Tôhoku Math. Journ.
39 (1987), 71–79.

ALMOST PERIODIC SOLUTIONS OF A SYSTEM OF INTEGRODIFFERENTIAL EQUATIONS

SATORU MURAKAMI

(Received January 25, 1986)

The purpose of this article is to discuss the existence of almost periodic solutions of a system of almost periodic integrodifferential equations

$$(E)\qquad \dot{x}_i(t) = h_i(x_i(t))\left\{b_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1\\ j\neq i}}^{k} a_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-u)G_i(x_j(u))du\right\}, \quad i = 1, 2, \cdots, k,$$

which describes a model of the dynamics of a k-species system in mathematical ecology when $h_i(s) = G_i(s) \equiv s$. When $h_i(s) = G_i(s) \equiv s$ and $a_{ij}(t)$, $b_i(t)$ are ω-periodic, Gopalsamy [2] has recently discussed the existence of ω-periodic solutions of System (E) under some conditions. In order to obtain an ω-periodic solution of System (E), he has investigated the existence of ω-periodic solutions of another system

$$(E_0)\qquad \dot{x}_i(t) = h_i(x_i(t))\left\{b_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1\\ j\neq i}}^{k} a_{ij}(t)\int_{t-\omega}^{t} \sum_{r=0}^{\infty} K_{ij}(t-u+r\omega)G_i(x_j(u))du\right\}, \quad i = 1, 2, \cdots, k,$$

instead of the original system (E), because any ω-periodic solution of System (E) is also an ω-periodic solution of System (E_0) and vice versa. As easily seen, however, we cannot directly employ Gopalsamy's idea when System (E) is almost periodic. In this article, we shall investigate some stability properties of a solution of System (E), and consequently obtain an almost periodic solution of System (E). We emphasize that our result contains Theorem 2.1 in [2] as a special case.

In what follows, we denote by R^k the k-dimensional real Euclidean space and by $|x|$ the norm of $x \in R^k$. Throughout this paper, we suppose that the functions h_i, b_i, a_{ij}, K_{ij} and G_i in System (E) are real-valued continuous functions on $R := R^1$ and that the following conditions are satisfied:

(H1) a_{ij} and b_i are almost periodic functions, and $\inf_{t\in R} a_{ij}(t) > 0$ and $\inf_{t\in R} b_i(t) > 0$ for $i, j = 1, \cdots, k$;

(H2) $h_i(s) > 0$ for $s > 0$, $h_i(0) = 0$ and $h_i(s)$ is Lipschitz continuous in s for $i = 1, \cdots, k$;

(H3) K_{ij} is nonnegative, $\int_0^\infty K_{ij}(s)ds = 1$ and $\int_0^\infty sK_{ij}(s)ds < \infty$ for $i, j = 1, \cdots, k$, $i \neq j$;

(H4) $G_i(t)$ is nondecreasing in t, $G_i(t) \geqq 0$ for $t \geqq 0$ and there exists a constant $N > 0$ satisfying $|G_i(t) - G_i(s)| \leqq N|t - s|$ for all $t, s \in R$ and all $i = 1, \cdots, k$;

(H5) $b_i^l > \sum_{\substack{j=1 \\ j\neq i}}^k a_{ij}^u G_i(b_j^u/a_{jj}^l)$ for $i = 1, \cdots, k$;

where

$$b_i^l = \inf_{t\in R} b_i(t)\ , \qquad b_j^u = \sup_{t\in R} b_j(t)$$

$$a_{jj}^l = \inf_{t\in R} a_{jj}(t)\ , \qquad a_{ij}^u = \sup_{t\in R} a_{ij}(t)\ , \quad i, j = 1, \cdots, k\ .$$

Let BC be the set of all bounded continuous functions from $(-\infty, 0]$ into R^k and set $\|\phi\| = \sup_{s\leq 0}|\phi(s)|$ for $\phi \in \mathrm{BC}$. From (H1)-(H4) it follows that for any $(t_0, \phi) \in R\times\mathrm{BC}$ there is a unique (local) solution $x(t) = (x_1(t), \cdots, x_k(t))$ of System (E) through (t_0, ϕ), which is continuable to $t = \infty$ if it remains bounded (cf. [1]). For each i we set

$$x_i^* = b_i^u/a_{ii}^l \quad \text{and} \quad x_{i*} = \min\left\{x_i^*, \left[b_i^l - \sum_{\substack{j=1 \\ j\neq i}}^k a_{ij}^u G_i(x_j^*)\right]\Big/a_{ii}^u\right\}\ .$$

From (H1) and (H5), x_i^* and x_{i*} are positive numbers for each i. We can prove the following lemma by repeating almost the same argument as in [2, p. 325].

