Tôhoku Math. Journ.
89 (1987), 89-94.

THE RANGE AND PSEUDO-INVERSE OF A PRODUCT

LU SHIJIE

(Received February 12, 1986)

Abstract. By definition the cosine of the angle between the two subspaces M and N is $\sup\{|\langle u, v\rangle|: u\in M,\ v\in N,\ \|u\|=1=\|v\|\}$. For operators A and B with closed range in Hilbert spaces, AB has closed range if and only if the angle between $\ker A$ and $B((\ker AB)^{\perp})$ is positive. Moreover, if we denote by $A^{\dagger}$ the pseudo-inverse of A, then $(AB)^{\dagger}=B^{\dagger}A^{\dagger}$ if and only if $B((\ker AB)^{\perp})\subset(\ker A)^{\perp}$ and $A^{*}((\ker B^{*}A^{*})^{\perp})\subset(\ker B^{*})^{\perp}$.

Let H, K, L be Hilbert spaces over complex field. For a subspace $M\subset H$, we denote by $M^{\perp}$ the orthogonal complement of M and by $\bar{M}$ the closure of M. For two subspaces M and N of H, we shall say that the angle between M and N is θ, if

$$\cos\theta = \sup\{|\langle u, v\rangle|: u\in M,\ v\in N,\ \|u\| = 1 = \|v\|\} .$$

For convenience, we denote the angle by $\theta(M, N)$. Let $C(H, K)$ (resp. $B(H, K)$) be the set of all closed linear operators (resp. bounded operators) from H to K. For $T\in C(H, K)$ we denote the domain of T by $D(T)$, the kernel of T by $\ker T$ and the range of T by $R(T)$. Each $T\in C(H, K)$ induces a one-to-one operator from $(\ker T)^{\perp}$ onto TH. This induced operator is invertible. Define $T^{\dagger}$ to be that inverse on TH and to be zero on $(TH)^{\perp}$. We call $T^{\dagger}$ the pseudo-inverse of T. $T^{\dagger}$ is bounded if and only if $R(T)$ is closed (cf. [3, Theorem 3.1.2]). If $H = K$, we write $B(H)$ instead of $B(H, H)$.

A basic problem in the theory of pseudo-inverse is to determine when the range of a product is closed and the pseudo-inverse of a product is the product of the pseudo-inverses. For $A, B\in B(H)$ with closed range, Bouldin [1] indicated that the simple geometric condition

$$(1)\qquad \theta(\ker A\cap(\ker A\cap BH)^{\perp}, BH) > 0$$

is both necessary and sufficient for AB to have closed range. Furthermore, he proved in [2] that $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ if and only if the following conditions hold:

$$(2)\qquad (AB)^{\dagger} \text{ is bounded ;}$$

$$(3)\qquad A^{*}H \text{ is invariant under } BB^{*} \text{ ;}$$

$$(4)\qquad A^{*}H\cap\ker B^{*} \text{ is invariant under } A^{*}A .$$

In this paper we shall show that the condition (1) can be replaced by

$$(1')\qquad \theta(\ker A, B((\ker AB)^{\perp})) > 0$$

and the conditions (2), (3), (4) can be replaced by

$$(3')\qquad B((\ker AB)^{\perp}) \subset (\ker A)^{\perp} ;$$

$$(4')\qquad A^*((\ker B^*A^*)^{\perp}) \subset (\ker B^*)^{\perp} .$$

Clearly, (1′), (3′), (4′) are not only simpler than (1)–(4), but also have a unified symmetric form and apparent geometric sense. Moreover, the proofs we present are much simpler and clearer.

The following theorem is important for our purpose.

THEOREM (cf. [4, IV, 5.2]). *Let X, Y be complex Banach spaces and $T \in C(X, Y)$. Then T has closed range if and only if there is a positive number δ such that*

$$(5)\qquad \| Tx \| \geqq \delta \operatorname{dist}(x, \ker T) .$$

For Hilbert space operators, the inequality (5) can be simplified as

$$(6)\qquad \| Tx \| \geqq \delta \| x \| , \qquad \text{for} \quad x \in (\ker T)^{\perp} .$$

With those notation and preliminaries we can prove the first main result directly.

THEOREM 1. *Assume that $A \in B(K, H)$ and $B \in B(L, K)$ have closed range. Then AB has closed range if and only if the angle between* $\ker A$ *and* $B((\ker AB)^{\perp})$ *is positive.*

PROOF. Sufficiency. Suppose $\theta(\ker A, B((\ker AB)^{\perp})) > 0$. Then there is a positive number $\delta < 1$ such that

$$|\langle y, Bz\rangle| \leqq \delta \| y \| \| Bz \| \qquad \text{for} \quad y \in \ker A,\ z \in (\ker AB)^{\perp} .$$

