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LUSIN FUNCTIONS ON PRODUCT SPACES

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1. Introduction. In [1] and [2], Calderón and Torchinsky introduced the parabolic H^p spaces associated with a group of linear transformations of $\boldsymbol{R}^d$ and obtained analogues of some results of Fefferman-Stein [8] in this context. Later Gundy-Stein [11] extended some of the results of [8] to the product spaces. (See also Gundy [10], M. P. and P. Malliavin [13].) On the other hand, it seems likely that some parts of the theory of Calderón-Torchinsky [1], [2] also extend to the product spaces. In fact, in the present note we prove the equivalence with respect to the L^p-"norms" of the Lusin functions and the nontangential maximal functions arising from certain two-parameter families of linear transformations of $\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}$ (see Theorem 1 and the corollary in § 3), which is an extension to the product spaces of a special case of a result of [1] and also is a generalization of a result of Gundy-Stein [11]. Combined with the argument of Fefferman-Stein [9], this enables us to extend Fefferman's weak type estimates (see [7]) to the case of the double singular integrals with mixed homogeneity (see Theorem 3 in § 3).

2. Preliminaries.

2.1. Let $x\in\boldsymbol{R}^n$ $(n\geqq 2)$. We write $x=(x^{(1)}, x^{(2)})$, where $x^{(1)}\in\boldsymbol{R}^{n_1}$, $x^{(2)}\in\boldsymbol{R}^{n_2}$ $(n_1, n_2\geqq 1,\ n_1+n_2=n)$ and $x^{(i)}=(x_1^{(i)},\cdots,x_{n_i}^{(i)})$ $(i=1, 2)$. If $X\in\boldsymbol{R}^{n_1+1}\times\boldsymbol{R}^{n_2+1}$, we write $X=(x^{(1)}, t_1; x^{(2)}, t_2)$; $x^{(i)}\in\boldsymbol{R}^{n_i}$, $t_i\in\boldsymbol{R}$. (We often write, for example, "$x^{(i)}\in\boldsymbol{R}^{n_i}$" instead of "$x^{(1)}\in\boldsymbol{R}^{n_1}$ and $x^{(2)}\in\boldsymbol{R}^{n_2}$" for simplicity. This abbreviation will be used throughout.) We also write $(x^{(1)}, t_1; x^{(2)}, t_2)=(x, t)$, where $x=(x^{(1)}, x^{(2)})$, $t=(t_1, t_2)$.

Set $\boldsymbol{R}_+^{n_i+1}=\{(x^{(i)}, t_i)\in\boldsymbol{R}^{n_i+1}: t_i>0\}$ $(i=1, 2)$ and $\boldsymbol{D}=\boldsymbol{R}_+^{n_1+1}\times\boldsymbol{R}_+^{n_2+1}$.

2.2. Let P_i be a linear transformation of $\boldsymbol{R}^{n_i}$ such that $(P_i x^{(i)}, x^{(i)})\geqq(x^{(i)}, x^{(i)})$ for all $x^{(i)}\in\boldsymbol{R}^{n_i}$, where $(x^{(i)}, y^{(i)})$ denotes the ordinary inner product in $\boldsymbol{R}^{n_i}$. We consider a group $A_{t_i}^{(i)}=t_i^{P_i}$ $(0<t_i<\infty)$ of linear transformations of $\boldsymbol{R}^{n_i}$.

For $x^{(i)}\in\boldsymbol{R}^{n_i}-\{0\}$, let us denote by $\rho^{(i)}(x^{(i)})$ the unique t_i such that

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$|A_{t_i}^{(i)^{-1}}x^{(i)}| = 1$, where $|x^{(i)}| = (x^{(i)}, x^{(i)})^{1/2}$, and we define $\rho^{(i)}(0) = 0$. $\rho^{(i)*}$ is defined similarly in terms of $A_{t_i}^{(i)*}$, where $A_{t_i}^{(i)*}$ is the transposed transformation of $A_{t_i}^{(i)}$.

2.3. Let $f^{(1)}$ and $f^{(2)}$ be functions defined on $\boldsymbol{R}^{n_1}$ and $\boldsymbol{R}^{n_2}$, respectively. We define a function $f^{(1)} \times f^{(2)}$ on $\boldsymbol{R}^n$ by

$$(f^{(1)} \times f^{(2)})(x^{(1)}, x^{(2)}) = f^{(1)}(x^{(1)}) f^{(2)}(x^{(2)}) .$$

An operator $T_{t_i}^{(i)}$ $(t_i > 0)$ is defined by

$$T_{t_i}^{(i)} f^{(i)}(x^{(i)}) = f^{(i)}(A_{t_i}^{(i)} x^{(i)}) .$$

We set

$$f_{t_i}^{(i)}(x^{(i)}) = t_i^{-\gamma_i}(T_{t_i}^{(i)^{-1}} f^{(i)})(x^{(i)}) = t_i^{-\gamma_i} f^{(i)}(A_{t_i}^{(i)^{-1}} x^{(i)}) ,$$

where $\gamma_i = \operatorname{trace} P_i$,

Set

$$A_{(t_1,t_2)}(x^{(1)}, x^{(2)}) = (A_{t_1}^{(1)} x^{(1)}, A_{t_2}^{(2)} x^{(2)}) ,$$
$$A_{(t_1,t_2)}^*(\xi^{(1)}, \xi^{(2)}) = (A_{t_1}^{(1)*} \xi^{(1)}, A_{t_2}^{(2)*} \xi^{(2)}) .$$

If f is a function on $\boldsymbol{R}^n$, an operator T_t is defined by

$$T_t f(x) = f(A_t x) .$$

We set

$$f_t(x) = t_1^{-\gamma_1} t_2^{-\gamma_2}(T_t^{-1} f)(x) = t_1^{-\gamma_1} t_2^{-\gamma_2} f(A_{t_1}^{(1)^{-1}} x^{(1)}, A_{t_2}^{(2)^{-1}} x^{(2)}) .$$

2.4. There is a unique strictly positive self-adjoint transformation B_i of $\boldsymbol{R}^{n_i}$ such that $P_i B_i + B_i P_i^* = I_i$, where I_i is the identity transformation of $\boldsymbol{R}^{n_i}$.