LEMMA 1. *Let* $\phi = (\phi_1, \cdots, \phi_k) \in \mathrm{BC}$ *satisfy* $x_{i*} \leqq \phi_i(s) \leqq x_i^*$ *for all* $s \leqq 0$ *and all* $i = 1, \cdots, k$, *and let* $x(t) = (x_1(t), \cdots, x_k(t))$ *be the solution of System* (E) *through* (t_0, ϕ). *Then* $x_{i*} \leqq x_i(t) \leqq x_i^*$ *for all* $t \geqq t_0$ *and all* $i = 1, \cdots, k$.

We denote by $S(E)$ the set of all solutions $x(t) = (x_1(t), \cdots, x_k(t))$ of System (E) on R satisfying $x_{i*} \leqq x_i(t) \leqq x_i^*$ for all $t \in R$ and all $i = 1, \cdots, k$. Then we have:

LEMMA 2. $S(E) \neq \varnothing$.

PROOF. By (H1) there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $b_i(t + t_n) \to b_i(t)$ and $a_{ij}(t + t_n) \to a_{ij}(t)$ as $n \to \infty$ uniformly on R.

Let $x(t)$ be a solution of System (E) through $(t_0, \phi) \in R \times \mathrm{BC}$ satisfying $x_{i*} \leqq x_i(t) \leqq x_i^*$ for all $t \geqq t_0$ and all $i = 1, \cdots, k$, whose existence was ensured by Lemma 1. Clearly, the sequence $\{x(t + t_n)\}$ is uniformly bounded and equicontinuous on each bounded subset of R. Therefore, by Ascoli's theorem and diagonalization procedure we may assume that the sequence $\{x(t + t_n)\}$ converges to a continuous function $p(t) = (p_1(t), \cdots, p_k(t))$ as $n \to \infty$ uniformly on each bounded subset of R. Let a $\tau \in R$ be given. We may assume that $t_n + \tau \geqq t_0$ for all n. For $t \geqq 0$, we have

$$\begin{aligned} & x_i(t + t_n + \tau) - x_i(t_n + \tau) \\ (1) \qquad & = \int_\tau^{t+\tau} \Big[h_i(x_i(s + t_n)) \Big\{ b_i(s + t_n) - a_{ii}(s + t_n) x_i(s + t_n) \\ & \quad - \sum_{\substack{j=1 \\ j \neq i}}^{k} a_{ij}(s + t_n) \int_{-\infty}^{0} K_{ij}(-v) G_i(x_j(v + s + t_n)) dv \Big\} \Big] ds . \end{aligned}$$

Note that $K_{ij}(-v)G_i(x_j(v + s + t_n)) \to K_{ij}(-v)G_i(p_j(v + s))$ as $n \to \infty$ and that $|K_{ij}(-v)G_i(x_j(v + s + t_n))| \leqq K_{ij}(-v)G_i(\|\phi\| + x_j^*)$ for $v \leqq 0$ and $s \in [\tau, t + \tau]$. Then, by (H3) and Lebesgue's dominated convergence theorem, we obtain

$$\int_{-\infty}^{0} K_{ij}(-v) G_i(x_j(v + s + t_n)) dv \to \int_{-\infty}^{0} K_{ij}(-v) G_i(p_j(v + s)) dv$$

as $n \to \infty$ for each $s \in [\tau, t + \tau]$. Moreover, from (H3),

$$\left| \int_{-\infty}^{0} K_{ij}(-v) G_i(x_j(v + s + t_n)) dv \right| \leqq G_i(\|\phi\| + x_j^*) .$$

