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SEMI-SYMMETRIC LORENTZIAN HYPERSURFACES

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Dedicated to Professor Dr. A. Lichnerowicz for his seventieth birthday

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1. Introduction. Nomizu [2] classified semi-symmetric hypersurfaces in Euclidean spaces. In this paper, we shall give a classification of semi-symmetric *Lorentzian hypersurfaces* in Minkowski spaces. We recall that a semi- or pseudo-Riemannian manifold M is said to be *semi-symmetric*, if it satisfies the condition $R\cdot R=0$, whereby R is the Riemmann-Christoffel curvature tensor of M and where the first tensor acts on the second one as a derivation. Semi-symmetry is a proper generalization of local symmetry, and was first studied by Cartan and Lichnerowicz. Recently, a general study of semi-symmetric Riemannian manifolds was made by Szabó [4].

The main results of this paper can be stated as follows.

THEOREM 1. *Let M^n be a Lorentzian hypersurface of dimension n in a Minkowski space $\boldsymbol{R}_1^{n+1}$. Suppose that the type number $k(x)$ is $\geqq 3$ at a point x of M^n. Then M^n is semi-symmetric at x if and only if the shape operator A_x of M^n at x has the form*

$$(1)\qquad A_x=\left[\begin{array}{c|c}\lambda I_{k(x)} & 0\\ \hline 0 & 0_{n-k(x)}\end{array}\right],\quad \lambda\in\boldsymbol{R}\setminus\{0\}$$

with respect to a suitable orthonormal frame of T_xM^n.

THEOREM 2. *Let M^n be a connected and complete Lorentzian hypersurface of dimension n in a Minkowski space $\boldsymbol{R}_1^{n+1}$. Suppose that the type number is $\geqq 3$ at least at one point of M^n. Then M^n is semi-symmetric if and only if*

(a) $M^n=S_1^k\times\boldsymbol{R}^{n-k}$

or

(b) $M^n=S^k\times\boldsymbol{R}_1^{n-k}$,

for some $k\geqq 3$. In case (a), *S_1^k is a Lorentzian hypersphere in a Minkowski subspace $\boldsymbol{R}_1^{k+1}$ of $\boldsymbol{R}_1^{n+1}$ and $\boldsymbol{R}^{n-k}$ is a Euclidean subspace of $\boldsymbol{R}_1^{n+1}$ orthogonal to $\boldsymbol{R}_1^{k+1}$. In case* (b), *S^k is a hypersphere in a Euclidean* *subspace* $\boldsymbol{R}^{k+1}$ *of* $\boldsymbol{R}_1^{n+1}$ *and* $\boldsymbol{R}_1^{n-k}$ *is a Minkowski subspace of* $\boldsymbol{R}_1^{n+1}$ *orthogonal to* $\boldsymbol{R}^{k+1}$.

2. Basic formulae. Let M^n be an n-dimensional Lorentzian hypersurface in a Minkowski space $\boldsymbol{R}_1^{n+1}$. The natural Lorentz metric on $\boldsymbol{R}_1^{n+1}$ with signature $(-, +, \cdots, +)$ and also the induced Lorentz metric on M^n will be denoted by $\langle\ ,\ \rangle$. The corresponding Levi Civita connection and Riemann-Christoffel curvature tensor of M^n will be denoted by ∇ and R, respectively. When D is the standard connection on $\boldsymbol{R}_1^{n+1}$, the second fundamental form h and the shape operator A with respect to a unit normal vector field ξ are defined by the formulas $D_X Y = \nabla_X Y + h(X, Y)$ and $D_X\xi = -AX$ of Gauss and Weingarten; X, Y, Z, W, V will always denote vector fields tangent to M^n. The rank of the shape operator A_x at a point x of M^n is called the type number $k(x)$ of M^n at x. The Gauss equation of M^n is given by

$$R(X, Y)Z = \langle AY, Z\rangle AX - \langle AX, Z\rangle AY\,. \tag{2}$$

M^n is said to be semi-symmetric if $R(X, Y)\cdot R = 0$ for all X and Y, where the curvature operator $R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ acts as a derivation on the tensor algebra at each point of M^n. Using (2) one may verify that