Let $G^{(i)}$ be the inverse Fourier transform of the function $\exp(-4\pi^2(B_i\xi^{(i)}, \xi^{(i)}))$. $\left(\text{If } f^{(j)} \in L^1(\boldsymbol{R}^{n_j})\ (j = 1, 2), \text{ the Fourier transform of } f^{(j)} \text{ is defined by } \hat{f}^{(j)}(\xi^{(j)}) = \int f^{(j)}(x^{(j)}) e^{-2\pi i (x^{(j)}, \xi^{(j)})} dx^{(j)}.\right)$ Set $G = G^{(1)} \times G^{(2)}$, $G^{(j,k)} = (\partial_j^{(1)} G^{(1)}) \times (\partial_k^{(2)} G^{(2)})$, where $\partial_j^{(i)} = \partial / \partial x_j^{(i)}$ $(j = 1, \cdots, n_i)$.

2.5. If $a_i > 0$, set

$$\Gamma_{a_i}^{(i)}(x^{(i)}) = \{(y^{(i)}, t_i) \in \boldsymbol{R}_+^{n_i+1} : \rho^{(i)}(x^{(i)} - y^{(i)}) < a_i t_i\} ,$$

and for $a = (a_1, a_2)$ set

$$\Gamma_a(x) = \Gamma_{a_1}^{(1)}(x^{(1)}) \times \Gamma_{a_2}^{(2)}(x^{(2)}) = \{(y^{(1)}, t_1; y^{(2)}, t_2) : (y^{(i)}, t_i) \in \Gamma_{a_i}^{(i)}(x^{(i)})\ (i = 1, 2)\}.$$

If $F^{(i)}$ is a function on $\boldsymbol{R}_+^{n_i+1}$, we define the nontangential maximal function by

$$N_{a_i}^{(i)}(F^{(i)})(x^{(i)}) = \sup\{|F^{(i)}(y^{(i)}, t_i)| : (y^{(i)}, t_i) \in \Gamma_{a_i}^{(i)}(x^{(i)})\}$$

and the Lusin function by

$$S_{a_i}^{(i)}(F^{(i)})(x^{(i)}) = \left(\int_{\Gamma_{a_i}^{(i)}(x^{(i)})} |F^{(i)}(y^{(i)}, t_i)|^2 t_i^{-\gamma_i} dy^{(i)} \frac{dt_i}{t_i}\right)^{1/2}.$$

For a function F on $\boldsymbol{D}$, we define the nontangential maximal function by

$$N_a(F)(x) = \sup\{|F(y, t)|: (y, t) \in \Gamma_a(x)\},$$

and the Lusin function by

$$S_a(F)(x) = \left(\int_{\Gamma_a(x)} |F(y, t)|^2 t_1^{-\gamma_1} t_2^{-\gamma_2} dy \frac{dt}{t_1 t_2}\right)^{1/2}.$$

Let $\Gamma_1^{(i)} = \Gamma^{(i)}$, $\Gamma_{(1,1)} = \Gamma$, $N_1^{(i)} = N^{(i)}$, $S_1^{(i)} = S^{(i)}$, $N_{(1,1)} = N$ and $S_{(1,1)} = S$.

For more details about 2.2, 2.3, 2.4 and 2.5 see [1].

2.6. $\mathscr{S}(\boldsymbol{R}^m)$ denotes the Schwartz class of infinitely differentiable and rapidly decreasing functions on $\boldsymbol{R}^m$. Let

$$\mathscr{S}_0(\boldsymbol{R}^{n_i}) = \{f \in S(\boldsymbol{R}^{n_i}): \hat{f}(0) = 0\},$$
$$\mathscr{S}_1(\boldsymbol{R}^{n_i}) = \{f \in S(\boldsymbol{R}^{n_i}): \hat{f}(0) = 1\}.$$

$\mathscr{S}'(\boldsymbol{R}^m)$ denotes the set of tempered distributions in $\boldsymbol{R}^m$. $\mathscr{S}_*'(\boldsymbol{R}^m)$ denotes the set of tempered distributions f such that $\hat{f}(\xi)(1 + |\xi|^2)^{-k} \in L^2(\boldsymbol{R}^m)$ for sufficiently large k.

2.7. If E is a set, χ_E denotes its characteristic function and $\complement E$ denotes its complement.

The letter c is used to denote a constant which need not be the same at each occurrence.

3. Statement of results. Let $f \in \mathscr{S}'(\boldsymbol{R}^n)$ and set $F(x, t) = f * G_t(x)$ (cf. 2.3, 2.4), where the symbol $*$ denotes the operation of convolution. We say that $f \in H^p_{n_1,n_2}$ $(0 < p < \infty)$ if $N(F) \in L^p(\boldsymbol{R}^n)$ and set $\|f\|_{H^p_{n_1,n_2}} = \|N(F)\|_p$, where $\|\cdot\|_p$ denotes the L^p-norm.

It is easy to see that $H^p_{n_1,n_2}$ coincides with L^p if $p > 1$ and $H^p_{1,1}$ $(0 < p < \infty)$ is independent of P_i.

Certain $H^p_{n_1,n_2}$ spaces are characterized in terms of the Lusin functions as we show in Theorem 1 and Corollary below.