Applying Lebesgue's dominated convergence theorem again, and letting $n \to \infty$ in (1), we have

$$\begin{aligned} p_i(t + \tau) - p_i(\tau) & = \int_\tau^{t+\tau} \Big[h_i(p_i(s)) \Big\{ b_i(s) - a_{ii}(s) p_i(s) \\ & \quad - \sum_{\substack{j=1 \\ j \neq i}}^{k} a_{ij}(s) \int_{-\infty}^{0} K_{ij}(-v) G_i(p_j(v + s)) dv \Big\} \Big] ds \\ & = \int_\tau^{t+\tau} \Big[h_i(p_i(s)) \Big\{ b_i(s) - a_{ii}(s) p_i(s) \\ & \quad - \sum_{\substack{j=1 \\ j \neq i}}^{k} a_{ij}(s) \int_{-\infty}^{s} K_{ij}(s - u) G_i(p_j(u)) du \Big\} \Big] ds \end{aligned}$$

for all $t \geqq 0$ and all $i = 1, \cdots, k$. Since $\tau \in R$ is arbitrarily given, $p(t) = (p_1(t), \cdots, p_k(t))$ is a solution of System (E) on R. It is clear that $x_{i*} \leqq p_i(t) \leqq x_i^*$ for all $t \in R$ and all $i = 1, \cdots, k$. Thus $p \in S(E)$. q.e.d.

By repeating almost the same argument as in the proof of Lemma 2, we also conclude:

LEMMA 3. *Let a $p \in S(E)$ and a sequence $\{t_n\}$, $t_n \geqq 0$, be given. If*

(2) $a_{ij}(t+t_n) \to \bar{a}_{ij}(t)$ *and* $b_i(t+t_n) \to \bar{b}_i(t)$ *as* $n \to \infty$ *uniformly on R for all* $i, j = 1, \cdots, k$, *and*

(3) $p(t+t_n) \to \bar{p}(t)$ *as* $n \to \infty$ *uniformly on each bounded subset of R*

for some functions $\bar{a}_{ij}$, $\bar{b}_i$ and $\bar{p}$, then $\bar{p} \in S(\bar{E})$, where $S(\bar{E})$ denotes the set of all solutions $y(t) = (y_1(t), \cdots, y_k(t))$ of the system

$$(\bar{E}) \qquad \dot{y}_i(t) = h_i(y_i(t))\left\{\bar{b}_i(t) - \bar{a}_{ii}(t)y_i(t) - \sum_{\substack{j=1 \\ j\neq i}}^{k} \bar{a}_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-u)G_i(y_j(u))du\right\}, \quad i = 1, 2, \cdots, k,$$

on R satisfying $x_{i*} \leqq y_i(t) \leqq x_i^*$ for all $t \in R$ and all $i = 1, \cdots, k$. (Henceforth, we denote $(\bar{p}, \bar{E}) \in \Omega(p, E)$ when (2) *and* (3) *hold.)*

Next, for any $\phi, \psi \in \mathrm{BC}$ we set

$$\rho_m(\phi, \psi) = \sup_{-m \leqq s \leqq 0} |\phi(s) - \psi(s)|,$$

$$\rho(\phi, \psi) = \sum_{m=1}^{\infty} \rho_m(\phi, \psi)/[2^m(1 + \rho_m(\phi, \psi))].$$

Clearly, $\rho(\phi_n, \phi) \to 0$ as $n \to \infty$ if and only if $\phi_n(s) \to \phi(s)$ as $n \to \infty$ uniformly on each bounded subset of $(-\infty, 0]$. For any function $x: R \to R^k$ and any $t \in R$, we define a function $x^t: (-\infty, 0] \to R^k$ by $x^t(s) = x(t+s)$ for $s \leqq 0$.

DEFINITION 1. A function $p \in S(E)$ is said to be relatively uniformly stable in $\Omega(E)$ (RUS in $\Omega(E)$, for short) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ with the property that for any $t_0 \geqq 0$, any $(\bar{p}, \bar{E}) \in \Omega(p, E)$ and any $\bar{z} \in S(\bar{E})$ satisfying $\rho(\bar{p}^{t_0}, \bar{z}^{t_0}) < \delta(\varepsilon)$ we have $\rho(\bar{p}^t, \bar{z}^t) < \varepsilon$ for all $t \geqq t_0$.

DEFINITION 2. A function $p \in S(E)$ is said to be relatively weakly uniformly asymptotically stable in $\Omega(E)$ (RWUAS in $\Omega(E)$, for short) if p is RUS in $\Omega(E)$, and if $\rho(\bar{p}^t, \bar{z}^t) \to 0$ as $t \to \infty$ for all $(\bar{p}, \bar{E}) \in \Omega(p, E)$ and all $\bar{z} \in S(\bar{E})$.