$$\begin{aligned}(R(X, Y)\cdot R)(Z, W)V = {} & (\langle AY, AZ\rangle\langle AW, V\rangle - \langle AY, AW\rangle\langle AZ, V\rangle)AX \\ & -(\langle AX, AZ\rangle\langle AW, V\rangle - \langle AX, AW\rangle\langle AZ, V\rangle)AY \\ & +(\langle AX, W\rangle\langle A^2Y, V\rangle - \langle AY, W\rangle\langle A^2X, V\rangle \\ & +\langle AW, AY\rangle\langle AX, V\rangle - \langle AW, AX\rangle\langle AY, V\rangle)AZ \\ & -(\langle AX, Z\rangle\langle A^2Y, V\rangle - \langle AY, Z\rangle\langle A^2X, V\rangle \\ & +\langle AZ, AY\rangle\langle AX, V\rangle - \langle AZ, AX\rangle\langle AY, V\rangle)AW \\ & +(\langle AZ, V\rangle\langle AY, W\rangle - \langle AW, V\rangle\langle AY, Z\rangle)A^2X \\ & -(\langle AZ, V\rangle\langle AX, W\rangle - \langle AW, V\rangle\langle AX, Z\rangle)A^2Y\,.\end{aligned} \tag{3}$$

Since the metric $\langle\ ,\ \rangle$ on M^n is of Lorentz type and A_x is a symmetric endomorphism of the tangent space T_xM^n of M^n at x, with respect to suitably chosen frames for T_xM, the shape operator A_x has one of the following forms [1]:

$$\text{(i)}\quad A_x = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix};$$

$$\text{(ii)}\quad A_x = \begin{bmatrix} a & b & & & & \\ -b & a & & & 0 & \\ & & a_3 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & 0 & & & & \cdot \\ & & & & & a_n \end{bmatrix}, \quad (b \neq 0)\ ;$$

(4)

$$\text{(iii)}\quad A_x = \begin{bmatrix} a & 0 & & & & \\ 1 & a & & & 0 & \\ & & a_3 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & 0 & & & & \cdot \\ & & & & & a_n \end{bmatrix};$$

$$\text{(iv)}\quad A_x = \begin{bmatrix} a & 0 & 0 & & & \\ 0 & a & 1 & & 0 & \\ -1 & 0 & a & & & \\ & & & a_4 & & \\ & & & & \cdot & \\ & 0 & & & & \cdot \\ & & & & & a_n \end{bmatrix}.$$

In cases (i) and (ii), A_x is represented with respect to an orthonormal frame $(e_1, e_2, \cdots, e_n)$; this means that $\langle e_1, e_1\rangle = -1$, $\langle e_i, e_j\rangle = \delta_{ij}$, $\langle e_1, e_j\rangle = 0$, $(2 \leqq i, j \leqq n)$ (in later considerations we permit ourselves sometimes to change the ordering of these vectors). In cases (iii) and (iv), A_x is represented with respect to a pseudo-orthonormal frame $(u_1, u_2, \cdots, u_n)$; this means that $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_1, u_i\rangle = \langle u_2, u_i\rangle = 0$, $\langle u_1, u_2\rangle = -1$, $\langle u_i, u_j\rangle = \delta_{ij}$, $(3 \leqq i, j \leqq n)$.

3. Proof of Theorem 1. Let M^n be semi-symmetric, i.e., let

(5) $$(R(X, Y)\cdot R)(Z, W)V = 0\ ,$$

for all X, Y, Z, W, V. The proof of Theorem 1 will be devided into four parts, according to the four possible forms of A. It is always assumed that $k(x) \geqq 3$.

(I) Suppose that A_x is of the form (i). Putting $X = e_i$, $Y = e_j$, $Z = e_i$, $W = e_k$ and $V = e_j$, (i, j and k being mutually distinct), from (3) and (5) we find that

$$a_k a_i a_j (a_i - a_j) = 0\ .$$

Thus, by the assumption on the type number, all non-zero eigenvalues are the same, which yields formula (1).