THEOREM 1. *Let $\phi^{(i)} \in \mathscr{S}_1(\boldsymbol{R}^{n_i})$ and $\psi_j^{(i)} \in \mathscr{S}_0(\boldsymbol{R}^{n_i})$ $(j = 1, \cdots, l_i)$. Suppose that*

$$\sup_{t_i>0} \sum_{j=1}^{l_i} |\hat{\psi}_j^{(i)}(A_{t_i}^{(i)*}\xi^{(i)})| > 0 \quad \text{if} \quad \xi^{(i)} \neq 0.$$

Set $\phi = \phi^{(1)} \times \phi^{(2)}$, $\psi^{(j,k)} = \psi_j^{(1)} \times \psi_k^{(2)}$, *and* $F(x, t) = f * \phi_t(x)$, $K_{jk}(x, t) = f *$

$\psi_t^{(j,k)}(x)$ *for* $f \in \mathcal{S}'_*(\boldsymbol{R}^n)$. *Then if* $1 < p < 2$, *or if* $0 < p \leqq 1$ *and* P_i *is diagonal, we have*

$$\| N(F) \|_p \leqq c \sum_{j=1}^{l_1} \sum_{k=1}^{l_2} \| S(K_{jk}) \|_p \, .$$

REMARK. This result also holds when $p \geqq 2$ and $f \in L^p$ by a duality argument and the results below.

THEOREM 2. *If* $f \in \mathcal{S}'_*(\boldsymbol{R}^n)$, *set* $F(x, t) = f * G_t(x)$ *and* $K_{jk}(x, t) = f * G_t^{(j,k)}(x)$. *Then*

$$\left| \left\{ x \colon S\left(\left(\sum_{j=1}^{n_1} \sum_{k=1}^{n_2} | K_{jk} |^2 \right)^{1/2} \right)(x) > 1 \right\} \right| \leqq c \int_{R^n} \{ N(F) \wedge 1 \}^2 dx \, ,$$

where the symbol $\wedge$ *denotes the operation of taking the minimum.*

We will prove this by the idea of [11] and [13].

COROLLARY. *Let* $\phi^{(i)} \in \mathcal{S}_1(\boldsymbol{R}^{n_i})$ *and* $\psi^{(i)} \in \mathcal{S}_0(\boldsymbol{R}^{n_i})$. *Set* $\phi = \phi^{(1)} \times \phi^{(2)}$, $\psi = \psi^{(1)} \times \psi^{(2)}$, *and* $F(x, t) = f * \phi_t(x)$, $K(x, t) = f * \psi_t(x)$ *for* $f \in \mathcal{S}'(\boldsymbol{R}^n)$. *Then*

$$\| S(K) \|_p \leqq c \| N(F) \|_p \quad (0 < p < 2) \, .$$

This follows immediately from Theorem 2 and Lemmas 4 and 5 in § 4 (See [1, Lemma 3.3]).

REMARK. The corollary also holds when $p \geqq 2$ as a consequence of the theory of singular integrals.

We give an application of the above results. Let $K^{(i)} \in C^\infty(\boldsymbol{R}^{n_i} - \{0\})$ be such that

$$\int_{|x^{(i)}|=1} K^{(i)}(x^{(i)})(P_i x^{(i)}, x^{(i)}) d\sigma(x^{(i)}) = 0 \, ,$$

$$K^{(i)}(A_{t_i}^{(i)} x^{(i)}) = t_i^{-\gamma_i} K^{(i)}(x^{(i)}) \quad \text{for all} \quad t_i > 0 \, ,$$

where $d\sigma(x^{(i)})$ is the area element of $S^{n_i - 1} = \{x^{(i)} \colon |x^{(i)}| = 1\}$. (See [6], [14].) Set

$$K_{\varepsilon_1, \varepsilon_2}(x^{(1)}, x^{(2)}) = \prod_{i=1,2} K^{(i)}(x^{(i)})(1 - \chi_{[0,1]}(\varepsilon_i^{-1} \rho^{(i)}(x^{(i)}))) \quad \text{for} \quad \varepsilon_1, \varepsilon_2 > 0 \, .$$

THEOREM 3. *Suppose* P_i *is diagonal. Let* A *and* B *be compact sets in* $\boldsymbol{R}^n$. *If* f *is a function on* $\boldsymbol{R}^n$ *vanishing outside* B *and if* $\int_B |f| \log(2 + |f|) dx < \infty$, *then we have*

$$| \{ x \in A \colon \sup_{\varepsilon_1, \varepsilon_2 > 0} | f * K_{\varepsilon_1, \varepsilon_2}(x) | > 1 \} | \leqq c \int_B |f| \log(2 + |f|) dx \, .$$

This is a generalization of the weak type estimates of [7].

4. Lemmas. In this section, we give several lemmas, which will be used in the proof of Theorem 1. We prove Lemmas 1, 2 and 4 in later sections.

Let $u_1^{(i)}(s_i) = s_i$ $(s_i > 0)$ and let $u_j^{(i)}(s_i)$ $(j = 2, \cdots, n_i)$ be positive increasing functions. Set

$$\Gamma^i = \{\xi^{(i)}: |\xi_2^{(i)}| \leqq u_2^{(i)}(\xi_1^{(i)}), \cdots, |\xi_{n_i}^{(i)}| \leqq u_{n_i}^{(i)}(\xi_1^{(i)}), \xi_1^{(i)} > 0\}$$

and

$$\Gamma_0 = \Gamma^1 \times \Gamma^2 = \{(\xi^{(1)}, \xi^{(2)}): \xi^{(1)} \in \Gamma^1, \xi^{(2)} \in \Gamma^2\} .$$

LEMMA 1. *There is $\Phi^{(i)} \in \mathscr{S}_1(\boldsymbol{R}^{n_i})$ such that if we denote by $\Phi_{s_1,s_2}(x)$ the function:*

$$\prod_{i=1,2} u_1^{(i)}(s_i) \cdots u_{n_i}^{(i)}(s_i)\Phi^{(i)}(u_1^{(i)}(s_i)x_1^{(i)}, \cdots, u_{n_i}^{(i)}(s_i)x_{n_i}^{(i)}) ,$$

then

$$\|\sup_{s_1,s_2>0} |f * \Phi_{s_1,s_2}|\|_p \leqq c\|f\|_p$$

$(0 < p < \infty)$ *for all $f \in L^2(\boldsymbol{R}^n)$ with $\hat{f}$ vanishing outside Γ_0.*

This is an analogue of Coifman-Dahlberg [4, Theorem I]. (See also Carleson [3] and Coifman-Weiss [5, p. 585].)