Write

$$Bz = y_1 + y_2 \in \ker A \oplus (\ker A)^{\perp} .$$

Since

$$\| y_1 \|^2 = |\langle y_1, Bz\rangle| \leqq \delta \| y_1 \| \| Bz \| ,$$

we must have

$$\| y_1 \| \leqq \delta \| Bz \| .$$

Since $R(A)$, $R(B)$ are closed, there are δ_1, $\delta_2 > 0$, such that

$$\| Ay \| \geqq \delta_1 \operatorname{dist}(y, \ker A) \qquad \text{for all} \quad y \in K ;$$
$$\| Bx \| \geqq \delta_2 \operatorname{dist}(x, \ker B) \qquad \text{for all} \quad x \in L .$$

Thus for $z \in (\ker AB)^{\perp}$ we have

$$\begin{aligned}\|ABz\| \geqq \delta_1 \|y_2\| = \delta_1 \|Bz - y_1\| &\geqq (1-\delta)\delta_1 \|Bz\| \\ &\geqq (1-\delta)\delta_1\delta_2 \operatorname{dist}(z, \ker B)(\text{via}(\ker AB)^{\perp} \subset (\ker B)^{\perp}) \\ &\geqq (1-\delta)\delta_1\delta_2 \operatorname{dist}(z, \ker AB) ,\end{aligned}$$

which shows that $R(AB)$ is closed.

Necessity. Assume that $R(AB)$ is closed. We suppose $\theta(\ker A, B((\ker AB)^{\perp})) = 0$. Then there exist two sequences $\{y_n\} \subset \ker A$ and $\{z_n\} \subset (\ker AB)^{\perp}$ such that

$$(7) \qquad \|y_n\| = \|Bz_n\| = 1 , \quad |\langle y_n, Bz_n \rangle| \to 1 .$$

Clearly, without loss of generality we may assume that $\|z_n\| \geqq \eta > 0$. Write

$$Bz_n = y_1^{(n)} + y_2^{(n)} \in \ker A \oplus (\ker A)^{\perp} .$$

By (7) we have $|\langle y_n, y_1^{(n)} \rangle| \to 1$, which implies $\|y_1^{(n)}\| \to 1$, and hence

$$\|y_2^{(n)}\| \to 0 .$$

Therefore

$$\|ABz_n\| = \|Ay_2^{(n)}\| \to 0 , \qquad \text{for} \quad \|z_n\| \geqq \eta > 0 ,$$

which contradicts the inequality (6).

By Theorem 1 we can obtain a sufficient condition for AB to have closed range:

COROLLARY 2. *Let $A \in B(K, H)$ and $B \in B(L, K)$ have closed range. If $\theta(\ker A, BL) > 0$, then AB has closed range.*

PROOF. It is a consequence of Theorem 1 and the obvious inequality

$$\theta(\ker A, BL) \leqq \theta(\ker A, B((\ker AB)^{\perp})) .$$

Also we can deduce the first corollary in [1]:

COROLLARY 3. *Suppose $A \in B(K, H)$ and $B \in B(L, K)$ have closed range and* $\ker A \cap BL = \{0\}$. *Then AB has closed range if and only if* $\theta(\ker A, BL) > 0$.

PROOF. Since $B(\ker AB) \subset \ker A \cap BL = \{0\}$, we have

$$BL = B((\ker AB)^{\perp}) .$$

Thus Corollary 3 results from Theorem 1.

COROLLARY 4. *Let $A \in B(H)$ have closed range. Then T^2 has closed range if and only if* $\theta(\ker T, T((\ker T^2)^{\perp})) > 0$.

COROLLARY 5. *Let $A \in B(K, H)$, $B \in B(L, K)$ and suppose $A^{\dagger}$, $B^{\dagger}$ are* *bounded. Then* $(AB)^\dagger$ *is bounded if and only if* $\theta(\ker A, B((\ker AB)^\perp)) > 0$.

Now we turn to the second main result, whose proof consists of three propositions.

PROPOSITION 6. *Let* $T \in B(H, K)$ *and suppose* $T^\dagger$ *is the* (*not necessarily bounded*) *pseudo-inverse of* T. *Then* $(T^\dagger)^*$ *is the pseudo-inverse of* T^*.

PROOF. By definition, $T^\dagger$ satisfies the equations

$$\ker T^\dagger = (TH)^\perp = \ker T^* ;$$

$$TT^\dagger = I \quad \text{on} \quad TH ; \tag{8}$$

$$R(T^\dagger) = (\ker T)^\perp . \tag{9}$$

Therefore $T^\dagger$ is densely defined and hence $(T^\dagger)^*$ exists. We shall prove that

$$(T^\dagger)^* T^* = I \quad \text{on} \quad \overline{TH} = (\ker T^*)^\perp . \tag{10}$$

Indeed, for fixed $f \in \overline{TH}$ and each $g = g_1 \oplus g_2 \in TH + (TH)^\perp$, (8) implies

$$|\langle T^*f, T^\dagger g\rangle| = |\langle f, TT^\dagger(g_1 + g_2)\rangle| = |\langle f, g_1\rangle| \leqq \|f\| \|g\| .$$

Hence $\langle T^*f, T^\dagger g\rangle$ is a continuous linear functional on $TH \oplus (TH)^\perp$. This implies that $T^*f \in D((T^\dagger)^*)$ and (10) holds. On the other hand, (9) implies that

$$\ker(T^\dagger)^* = R(T^\dagger)^\perp = \ker T = R(T^*)^\perp . \tag{11}$$

(10) and (11) indicate that $(T^\dagger)^*$ is the pseudo-inverse of T^*.