(II) Suppose that A_x is of the form (ii). Then we have:

$$Ae_j = a_j e_j\ , \qquad (j = 3, \cdots, n)\ ,$$
$$Ae_1 = ae_1 - be_2\ ,$$
$$Ae_2 = be_1 + ae_2\ .$$

Putting $X=Z=e_1$, $Y=V=e_j$, $(j=3, \cdots, n)$, from (3) and (5) we find that

$$a_j b(a^2+b^2)e_1 + (a^2+b^2)a_j(a_j - a) = 0\ .$$

In particular, this implies that

$$a_j b(a^2+b^2) = 0\ ,$$

which in turn implies $a_j = 0$ because $b \neq 0$. This, however, contradicts the assumption on the type number. Thus this case cannot occur.

(III) Suppose next that A_x is of the form (iii). Then, with respect to a pseudo-orthonormal frame $(u_1, u_2, \cdots, u_n)$, we have:

$$Au_1 = au_1 + u_2\ ,$$
$$Au_2 = au_2\ ,$$
$$Au_j = a_j u_j\ , \qquad (j = 3, \cdots, n)\ .$$

Putting $X = Z = u_1$, $W = u_2$, $Y = V = u_j$, $(j = 3, \cdots, n)$, from (3) and (5) we find that

$$a^2 a_j = 0\ .$$

In case $a \neq 0$, this implies that $a_j = 0$ for all $j = 3, \cdots, n$. This contradicts the assumption on the type number. So we may assume that a $\neq 0$. Then, putting $X = Z = u_1$, $Y = V = u_i$, $W = u_j$, $(i, j = 3, \cdots, n;$ $i \neq j)$, from (3) and (5) we obtain:

$$a_i^2 a_j = 0\ .$$

Thus, in this case, at most one of the numbers $a_3, \cdots, a_n$ can be different from zero. This again contradicts the assumption on the type number. Consequently, also the form (iii) for the shape operator cannot occur.

(IV) Finally, suppose that A_x is of the form (iv). Then, with respect to a pseudo-orthonormal frame $(u_1, u_2, \cdots, u_n)$, we have:

$$Au_1 = au_1 - u_3\ ,$$
$$Au_2 = au_2\ ,$$
$$Au_3 = u_2 + au_3\ ,$$
$$Au_j = a_j u_j\ , \qquad (j = 4, \cdots, n)\ .$$

Putting $X = Z = u_1$, $W = V = u_2$, $Y = u_3$, from (3) and (5) it follows that $a = 0$. Next, putting $X = Z = u_1$, $Y = V = u_i$, $W = u_j$, $(i, j = 4, \cdots, n; i \neq j)$, from (3) and (5) it follows that

$$a_i a_j = 0 .$$

Thus at most one of the numbers $a_4, \cdots, a_n$ can be different from zero. Assuming, however, that $a_4 = \cdots = a_{n-1} = 0$, for instance, we find that also $a_n = 0$ from (3) and (5), where we put $X = Z = u_1$, $Y = u_3$, $W = V = u_n$. Then, clearly rank $A_x = 2$, which is a contradiction.

The converse statement of Theorem 1, the fact that $R \cdot R = 0$ when A_x is given by the expression (1), can readily be verified by a straightforward calculation. □

4. **Proof of Theorem 2.** The main part of the proof of Theorem 2 can be taken over without any changes from Nomizu's classification of the semi-symmetric hypersurfaces of Euclidean spaces in [2].

First, we assume that the type number $k(x) \geqq 3$, everywhere on the Lorentzian hypersurface M^n. Without loss of generality, we may suppose that M^n is orientable and thus that there exists a unit normal vector field ξ defined on the entire hypersurface ([3, p. 189]); (in case that M^n is not orientable we can always work with the universal covering). From the fact that M^n is semi-symmetric, we know by Theorem 1 that the shape operator A of M^n corresponding to ξ is given by formula (1) for every point x of the hypersurface. Precisely as in [2], it then follows that the type number $k(x)$ is constant on M^n, say $k(x) = k$, and that the only eigenvalue $\lambda(x)$ of A_x which is non-zero defines a differentiable function λ on M^n. Next, we consider the distributions T_0 and T_1 which are defined by

$$T_0(x) = \{X \in T_x M^n \mid AX = 0\} ,$$
$$T_1(x) = \{X \in T_x M^n \mid AX = \lambda(x) X\} .$$

It is easy to see that both these distributions on M are differentiable and involutive, and that $T_x M^n = T_0(x) \oplus T_1(x)$ at each $x \in M^n$. Also, as in [2], we have the following result.