Let $\{\omega_j^{(i)}: j = 1, \cdots, 2n_i\}$ be a C^∞-partition of unity on $\boldsymbol{R}^{n_i} - \{0\}$ such that

$$\mathrm{Cl}\{\xi^{(i)}: \omega_j^{(i)}(\xi^{(i)}) \neq 0, |\xi^{(i)}| = 1\} \subset \{\xi^{(i)}: \xi_j^{(i)} > 0, |\xi^{(i)}| = 1\}$$
$$\mathrm{Cl}\{\xi^{(i)}: \omega_{n_i+j}^{(i)}(\xi^{(i)}) \neq 0, |\xi^{(i)}| = 1\} \subset \{\xi^{(i)}: \xi_j^{(i)} < 0, |\xi^{(i)}| = 1\}$$

for $j = 1, \cdots, n_i$ (where for a set E, $\mathrm{Cl}\, E$ denotes its closure) and such that $\omega_j^{(i)}(\xi^{(i)}) = \omega_j^{(i)}(A_{t_i}^{(i)*}\xi^{(i)})$ for all $t_i > 0$. Define an operator T_{jk} by

$$(T_{jk}f)^\wedge(\xi) = \omega_j^{(1)}(\xi^{(1)})\omega_k^{(2)}(\xi^{(2)})\hat{f}(\xi)$$

for $f \in L^2(\boldsymbol{R}^n)$.

LEMMA 2. *Let $\phi^{(i)} \in \mathscr{S}_1(\boldsymbol{R}^{n_i})$ and set $\phi = \phi^{(1)} \times \phi^{(2)}$. Suppose that P_i is diagonal. Then*

$$\|\sup_t |f * \phi_t|\|_p \leqq c \sum_{j=1}^{2n_1} \sum_{k=1}^{2n_2} \|T_{jk}f\|_p ,$$

for $0 < p < \infty$.

This is a consequence of Lemma 1.

Let W be a measurable subset of $\boldsymbol{R}^m$ and ω be a positive function on W. Let $\phi^{(1)} \in \mathscr{S}_1(\boldsymbol{R}^{n_1})$ and $\psi^{(1)} \in \mathscr{S}_0(\boldsymbol{R}^{n_1})$ and suppose

$$\sup_{t_1>0} |\hat{\psi}^{(1)}(A_{t_1}^{(1)*}\xi^{(1)})| > 0 \quad \text{if} \quad \xi^{(1)} \neq 0 .$$

If f is a function on $\boldsymbol{R}^{n_1}\times W=\{(x^{(1)}, w): x^{(1)}\in\boldsymbol{R}^{n_1}, w\in W\}$ such that

$$\int_{\boldsymbol{R}^{n_1}\times W}|f(x^{(1)}, w)|^2\omega(w)dx^{(1)}dw<\infty ,$$

then we set

$$F(y^{(1)}, t_1; w)=\int\phi_{t_1}^{(1)}(y^{(1)}-z^{(1)})f(z^{(1)}, w)dz^{(1)} ,$$

$$K(y^{(1)}, t_1; w)=\int\psi_{t_1}^{(1)}(y^{(1)}-z^{(1)})f(z^{(1)}, w)dz^{(1)} .$$

LEMMA 3. *Set*

$$|F|_W(y^{(1)}, t_1)=\left(\int_W|F(y^{(1)}, t_1; w)|^2\omega(w)dw\right)^{1/2} ,$$

$$|K|_W(y^{(1)}, t_1)=\left(\int_W|K(y^{(1)}, t_1; w)|^2\omega(w)dw\right)^{1/2} .$$

Then we have

$$\|N^{(1)}(|F|_W)\|_p\leqq c\|S^{(1)}(|K|_W)\|_p\quad(0<p<2) .$$

This is a vector-valued analogue of Calderón-Torchinsky [1, Theorem 6.9] and can be proved along the same line.

LEMMA 4. *Let $f\in\mathscr{S}_*'(\boldsymbol{R}^n)$, $\eta^{(i)}\in\mathscr{S}_0(\boldsymbol{R}^{n_i})$ and let $\psi^{(j,k)}$ be the same as in Theorem* 1. *Set $L(x, t)=f*\eta_t(x)$ $(\eta=\eta^{(1)}\times\eta^{(2)})$, $K_{jk}(x, t)=f*\psi_t^{(j,k)}(x)$. Then if $0<p\leqq 2$, we have*

$$\|S_a(L)\|_p\leqq c\sum_{j,k}\|S_b(K_{jk})\|_p .$$

If F is a function on $\boldsymbol{D}$, we set

$$F^+(x)=\sup_t|F(x, t)| .$$

LEMMA 5. *Let $f\in\mathscr{S}'(\boldsymbol{R}^n)$ and $\phi^{(i)}, \psi^{(i)}\in\mathscr{S}_1(\boldsymbol{R}^{n_i})$. Set $\phi=\phi^{(1)}\times\phi^{(2)}$, $\psi=\psi^{(1)}\times\psi^{(2)}$ and $F(x, t)=f*\phi_t(x)$, $H(x, t)=f*\psi_t(x)$. Then*

$$\|N_a(H)\|_p\leqq c\|F^+\|_p\quad for\quad 0<p<\infty .$$

The arguments of [1] and [8] also apply to the proof of Lemma 5.
Let $\lambda=(\lambda_1, \lambda_2)$; $\lambda_1, \lambda_2>0$. If F is a function on $\boldsymbol{D}$, we set

$$G_\lambda(F)(x)=\left[\int_D|F(y, t)|^2\prod_{i=1,2}\left\{\left(1+\frac{\rho^{(i)}(x^{(i)}-y^{(i)})}{t_i}\right)^{-2\lambda_i}t_i^{-\gamma_i}\right\}dy\frac{dt}{t_1t_2}\right]^{1/2} .$$