Denote by I_H the identity on H and by P_A the orthogonal projection on $\ker A$.

PROPOSITION 7. *Let* $A \in B(K, H)$ *and* $B \in B(L, K)$ *have closed range. Then*

$$B^\dagger A^\dagger AB = I_L - P_{AB} \tag{12}$$

if and only if the condition (3′) *holds.*

PROOF. First we have, by definition, that

$$A^\dagger A = I_K - P_A , \quad B^\dagger B = I_L - P_B .$$

For $z \in \ker AB$ the equation (12) is trivial. If $z \in (\ker AB)^\perp$, then $P_B z = 0$. Hence

$$B^\dagger A^\dagger ABz = B^\dagger(I_K - P_A)Bz = (I_L - P_B - B^\dagger P_A B)z = z - B^\dagger P_A Bz . \tag{13}$$

Suppose that (3′) holds. Then $P_A Bz = 0$. By (13) we have

$$B^\dagger A^\dagger ABz = z = (I_L - P_{AB})z .$$

Thus $(3') \Rightarrow (12)$ is proved.

Conversely, if (12) holds, then for $z \in (\ker AB)^{\perp}$ we have

$$B^{\dagger}A^{\dagger}ABz = z - B^{\dagger}P_A Bz = z .$$

Thus $B^{\dagger}P_A Bz = 0$ and hence $P_A Bz \in \ker B^{\dagger} = (BL)^{\perp}$. Write

$$Bz = u + v \in \ker A \oplus (\ker A)^{\perp} .$$

Since $u = P_A Bz \in (BL)^{\perp}$, we have

$$(u, u) = (u, Bz) - (u, v) = 0 \cdot$$

This shows that $Bz \in (\ker A)^{\perp}$.

PROPOSITION 8. *Let* $A \in B(K, H)$ *and* $B \in B(L, K)$ *have closed range. Then*

$$AB\,B^{\dagger}A^{\dagger} = I_H - P_{(AB)^*} \tag{14}$$

if and only if the condition (4′) *holds.*

PROOF. By Lemma 6 we have

$$(B^{\dagger})^*B^* = I_K - P_{B^*} , \quad (A^{\dagger})^*A^* = I_H - P_{A^*} .$$

By Lemma 7 the equation

$$(A^{\dagger})^*(B^{\dagger})^*B^*A^* = I_H - P_{B^*A^*} \tag{15}$$

holds if and only if (4′) holds. By considering conjugate operators we see that (15) holds if and only if (14) holds.

Combining Propositions 7 and 8 we establish the following:

THEOREM 9. *Let* $A \in B(K, H)$ *and* $B \in B(L, K)$ *have closed range. Then*

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger}$$

if and only if (3′) *and* (4′) *hold.*

REMARK 10. Note that the condition (3′) is equivalent to (3). Indeed, if (3′) holds, then, by Theorem 1, $R(AB)$ is closed and hence $R((AB)^*) = (\ker AB)^{\perp}$. Therefore

$$B((\ker AB)^{\perp}) = B(AB)^*H = BB^*A^*H .$$

Since $R(A)$ is closed, we have $(\ker A)^{\perp} = A^*H$. Thus (3′) implies (3).

Conversely, suppose that (3) holds. Since A^*H is closed, we have

$$B((\ker AB)^{\perp}) = B(\overline{B^*A^*H}) \subset A^*H = (\ker A)^{\perp} .$$

Thus (3′) holds.

REMARK 11. Similarly (4′) is equivalent to (4). Indeed, by the above argument, (4′) is equivalent to

$$A^*ABL \subset BL \ . \tag{16}$$

Since A^*A is self-adjoint, (16) is equivalent to

$$A^*A(BL)^{\perp} \subset (BL)^{\perp} \ , \tag{17}$$

namely, $A^*A \ker B^* \subset \ker B^*$. Since A^*H is always invariant under A^*A, we see that (17) is equivalent to (4).

REMARK 12. The condition (2) can be removed, because (3) implies (2). For if (3) holds, then (3′) holds and hence the angle between $\ker A$ and $B((\ker AB)^{\perp})$ is the right angle. By Theorem 1, $R(AB)$ is closed and hence $(AB)^{\dagger}$ is bounded.

REFERENCES

[1] R. H. BOULDIN, The product of two operators with closed range, Tôhoku Math. J. 25 (1973), 359–63.

[2] R. H. BOULDIN, The pseudo-inverse of a product, SIAM J. Appl. Math. 24 (1973), 489–95.

[3] C. W. GROETSCH, Generalized Inverses of Linear Operators, Pure and Applied Mathematics 37, Marcel Dekker, Inc., New York, Basel, 1977.

[4] T. KATO, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1966.

DEPARTMENT OF MATHEMATICS
ZHEJIANG UNIVERSITY
HANGZHOU
PEOPLE'S REPUBLIC OF CHINA