LEMMA 1. *If* $Y\lambda = 0$ *for every* $Y \in T_0$, $\nabla_X T_0 \subset T_0$ *and* $\nabla_X T_1 \subset T_1$ *for every vector* X *which is tangent to* M^n.

Further, by $M_0(x)$ and $M_1(x)$ we will denote the maximal integral submanifolds of M^n corresponding respectively to T_0 and T_1, and which pass through the point x. We then have the following result.

THEOREM 2a. (i) $M_0(x)$ *is a complete totally geodesic submanifold* *of* M^n.

(ii) $f|_{M_0}$, *the restriction of the isometrical immersion* f *of* M^n *in* $\boldsymbol{R}_1^{n+1}$ *to* M_0, *is an isometry of* $M_0(x)$ *to* $\boldsymbol{R}^{n-k}(x)$ *or to* $\boldsymbol{R}_1^{n-k}(x)$.

PROOF. (i) See [2].

(ii) It is clear that the second fundamental form h of $M_0(x)$ in $\boldsymbol{R}_1^{n+1}$ vanishes identically (for all X, Y tangent to $M_0(x)$, we have $h(X, Y) = \langle Y, AX\rangle\xi$), and so $M_0(x)$ is also a totally geodesic submanifold of the Minkowski space $\boldsymbol{R}_1^{n+1}$. Consequently, by the immersion f, every geodesic of $M_0(x)$ is mapped upon a straight line in $\boldsymbol{R}_1^{n+1}$. The restriction of the metric on M to $M_0(x)$ is either Euclidean or Lorentzian. Accordingly, by the completeness of $M_0(x)$, either $f(M_0(x)) = \boldsymbol{R}^{n-k}(x)$ or $f(M_0(x)) = \boldsymbol{R}_1^{n-k}(x)$. It follows that f is a covering map ([3, p. 202]), and so it is an isometry of $M_0(x)$ to $\boldsymbol{R}^{n-k}(x)$ or to $\boldsymbol{R}_1^{n-k}(x)$ respectively. □

In the next theorem we will need the following.

LEMMA 2. *For every* $Y \in T_0$, *we have* $Y\lambda = 0$.

PROOF. Following Theorem 2a, we have to consider two cases, according as $M_0(x)$ is isometric to a Euclidean space $\boldsymbol{R}^{n-k}(x)$ or to a Minkowski space $\boldsymbol{R}_1^{n-k}(x)$. In the first case, the proof of this lemma can be carried over completely from [2]. We now give a proof for the second case. Let $\{y^1, \cdots, y^k, y^{k+1}, \cdots, y^n\}$ be a coordinate system in a neighbourhood U of M with origin x such that $\{\partial/\partial y^1, \cdots, \partial/\partial y^k\}$ and $\{\partial/\partial y^{k+1}, \cdots, \partial/\partial y^n\}$ are local frames for T_1 and T_0, respectively, and such that the restriction of $\{y^{k+1}, \cdots, y^n\}$ to $M_0(x) \cap U$ is rectangular in the Lorentzian sense, i.e.,

$$\langle \partial/\partial y^\alpha, \partial/\partial y^\beta\rangle = \varepsilon_\alpha \delta_{\alpha\beta} ,$$

for $k+1 \leqq \alpha, \beta \leqq n$, where $\varepsilon_\gamma = 1$ for $\gamma = k+2, \cdots, n$ and $\varepsilon_{k+1} = -1$. Then, precisely as in [2], we find that $Y^2(1/\lambda) = 0$ for all $Y = \partial/\partial y^\alpha$. Consequently, λ is constant on all straight lines in $M_0(x)$ which pass through x and which are not lying on the null-cone through x. From this and the fact that λ is continuous on M^n (even differentiable), it follows that λ is a constant function on the whole of $M_0(x)$. □

The proof of Theorem 2, under the assumption that the type number $k(x) \geqq 3$ at every point x of M^n, is completed by the following.