LEMMA 6. *Let $f\in\mathscr{S}_*'(\boldsymbol{R}^n)$, $\phi^{(i)}\in\mathscr{S}_0(\boldsymbol{R}^{n_i})$, $\eta^{(i)}\in\mathscr{S}(\boldsymbol{R}^{n_i})$ and let $\phi=\phi^{(1)}\times\phi^{(2)}$, $\eta=\eta^{(1)}\times\eta^{(2)}$, $\hat\psi=\hat\phi\hat\eta$. If $k\in L^\infty(\boldsymbol{R}^n)$, define Tf by $(Tf)\hat{}=k\hat f$ and $h^{(t)}$ by $\hat h^{(t)}(\xi)=\hat\eta(\xi)k(A_t^*\xi)$. Set $F(x, t)=f*\phi_t(x)$ and $H(x, t)=Tf*\psi_t(x)$. Suppose that $l^{(t)}(x)=h^{(t)}(x)\prod_{i=1,2}(1+\rho^{(i)}(x^{(i)}))^{\lambda_i}$ $(\lambda_i>0)\in L^2(\boldsymbol{R}^n)$*

and $\sup_t \|l^{(t)}\|_2 < \infty$. *Then if* $\mu = (\mu_1, \mu_2)$, $\lambda = (\lambda_1, \lambda_2)$ *and* $\mu_1 - \lambda_1 > \gamma_1/2$, $\mu_2 - \lambda_2 > \gamma_2/2$, *we have*

$$G_\mu(H) \leqq cG_\lambda(F) .$$

The proof of Lemma 6 is similar to that of Theorem 5.3 of [1], and is omitted.

5. Proof of Lemma 1. Let $\phi \in \mathscr{S}(\boldsymbol{R}^1)$ be such that

$$\hat{\phi}(\zeta) = \begin{cases} 1 & \text{if } |\zeta| \leqq 1/2 \\ 0 & \text{if } |\zeta| \geqq 1 . \end{cases}$$

Define $\Phi^{(i)} \in \mathscr{S}(\boldsymbol{R}^{n_i})$ by

$$\hat{\Phi}^{(i)}(\xi^{(i)}) = \hat{\phi}(\xi_1^{(i)})\hat{\phi}\Big(\frac{\xi_2^{(i)}}{2}\Big)\cdots\hat{\phi}\Big(\frac{\xi_{n_i}^{(i)}}{2}\Big)$$

and let Φ_{s_1,s_2} be the same as in the statement of Lemma 1.

Suppose that $f \in L^2(\boldsymbol{R}^n)$ and $\hat{f}$ vanishes outside Γ_0. Then note that

$$\hat{\phi}\Big(\frac{\xi_1^{(1)}}{s_1}\Big)\hat{\phi}\Big(\frac{\xi_1^{(2)}}{s_2}\Big)\hat{f}(\xi) = \hat{\Phi}_{s_1,s_2}(\xi)\hat{f}(\xi) . \tag{5.1}$$

The proof of Lemma 1 is based on the observation (5.1) and the following lemma.

LEMMA 7. *Let* $f \in L^2(\boldsymbol{R}^2)$ *and suppose that* $\operatorname{supp}\hat{f} \subset \Gamma_* = \{(y_1, y_2) \in \boldsymbol{R}^2\colon y_1 \geqq 0,\ y_2 \geqq 0\}$. *Then if* $f \in L^p(\boldsymbol{R}^2)$ $(0 < p < \infty)$, *it follows that* $f \in H^p_{1,1}$ *and*

$$\|f\|_{H^p_{1,1}} \leqq c\|f\|_{L^p(\boldsymbol{R}^2)} .$$

(This can be proved by the argument of Stein-Weiss [17, pp. 116–117] and the theory of Fefferman-Stein [8].)

Set $F_{s_1,s_2}(x) = f * \Phi_{s_1,s_2}(x)$. If $x^{(1)} = (x_1^{(1)}, x^{(1)\prime})$, $x^{(2)} = (x_1^{(2)}, x^{(2)\prime})$ and if we consider F_{s_1,s_2} and f as functions of $(x_1^{(1)}, x_1^{(2)})$, fixing $x^{(1)\prime}$ and $x^{(2)\prime}$, then we write

$$F_{s_1,s_2}(x_1^{(1)}, x^{(1)\prime}; x_1^{(2)}, x^{(2)\prime}) = \tilde{F}_{s_1,s_2}(x_1^{(1)}, x_1^{(2)}) , \qquad f(x_1^{(1)}, x^{(1)\prime}; x_1^{(2)}, x^{(2)\prime}) = \tilde{f}(x_1^{(1)}, x_1^{(2)}) .$$

When $\int |F_{s_1,s_2}(x)|^p dx < \infty$, by Lemma 7 we have for almost every $x^{(1)\prime}$ and $x^{(2)\prime}$

$$\|\sup_{s_1,s_2>0} |(\phi_{s_1}\times\phi_{s_2}) * \tilde{f}|\|_{L^p(\boldsymbol{R}^2)} \leqq c\|\tilde{f}\|_{L^p(\boldsymbol{R}^2)} , \tag{5.2}$$

where $\phi_{s_i}(x_1^{(i)}) = s_i\phi(s_i x_1^{(i)})$.