DEFINITION 3. A function $p \in S(E)$ is said to be relatively totally stable for (E) (RTS for (E), for short) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ with the property that if $t_0 \geqq 0$, $\rho(x^{t_0}, p^{t_0}) < \delta(\varepsilon)$ and $g(t) = (g_1(t), \cdots, g_k(t)): R \to R^k$ is any continuous function satisfying $\sup_{t \in R}|g(t)| < \delta(\varepsilon)$, then we have $\rho(x^t, p^t) < \varepsilon$ for all $t \geqq t_0$, where x is any solution of the system

$$(E_g) \qquad \dot{x}_i(t) = h_i(x_i(t))\Big\{b_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{k} a_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-u)G_i(x_j(u))du\Big\} + g_i(t)\ , \quad i = 1, \cdots, k\ ,$$

on R satisfying $x_{i*} \leqq x_i(t) \leqq x_i^*$ for all $t \in R$ and all $i = 1, \cdots, k$.

LEMMA 4. *If* $p \in S(E)$ *is* RWUAS *in* $\Omega(E)$, *then it is* RTS *for* (E).

PROOF. We give the proof for completeness, although it is essentially the same as the one for [3, Theorem] (cf. [4, Proposition 4.1]). Suppose the contrary. Then there exist an $\varepsilon > 0$, sequences $\{\varepsilon_n\}$, $0 < \varepsilon_n < \varepsilon$ and $\varepsilon_n \to 0$ as $n \to \infty$, $\{s_n\}$, $\{t_n\}$, $t_n \geqq s_n \geqq 0$, $\{g_n\}$ and $\{x^n\}$ such that $g_n: R \to R^k$ is a continuous function satisfying $\sup_{t \in R}|g_n(t)| < \varepsilon_n$ and that

$$(4) \qquad \rho(p^{s_n}, (x^n)^{s_n}) < \varepsilon_n\ , \qquad \rho(p^{t_n}, (x^n)^{t_n}) = \varepsilon \quad \text{and} \quad \rho(p^t, (x^n)^t) < \varepsilon \quad \text{on} \quad [s_n, t_n)\ ,$$

where x^n is a solution of (E_{g_n}) on R satisfying $x_{i*} \leqq (x^n)_i(t) \leqq x_i^*$ on R for all $i = 1, \cdots, k$. Furthermore, by (4) we can choose a sequence $\{\tau_n\}$, $s_n < \tau_n < t_n$, so that

$$(5) \qquad \rho(p^{\tau_n}, (x^n)^{\tau_n}) = \delta(\varepsilon/2)/2$$

and

$$(6) \qquad \delta(\varepsilon/2)/2 \leqq \rho(p^t, (x^n)^t) \leqq \varepsilon \quad \text{on} \quad [\tau_n, t_n]\ ,$$

where $\delta(\cdot)$ is the number given in Definition 1. We may assume that $p(\tau_n + t) \to \bar{p}(t)$ as $n \to \infty$ on each bounded subset of R for a continuous function $\bar{p}$ and that $(\bar{p}, \bar{E}) \in \Omega(p, E)$. Moreover, we may assume that $x^n(\tau_n + t) \to \bar{z}(t)$ as $n \to \infty$ uniformly on any bounded subset of R for a continuous function $\bar{z}$, since the sequence $\{x^n(\tau_n + t)\}$ is uniformly bounded and equicontinuous on R. Then, the same argument as in the proof of Lemma 2 shows that $\bar{z} \in S(\bar{E})$. Now, suppose that $t_n - \tau_n \to \infty$ as $n \to \infty$. Letting $n \to \infty$ in (6) we have $\delta(\varepsilon/2)/2 \leqq \rho(\bar{p}^t, \bar{z}^t) \leqq \varepsilon$ for all $t \geqq 0$. On the other hand, $\rho(\bar{p}^t, \bar{z}^t) \to 0$ as $t \to \infty$, since p is RWUAS in $\Omega(E)$. This is a contradiction. Thus, $t_n - \tau_n \not\to \infty$ as $n \to \infty$. Taking a subsequence if necessary, we may assume $t_n - \tau_n \to r < \infty$ as $n \to \infty$. Letting $n \to \infty$ in (5), we obtain $\rho(\bar{p}^0, \bar{z}^0) = \delta(\varepsilon/2)/2 < \delta(\varepsilon/2)$, and hence $\rho(\bar{p}^t, \bar{z}^t) < \varepsilon/2$ for all $t \geqq 0$, because p is RUS in $\Omega(E)$. On the other hand, from (4) we have $\rho(\bar{p}^r, \bar{z}^r) = \varepsilon$, which is a contradiction. This completes the proof.