THEOREM 2b. (i) $M_1(x)$ *is a complete totally geodesic submanifold of* M^n.

(ii) M^n *is isometric with* $M_0 \times M_1$ *for every point* $m \in M^n$, *where* $M_0 = M_0(m)$ *and* $M_1 = M_1(m)$.

(iii) *For every point* $m \in M^n$, *the spaces* $f(M_0(m)) = \boldsymbol{R}^{n-k}(m)$, *respec-*

tively $f(M_0(m)) = \boldsymbol{R}_1^{n-k}(m)$, *are parallel.*

(iv) *In case* M_0 *is isometric to* $\boldsymbol{R}^{n-k}$, *the restriction* $f|_{M_1}$ *of* f *to* M_1 *is an isometry of* M_1 *to* $S_1^k \subset \boldsymbol{R}_1^{k+1}$, *where* $\boldsymbol{R}_1^{k+1}$ *is orthogonal to* $\boldsymbol{R}^{n-k}$ *in* $\boldsymbol{R}_1^{n+1}$.

(v) *In case* M_0 *is isometric to* $\boldsymbol{R}_1^{n-k}$, *the restriction* $f|_{M_1}$ *of* f *to* M_1 *is an isometry of* M_1 *to* $S^k \subset \boldsymbol{R}^{k+1}$, *where* $\boldsymbol{R}^{k+1}$ *is orthogonal to* $\boldsymbol{R}_1^{n-k}$ *in* $\boldsymbol{R}_1^{n+1}$.

(vi) $f = f|_{M_0} \times f|_{M_1}$, *i.e.*, $f(m_0, m_1) = (f|_{M_0}(m_0), f|_{M_1}(m_1))$ *for every point* $(m_0, m_1) \in M_0 \times M_1 = M$.

PROOF. (i) See [2].

(ii) From Lemmas 1 and 2, it follows that both distributions T_0 and T_1 are parallel. Thus $T_0(M_0(m))$ and $T_0(M_1(m))$ are invariant under the action of the holonomy group of M^n at any point $m \in M^n$. Since, moreover, the restrictions of the metric on $T_m M^n$ to $T_0(m)$ and $T_1(m)$ are non-degenerate, by Wu's extension of de Rham's decomposition theorem to indefinite metrics [5], we can conclude that M^n is isometric to $M_0 \times M_1$.

(iii) See [2].

(iv) We consider the function $x \mapsto \xi_x + \lambda f(x)$. Since $D_{f_* X}(\xi + \lambda f) = 0$ for every vector X tangent to M_1, we obtain that $f(M_1)$ is part of a hypersphere S_1^n in $\boldsymbol{R}_1^{n+1}$ with radius $|1/\lambda|$. Since $f(M_1)$ is orthogonal to $f(M_0) = \boldsymbol{R}^{n-k}$ at each point and since these spaces $\boldsymbol{R}^{n-k}$ are all parallel, it follows that $f(M_1)$ is also contained in the linear subspace $\boldsymbol{R}_1^{k+1}$ of $\boldsymbol{R}_1^{n+1}$ which passes through $f(x)$ and which is orthogonal to $\boldsymbol{R}^{n-k}$. Consequently, $f(M_1)$ is a part of the sphere $S_1^k = S_1^n \cap \boldsymbol{R}_1^{k+1}$. Finally, since M_1 is complete and f is a covering map, f is an isometry of M_1 onto S_1^k.

(v) This proof is similar to the one in (iv).

(vi) See [2]. □

So far, we proved Theorem 2 under the assumption that the type number is greater than 2 at every point of the Lorentz hypersurface M^n. The proof under the weaker assumption that there exists a point $x \in M^n$ where $k(x) \geqq 3$ can be adapted from [2] using arguments similar to those given above. □

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