From (5.1) it follows that

$$\tilde{F}_{s_1,s_2} = (\phi_{s_1}\times\phi_{s_2}) * \tilde{f} .$$

Thus by (5.2) we have

$$\int_{R^2} \sup_{s_1,s_2>0} |\tilde{F}_{s_1,s_2}|^p dx_1^{(1)} dx_1^{(2)} = \int_{R^2} \sup_{s_1,s_2>0} |(\phi_{s_1} \times \phi_{s_2}) * \tilde{f}|^p dx_1^{(1)} dx_1^{(2)}$$
$$\leqq c \int_{R^2} |\tilde{f}|^p dx_1^{(1)} dx_1^{(2)} .$$

Integrating this with respect to $x^{(1)\prime}$ and $x^{(2)\prime}$, we obtain

$$\int \sup_{s_1,s_2>0} |F_{s_1,s_2}(x)|^p dx \leqq c \int |f|^p dx ,$$

which proves Lemma 1.

6. Proof of Lemma 2. If P_i is diagonal, then

$$A_{t_i}^{(i)*} \xi^{(i)} = (t_i^{\alpha_1^{(i)}} \xi_1^{(i)}, \cdots, t_i^{\alpha_{n_i}^{(i)}} \xi_{n_i}^{(i)}) \quad \text{for some} \quad \alpha_j^{(i)} \geqq 1 .$$

Set

$$\Gamma_j^i = \{\xi^{(i)} : |\xi_k^{(i)}|^{\alpha_j^{(i)}} \leqq c_0 |\xi_j^{(i)}|^{\alpha_k^{(i)}} \ (1 \leqq k \leqq n_i),\ \xi_j^{(i)} \geqq 0\} \quad \text{for} \quad j = 1, \cdots, n_i$$

and

$$\Gamma_{n_i+j}^i = \{-\xi^{(i)} : \xi^{(i)} \in \Gamma_j^i\} \quad (j = 1, \cdots, n_i) .$$

Since $\mathrm{supp}(T_{jk}f)^\wedge \subset \Gamma_j^1 \times \Gamma_k^2$ for some $c_0 > 0$, by Lemma 1 there are $\phi_j^{(1)} \in \mathscr{S}_1(\boldsymbol{R}^{n_1})$ and $\phi_k^{(2)} \in \mathscr{S}_1(\boldsymbol{R}^{n_2})$ such that

$$\| \sup_t | T_{jk} f * \phi_t^{(j,k)} | \|_p \leqq c \| T_{jk} f \|_p ,$$

where $\phi^{(j,k)} = \phi_j^{(1)} \times \phi_k^{(2)}$. Since $\sum_{j,k} T_{jk} f = f$, this, combined with Lemma 5, proves Lemma 2.

7. Proof of Lemma 4. If F is a function on $\boldsymbol{D}$, then clearly we have $S_\alpha(F) \leqq c G_\lambda(F)$. On the other hand, the following result holds.

LEMMA 8. *If* $0 < p \leqq 2$ *and* $\lambda = (\lambda_1, \lambda_2)$ *with* $\lambda_1 > \gamma_1/p$, $\lambda_2 > \gamma_2/p$, *then*

$$\| G_\lambda(F) \|_p \leqq c \| S_\alpha(F) \|_p .$$

Thus Lemma 4 follows from the following lemma.

LEMMA 9. *Let* L *and* K_{jk} *be the same as in Lemma* 4. *Then if* $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$ *with* $\mu_1 - \lambda_1 > \gamma_1$ *and* $\mu_2 - \lambda_2 > \gamma_2$, *we have*

$$G_\mu(L) \leqq c \sum_{j=1}^{l_1} \sum_{k=1}^{l_2} G_\lambda(K_{jk}) .$$

(We can prove Lemma 8 and Lemma 9 by using [1, Theorem 3.5] and [1, Theorem 5.5], respectively.)

8. Proof of Theorem 1. Let $\eta^{(i)} \in \mathscr{S}(\boldsymbol{R}^{n_i})$, $\phi^{(i)} \in \mathscr{S}_1(\boldsymbol{R}^{n_i})$ and set $\eta = \eta^{(1)} \times \eta^{(2)}$, $\phi = \phi^{(1)} \times \phi^{(2)}$. Suppose that

$$\sup_{t_i > 0} |\hat{\eta}^{(i)}(A_{t_i}^{(i)*} \xi^{(i)})| > 0 \quad \text{if} \quad \xi^{(i)} \neq 0$$

and

$$\operatorname{supp} \hat{\eta}^{(i)} \subset \{\xi^{(i)} \colon 1 \leqq \rho^{(i)*}(\xi^{(i)}) \leqq 2\} .$$

To prove the theorem, we first assume that $f \in L^2(\boldsymbol{R}^n)$. Set $H(y, t) = f * \eta_t(y)$. Then arguing as in [11], by using Lemma 3, we have

$$\sup_t \| f * \phi_t \|_p^p \leqq c \int S^p(H) dx . \tag{8.1}$$

This proves the theorem when $p > 1$. When $0 < p \leqq 1$, suppose that P_i is diagonal, and let T_{jk} be the same as in Lemma 2. Then by Lemma 2 and (8.1)

$$\| \sup_t | f * \phi_t | \|_p^p \leqq c \sum_{j,k} \| T_{jk} f \|_p^p \leqq c \sum_{j,k} \int S^p(L_{jk}) dx ,$$

where $L_{jk}(y, t) = T_{jk} f * \eta_t(y)$. Note that $L_{jk}(y, t) = f * \theta_t^{(j,k)}(y)$ for some $\theta^{(j,k)} = \theta_j^{(1)} \times \theta_k^{(2)}$ with $\theta_j^{(1)} \in \mathscr{S}_0(\boldsymbol{R}^{n_1})$ and $\theta_k^{(2)} \in \mathscr{S}_0(\boldsymbol{R}^{n_2})$. Thus the theorem follows from Lemma 4.