Now, our main result on the existence of an almost periodic solution of System (E) is the following:

THEOREM. *In addition to* (H1)–(H5), *suppose that*

(H6) *there exists a positive constant* M *such that*

$$a_{ii}^l > N\cdot\sum_{\substack{j=1\\ j\neq i}}^{k} a_{ji}^u + M \quad \textit{for all}\ \ i = 1, \cdots, k$$

(*here,* N *is the number in* (H4)). *Then System* (E) *has a unique almost periodic solution* $q(t)$ *in* $S(E)$. *Moreover, the module of* $q(t)$ *is contained in the module of* $\{a_{ij}(t), b_i(t);\ i, j = 1, \cdots, k\}$.

PROOF. Let p be an element in $S(E)$. First of all, we shall prove that p is RTS for (E). By Lemma 4 it suffices to show that p is RWUAS in $\Omega(E)$. For arbitrary $(\bar{p}, \bar{E}) \in \Omega(p, E)$ and $\bar{z} \in S(\bar{E})$, let

$$(7)\qquad v(t) = V(t, \bar{p}(\cdot), \bar{z}(\cdot)) = \sum_{i=1}^{k}\Big[|H_i(\bar{p}_i(t)) - H_i(\bar{z}_i(t))| + \sum_{\substack{j=1\\ j\neq i}}^{k}\int_0^\infty K_{ij}(s)\Big\{\int_{t-s}^{t}\bar{a}_{ij}(s+u)|G_i(\bar{p}_j(u)) - G_i(\bar{z}_j(u))|\,du\Big\}ds\Big],$$

where

$$H_i(s) := \int_{x_{i*}}^{s} du/h_i(u)\ .$$

Note that the integrand in (7) converges by (H3) and that $v(t)$ is continuous in t. An easy computation shows that

$$(8)\qquad D^+v(t) \leqq \sum_{i=1}^{k}\Big\{-\bar{a}_{ii}(t)|\bar{p}_i(t) - \bar{z}_i(t)| + N\cdot\sum_{\substack{j=1\\ j\neq i}}^{k} a_{ij}^u|\bar{z}_j(t) - \bar{p}_j(t)|\Big\} \leqq -M\cdot\sum_{i=1}^{k}|\bar{p}_i(t) - \bar{z}_i(t)| \leqq 0$$

by (H3), (H4) and (H6). Hence we have

$$v(t) - v(0) \leqq -M\cdot\sum_{i=1}^{k}\int_0^t|\bar{p}_i(s) - \bar{z}_i(s)|ds \quad \text{for} \quad t \geqq 0\ .$$

Consequently $\sum_{i=1}^{k}\int_0^\infty|\bar{p}_i(s) - \bar{z}_i(s)|ds < \infty$, hence $\sum_{i=1}^{k}|\bar{p}_i(t) - \bar{z}_i(t)| \to 0$ as $t \to \infty$, since the function $\sum_{i=1}^{k}|\bar{p}_i(t) - \bar{z}_i(t)|$ is uniformly continuous on $[0, \infty)$. Thus $\rho(\bar{p}^t, \bar{z}^t) \to 0$ as $t \to \infty$. Moreover, from (7) and (8) it follows that

$$(9)\qquad \sum_{i=1}^{k}|H_i(\bar{p}_i(t)) - H_i(\bar{z}_i(t))| \leqq v(t) \leqq v(t_0) \leqq \sum_{i=1}^{k}\Big[|H_i(\bar{p}_i(t_0)) - H_i(\bar{z}_i(t_0))| + N\cdot\sum_{\substack{j=1\\ j\neq i}}^{k} a_{ij}^u x_j^*\int_L^\infty sK_{ij}(s)ds$$

$$+ N \cdot \sum_{\substack{j=1 \\ j \neq i}}^{k} a_{ij}^u \cdot \int_0^\infty sK_{ij}(s)ds \cdot \sup_{t_0 - L \leqq u \leqq t_0} |\bar{p}_j(u) - \bar{z}_j(u)| \Bigg]$$

for all $t \geqq t_0 \geqq 0$ and all $L \geqq 0$. For each $\varepsilon > 0$ we set

$$\tilde{\delta}(\varepsilon) = \inf \left\{ \sum_{i=1}^{k} |H_i(x_i) - H_i(y_i)| \colon |x - y| \geqq \varepsilon \text{ and } x_{i*} \leqq x_i ,\right. \\ \left. y_i \leqq x_i^* \text{ for all } i = 1, \cdots, k \right\} . \tag{10}$$

