N , then

$$\begin{aligned} 0 &= \xi(\langle x, x \rangle) = 2\langle \nabla_\xi^N x, x \rangle = 2\langle \nabla_x^N \xi + [\xi, x], x \rangle = 2\langle \nabla_x^N \xi, x \rangle \\ &= -2\langle \xi, \nabla_x^N x \rangle = -2\langle \xi, C(x, x) \rangle . \end{aligned}$$

Hence, $C(x, x) = 0$, i.e., C is skew-symmetric. From

$$[x, y] = \nabla_x^N y - \nabla_y^N x ,$$

we obtain

$$[x, y]^v = C(x, y) - C(y, x) = 2C(x, y) .$$

For a basic vector field x and a vertical vector field ξ of N , we set

$$(3.7) \quad \nabla_x^N \xi = (\nabla_x^N \xi)^v + A_x \xi , \quad \text{where } A_x \xi = (\nabla_x^N \xi)^h .$$

It is easy to check that A_x defines a bilinear map

$$A: T^h N \times T^v N \rightarrow T^h N , \quad (x, \xi) \mapsto A_x \xi .$$

Since

$$\nabla_\xi^N x - \nabla_x^N \xi = [\xi, x]$$

and since $[\xi, x]$ is vertical, we obtain

$$(3.8) \quad (\nabla_\xi^N x)^h = (\nabla_x^N \xi)^h = A_x \xi .$$

Now, the two tensor fields A and C are related by

$$(3.9) \quad \langle A_x \xi, w \rangle = \langle \nabla_x^N \xi, w \rangle = -\langle \xi, \nabla_x^N w \rangle = -\langle \xi, C(x, w) \rangle .$$

By straightforward calculation making use of (3.3) through (3.8), we obtain

$$\begin{aligned}
 (3.10) \quad \nabla_x^N \nabla_y^N z &= \tilde{\nabla}_x^W \tilde{\nabla}_y^W z + A_x(C(y, z)) + \text{vertical} , \\
 \nabla_y^N \nabla_x^N z &= \tilde{\nabla}_y^W \tilde{\nabla}_x^W z + A_y(C(x, z)) + \text{vertical} , \\
 \nabla_{[x, y]}^N z &= \tilde{\nabla}_{[x, y]}^W z + 2A_z(C(x, y)) + \text{vertical} .
 \end{aligned}$$

Hence,

$$\begin{aligned}
 (3.11) \quad R^N(x, y)z &= (R^W(x_*, y_*)z_*)^* + A_x(C(y, z)) \\
 &\quad - A_y(C(x, z)) - 2A_z(C(x, y)) + \text{vertical} ,
 \end{aligned}$$

where $(R^W(x_*, y_*)z_*)^*$ denotes the basic vector field of N corresponding to $R^W(x_*, y_*)z_*$. Taking the inner product of (3.11) with a basic vector field w and making use of (3.9), we obtain

$$\begin{aligned}
 (3.12) \quad R^N(w, z, x, y) &= R^W(w_*, z_*, x_*, y_*) - \langle C(y, z), C(x, w) \rangle \\
 &\quad + \langle C(x, z), C(y, w) \rangle + 2\langle C(x, y), C(z, w) \rangle ,
 \end{aligned}$$

4. Submersed CR submanifold N . Let V be a Kähler manifold and N a CR submanifold of V . Let W be an almost Hermitian manifold and $\pi: N \rightarrow W$ a Riemannian submersion such that $TN \cap J(TN)$ is the horizontal part of TN and, at each point $p \in N$, π_* is a complex isometry of $T_p^h N = T_p N \cap J(T_p N)$ onto $T_{\pi(p)} W$. First we find the relationship between the bilinear maps B and C . Throughout this section, x, y, z and w shall denote basic vector fields of N . We have the following orthogonal decomposition of $TV|_N$.

$$(4.1) \quad TV|_N = T^h N \oplus T^v N \oplus T^\perp N .$$

Accordingly,

$$(4.2) \quad \nabla_x^V y = \tilde{\nabla}_x^W y + C(x, y) + B(x, y) .$$

Hence,

$$(4.3) \quad J\nabla_x^V y = J\tilde{\nabla}_x^W y + J(C(x, y)) + J(B(x, y)) .$$

Since $\pi_*: T_p^h N \rightarrow T_{\pi(p)} W$ preserves J , Jy is also basic. Applying (4.2) to Jy , we have

$$(4.4) \quad \nabla_x^V Jy = \tilde{\nabla}_x^W Jy + C(x, Jy) + B(x, Jy) .$$

Since V is Kähler, $\nabla_x^V Jy = J\nabla_x^V y$. Assume that J interchanges $T^v N$ and $T^\perp N$. Then equating (4.3) and (4.4), we obtain

$$(4.5) \quad \tilde{\nabla}_x^W Jy = J\tilde{\nabla}_x^W y \in T^h N ,$$

$$(4.6) \quad C(x, Jy) = J(B(x, y)) \in T^v N ,$$

$$(4.7) \quad B(x, Jy) = J(C(x, y)) \in T^\perp N .$$

From (4.5) we see that the almost complex structure of W is parallel and,

hence W is Kähler. From (4.6) and (4.7) we obtain

$$(4.8) \quad B(Jx, Jy) = B(x, y), \quad C(Jx, Jy) = C(x, y).$$

In order to compare the holomorphic bisectional curvature of V with that of W , we set $z = Jw$ and $y = Jx$ in (2.4) and (3.12). Making use of (4.8), we obtain

$$(4.9) \quad R^V(w, Jw, x, Jx) = R^N(w, Jw, x, Jx) - |B(x, Jw)|^2 - |B(x, w)|^2,$$

$$(4.10) \quad R^N(w, Jw, x, Jx) = R^W(w_*, Jw_*, x_*, Jx_*) - |C(x, w)|^2 \\ - |C(x, Jw)|^2 - 2\langle C(x, Jx), C(w, Jw) \rangle.$$

Making use of (4.6) and (4.7), we combine (4.9) and (4.10) to obtain

$$(4.11) \quad R^V(w, Jw, x, Jx) = R^W(w_*, Jw_*, x_*, Jx_*) - 2|C(x, w)|^2 \\ - 2|C(x, Jw)|^2 - 2\langle C(x, Jx), C(w, Jw) \rangle.$$

Setting $x = w$ in (4.11) we obtain the holomorphic sectional curvature.

$$(4.12) \quad R^V(x, Jx, x, Jx) = R^W(x_*, Jx_*, x_*, Jx_*) - 4|C(x, Jx)|^2 \\ = R^W(x_*, Jx_*, x_*, Jx_*) - 4|B(x, x)|^2.$$

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