Next we remove the assumption that $f \in L^2$. Let $f \in \mathscr{S}'_*(\boldsymbol{R}^n)$ and for $\delta = (\delta_1, \delta_2)$ $(\delta_i > 0)$, set $f^{(\delta)} = f * G_\delta$. Then $f^{(\delta)} \in L^2$. Let $\psi^{(j,k)}$ be the same as in the statement of Theorem 1 and set

$$F^{(\delta)}(y, t) = f^{(\delta)} * \phi_t(y) , \qquad K_{jk}^{(\delta)}(y, t) = f^{(\delta)} * \psi_t^{(j,k)}(y) .$$

Then from what we have already proved, it follows that

$$\| N_a(F^{(\delta)}) \|_p \leqq c \sum_{j,k} \| S_b(K_{jk}^{(\delta)}) \|_p . \tag{8.2}$$

Let η be as above and set $I^{(\delta)}(y, t) = f^{(\delta)} * \eta_t * \eta_t(y)$. Then using Lemma 6 and Lemma 8, we have

$$\| S_b(I^{(\delta)}) \|_p \leqq c \| S_b(J) \|_p , \tag{8.3}$$

where $J(y, t) = f * \eta_t(y)$. By (8.2), (8.3) and Lemma 4, we have

$$\| N_a(F^{(\delta)}) \|_p \leqq c \sum_{j,k} \| S_b(K_{jk}) \|_p .$$

Letting $\delta_1 \to 0$, $\delta_2 \to 0$, we conclude the proof.

9. Preliminaries for the proof of Theorem 2. Recall that

$$G^{(j,k)} = (\partial_j^{(1)} G^{(1)}) \times (\partial_k^{(2)} G^{(2)}) \quad \text{for} \quad 1 \leqq j \leqq n_1 , \quad 1 \leqq k \leqq n_2 ;$$

and let

$$G^{(n_1+1,k)} = (\Delta_1 G^{(1)})\times(\partial_k^{(2)}G^{(2)}) , \qquad G^{(0,k)} = G^{(1)}\times(\partial_k^{(2)}G^{(2)}) \quad \text{for} \quad k = 1, \cdots, n_2$$

(where $\Delta_i = \sum_{j=1}^{n_i}(\partial_j^{(i)})^2$ is the Laplacian);

$$G^{(j,n_2+1)} = (\partial_j^{(1)}G^{(1)})\times(\Delta_2 G^{(2)}) , \qquad G^{(j,0)} = (\partial_j^{(1)}G^{(1)})\times G^{(2)} \quad \text{for} \quad j = 1, \cdots, n_1 ;$$

$$G^{(0,0)} = G^{(1)}\times G^{(2)} , \qquad G^{(0,n_2+1)} = G^{(1)}\times(\Delta_2 G^{(2)}) ,$$

$$G^{(n_1+1,0)} = (\Delta_1 G^{(1)})\times G^{(2)} , \qquad G^{(n_1+1,n_2+1)} = (\Delta_1 G^{(1)})\times(\Delta_2 G^{(2)}) .$$

Let f $(\in \mathcal{S}_*'(\boldsymbol{R}^n))$ be real-valued. Set $K_{jk}(y, t) = f * G_t^{(j,k)}$, $F = K_{00}$. Suppose $\int\{N(F)\wedge 1\}^2 dx < \infty$. Let

$$E = \{x: N(F)(x) \leqq 1\} ,$$

and set $v(y, t) = \chi_E * G_t(y)$, $w_{jk}(y, t) = \chi_{\complement E} * G_t^{(j,k)}(y)$, $w = w_{00}$. Note $v + w = 1$ and therefore $\partial_j^{(i)}v = -\partial_j^{(i)}w$. In the proof of Theorem 2, we will use the following equations:

$$\sum_{j=1}^{n_1} T_{t_1}^{(1)^{-1}}\partial_j^{(1)} T_{t_1}^{(1)} K_{jk} = K_{n_1+1,k} , \qquad \sum_{k=1}^{n_2} T_{t_2}^{(2)^{-1}}\partial_k^{(2)} T_{t_2}^{(2)} K_{jk} = K_{j,n_2+1} ,$$

$$T_{t_1}^{(1)^{-1}}\partial_j^{(1)} T_{t_1}^{(1)} K_{0k} = K_{jk} , \qquad T_{t_2}^{(2)^{-1}}\partial_k^{(2)} T_{t_2}^{(2)} K_{j0} = K_{jk} ,$$

$$t_1\frac{\partial}{\partial t_1}K_{0k} = K_{n_1+1,k} , \qquad t_2\frac{\partial}{\partial t_2}K_{j0} = K_{j,n_2+1} .$$

(See [1].) The same equations hold for w_{jk}.

It is easy to see the following two lemmas.