Clearly, $\tilde{\delta}(\varepsilon) > 0$ by (H2). We select a number $L > 0$ so large that

$$\sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} a_{ij}^u x_j^* \int_L^\infty sK_{ij}(s)ds < \tilde{\delta}(\varepsilon)/(2N) ,$$

which is possible by (H3). Moreover, we select a $\delta(\varepsilon) \in (0, \varepsilon)$ so that

$$\sum_{i=1}^{k} \left\{ |H_i(\phi_i(0)) - H_i(\psi_i(0))| + N \cdot \sum_{\substack{j=1 \\ j \neq i}}^{k} a_{ij}^u \int_0^\infty sK_{ij}(s)ds \cdot \sup_{t_0 - L \leqq u \leqq t_0} |\phi(u) - \psi(u)| \right\} \\ < \tilde{\delta}(\varepsilon)/2 ,$$

whenever $\rho(\phi, \psi) < \delta(\varepsilon)$. Hence, if $\rho(\bar{p}^{t_0}, \bar{z}^{t_0}) < \delta(\varepsilon)$, we have

$$\sum_{i=1}^{k} |H_i(\bar{p}_i(t)) - H_i(\bar{z}_i(t))| < \tilde{\delta}(\varepsilon)$$

by (9), and consequently, $|\bar{p}(t) - \bar{z}(t)| < \varepsilon$ for all $t \geqq t_0$ by (10). Thus, if $\rho(\bar{p}^{t_0}, \bar{z}^{t_0}) < \delta(\varepsilon)$, then

$$\begin{aligned} \rho(\bar{p}^t, \bar{z}^t) &\leqq \sum_{n=1}^{\infty} (\rho_n(\bar{p}^{t_0}, \bar{z}^{t_0}) + \varepsilon)/[2^n(1 + \rho_n(\bar{p}^{t_0}, \bar{z}^{t_0}) + \varepsilon)] \\ &\leqq \sum_{n=1}^{\infty} 2^{-n} \{\rho_n(\bar{p}^{t_0}, \bar{z}^{t_0})/[1 + \rho_n(\bar{p}^{t_0}, \bar{z}^{t_0})] + \varepsilon/(1 + \varepsilon)\} \\ &< \delta(\varepsilon) + \varepsilon < 2\varepsilon \end{aligned}$$

for all $t \geqq t_0$. Note that the number $\delta(\cdot)$ is independent of the particular choice of $\bar{p}, \bar{z} \in S(\bar{E})$. Therefore, each $p \in S(E)$ is RWUAS in $\Omega(E)$.

Next, we shall prove that each $p \in S(E)$ is asymptotically almost periodic. Let $\{t_n\}$ be any sequence satisfying $t_n \to \infty$ as $n \to \infty$. We may assume that the sequence $\{p(t + t_n)\}_{n=1}^{\infty}$ is uniformly convergent on each bounded subset of R and that the sequences $\{a_{ij}(t + t_n)\}_{n=1}^{\infty}$ and $\{b_i(t + t_n)\}_{n=1}^{\infty}$ are uniformly convergent on R. Set $p^m(t) = p(t + t_m)$, $t \in R$, for each positive integer m. Clearly, p^m is a solution of the system

$$(E^m) \qquad \dot{x}_i(t) = h_i(x_i(t)) \Big\{ b_i(t + t_m) - a_{ii}(t + t_m)x_i(t)$$

$$- \sum_{\substack{j=1 \\ j \neq i}}^{k} a_{ij}(t + t_m) \int_{-\infty}^{t} K_{ij}(t-u)G_i(x_j(u))du \Big\} , \quad i = 1, \cdots, k ,$$

on R and it is RTS for System (E^m) with the common number $\delta(\cdot)$, since p is RTS for (E) with the number $\delta(\cdot)$. For any positive integers m and n, we define a continuous function $g_{mn}: R \to R^k$ by $g_{mn}(t) = (g_{mn1}(t), \cdots, g_{mnk}(t))$, where

$$g_{mni}(t) := h_i(p_i(t + t_n)) \Big[b_i(t + t_n) - b_i(t + t_m) - (a_{ii}(t+t_n) - a_{ii}(t + t_m))p_i(t + t_n)$$
$$- \sum_{\substack{j=1 \\ j \neq i}}^{k} \{a_{ij}(t + t_n) - a_{ij}(t + t_m)\} \int_{-\infty}^{t} K_{ij}(t-u)G_i(p_j(u + t_n))du \Big] ,$$

for $i = 1, \cdots, k$. Now, for any $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that $\sup_{t \in R} |g_{mn}(t)| < \delta(\varepsilon)$ and $\rho((p^m)^0, (p^n)^0) < \delta(\varepsilon)$ if $m, n \geqq n_0(\varepsilon)$. Then, the fact that p^m is RTS for (E^m) implies that $\rho((p^m)^t, (p^n)^t) < \varepsilon$ for all $t \geqq 0$ if $m, n \geqq n_0(\varepsilon)$, since p^n is a solution of System $(E^m_{g_{mn}})$ on R and $x_{i*} \leqq (p^n)_i(t) \leqq x_i^*$ for all $t \in R$ and all $i = 1, \cdots, k$. Thus the sequence $\{p(t + t_n)\}_{n=1}^{\infty}$ is uniformly convergent on $[0, \infty)$, which shows that $p(t)$ is asymptotically almost periodic, that is, $p(t)$ is the sum of an almost periodic function $q(t)$ and a continuous function $r(t)$ defined on R such that $p(t) = q(t) + r(t)$, $t \in R$, and $r(t) \to 0$ as $t \to \infty$ (see [6]).

Finally, we shall show that $q(t)$ is a unique almost periodic solution in $S(E)$. We choose a sequence $\{s_n\}$, $s_n \to \infty$ as $n \to \infty$, such that $q(t + s_n) \to q(t)$, $a_{ij}(t + s_n) \to a_{ij}(t)$ and $b_i(t + s_n) \to b_i(t)$ as $n \to \infty$ uniformly on R. Then, $q \in S(E)$ by Lemma 3. Let $\tilde{q}$ be another almost periodic solution in $S(E)$. Since $q \in S(E)$ is RWUAS in $\Omega(E)$, as was shown in the first paragraph of the proof of the theorem, we obtain $\rho(q^t, \tilde{q}^t) \to 0$ as $t \to \infty$ and hence $|q(t) - \tilde{q}(t)| \to 0$ as $t \to \infty$. Hence $q(t) \equiv \tilde{q}(t)$ on R, because q and $\tilde{q}$ are almost periodic. Thus, System (E) has $q(t)$ as a unique almost periodic solution in $S(E)$. The assertion on the module of $q(t)$ can be proved by standard argument (see, for instance, [5, Lemma 5.1]).

As an immediate consequence of our theorem, we obtain the following result, which was proved by Gopalsamy in [2, Theorem 2.1] when $h_i(s) \equiv G_i(s) \equiv s$.

COROLLARY. *Under the assumptions* (H1)–(H6), *suppose that* $a_{ij}(t)$ *and* $b_i(t)$ *are* ω*-periodic for all* $i, j = 1, \cdots, k$. *Then System* (E) *has a unique* ω*-periodic solution in* $S(E)$.

REFERENCES

[1] R. D. DRIVER, Existence and stability of solutions of a delay-differential system, Arch. Rat. Mech. Anal. 10 (1962), 401–426.

[2] K. GOPALSAMY, Global asymptotic stability in a periodic integrodifferential system, Tôhoku Math. J. 37 (1985), 323–332.

[3] Y. HINO, Stability properties for functional differential equations with infinite delay, Tôhoku Math. J. 35 (1983), 597–605.

[4] S. MURAKAMI, Perturbation theorems for functional differential equations with infinite delay via limiting equations, J. Differential Equations 59 (1985), 314–335.

[5] S. MURAKAMI, Linear periodic functional differential equations with infinite delay, to appear in Funkcial Ekvac.

[6] T. YOSHIZAWA, Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions, Applied Math. Sciences 14, Springer-Verlag, New York-Heidelberg-Berlin, 1975,

DEPARTMENT OF LIBERAL ARTS AND ENGINEERING SCIENCES
HACHINOHE NATIONAL COLLEGE OF TECHNOLOGY
TAMONOKI, HACHINOHE 031
JAPAN