LEMMA 10. *There is a number α_1 such that $1/2 < \alpha_1 < 1$ and*

$$\sup\{v(y, t): (y, t) \notin \bigcup_{x\in E}\Gamma(x)\} \leqq \alpha_1 \quad (\textit{if } \boldsymbol{D} \neq \bigcup_{x\in E}\Gamma(x)) .$$

LEMMA 11. *Let α_1 be the same as in Lemma 10, and let $\alpha_1 < \alpha_2 < 1$. Set $E' = \{x \in \boldsymbol{R}^n: N(w)(x) \leqq 1 - \alpha_2\}$. Then*

$$|\complement E'| \leqq c|\complement E| ,$$

$$\inf\{v(y, t): (y, t) \in \bigcup_{x\in E'}\Gamma(x)\} \geqq \alpha_2 .$$

Let α_1, α_2 be as above and put $\alpha_3 = (\alpha_1 + \alpha_2)/2$. Let $r \in C^\infty(\boldsymbol{R}^1)$ be such that

$$r(u) = \begin{cases} 1 & \text{if} \quad u \geqq \alpha_2 \\ 0 & \text{if} \quad u \leqq \alpha_3 , \end{cases}$$

$$|r'(u)|^2 \leqq cr(u) \quad \text{for all} \quad u \in \boldsymbol{R}^1 .$$

Then by Lemma 11 we have

$$\int_{E'} S^2\Big(\Big(\sum_{j=1}^{n_1}\sum_{k=1}^{n_2} K_{jk}^2\Big)^{1/2}\Big)dx \leqq c\int_D \Big(\sum_{j,k} K_{jk}^2\Big) r(v)\, dy\frac{dt}{t_1 t_2} . \tag{9.1}$$

For $0 < \varepsilon < 1/2$, set

$$I = I^{(\varepsilon)} = \int_{\varepsilon}^{\varepsilon^{-1}} \int_{\varepsilon}^{\varepsilon^{-1}} \int_{R^n} \Big(\sum_{j=1}^{n_1} \sum_{k=1}^{n_2} K_{jk}^2 \Big) r(v) dy \frac{dt}{t_1 t_2} .$$

In order to estimate $I^{(\varepsilon)}$, we need the following result.

LEMMA 12. *Let* $\phi^{(1)}, \psi^{(1)} \in S(\boldsymbol{R}^{n_1})$; $\phi^{(2)}, \psi^{(2)} \in S(\boldsymbol{R}^{n_2})$ *and suppose* $\hat{\phi}^{(1)}(0) = \hat{\psi}^{(2)}(0) = 0$. *Set* $\phi = \phi^{(1)} \times \phi^{(2)}$, $\psi = \psi^{(1)} \times \psi^{(2)}$. *Then if* $f \in L^2(\boldsymbol{R}^n)$ *and* $g \in L^\infty(\boldsymbol{R}^n)$, *we have*

$$\int_D |f * \phi_t(y)|^2 |g * \psi_t(y)|^2 dy \frac{dt}{t_1 t_2} \leqq c \|f\|_2^2 \|g\|_\infty^2 .$$

This follows from the argument about a Carleson measure. For this argument see Stein [16, § 6].

10. Estimate for I. We begin the proof of Theorem 2. Set $\bar{d}t = dt/(t_1 t_2)$. Then by integration by parts we obtain

$$\begin{aligned} I &= -\sum_k \int K_{0k} K_{n_1+1,k} r(v) dy \bar{d}t + \sum_{j,k} \int K_{0k} K_{jk} w_{j0} r'(v) dy \bar{d}t \\ &= -I_1 + I_2 , \quad \text{say} , \end{aligned}$$

where (and hereafter) j and k run through $\{1, \cdots, n_1\}$ and $\{1, \cdots, n_2\}$, respectively. For $0 < \delta < 1/4$, we have

$$\begin{aligned} |I_2| &\leqq \delta \int \big(\sum_{j,k} K_{jk}^2 \big) r(v) dy \bar{d}t + c \int \big(\sum_{j,k} K_{0k}^2 w_{j0}^2 \big) s(v) dy \bar{d}t \\ &= \delta I + c I_3 , \quad \text{say} , \end{aligned}$$

where s is a C^∞-function on $\boldsymbol{R}^1$ such that

$$s(u) = \begin{cases} 1 & \text{if} \quad u \geqq \alpha_3 \\ 0 & \text{if} \quad u \leqq \alpha_1 , \end{cases}$$

$$|s'(u)|^2 \leqq c s(u) \quad \text{for all} \quad u \in \boldsymbol{R}^1 .$$

(For α_1 and α_3 see § 9.)

Integration by parts gives

$$\begin{aligned} I_1 &= \sum_k \int K_{0k} t_1 \Big(\frac{\partial}{\partial t_1} K_{0k} \Big) r(v) dy \bar{d}t \\ &= \sum_k \int [K_{0k}^2(y, \varepsilon^{-1}, t_2) r(v(y, \varepsilon^{-1}, t_2)) - K_{0k}^2(y, \varepsilon, t_2) r(v(y, \varepsilon, t_2))] dy \frac{dt_2}{t_2} \\ &\quad - \sum_k \int t_1 \Big(\frac{\partial}{\partial t_1} K_{0k} \Big) K_{0k} r(v) dy \bar{d}t + \sum_k \int K_{0k}^2 r'(v) w_{n_1+1,0} dy \bar{d}t \\ &= I_4 - I_1 + I_5 , \quad \text{say} . \end{aligned}$$

Note that

$$I_5 = -\sum_{j,k}\int 2K_{0k}K_{jk}r'(v)w_{j0}dy\bar{d}t + \sum_{j,k}\int K_{0k}^2 r''(v)w_{j0}^2 dy\bar{d}t .$$

Consequently

$$|I_5| \leqq \delta I + cI_3 .$$

Thus

$$I \leqq |I_1| + |I_2| \leqq \frac{1}{2}|I_4| + \frac{1}{2}|I_5| + |I_2| \leqq \frac{1}{2}|I_4| + \frac{3}{2}\delta I + cI_3 .$$

This implies that

$$I \leqq cI_3 + c|I_4| .$$

In the following, we will prove that

$$I_3 \leqq c|\complement E| , \tag{10.1}$$

$$|I_4| \leqq c\int \{N(F)\wedge 1\}^2 dy , \tag{10.2}$$

uniformly in ε. By Lemma 11 and (9.1), this proves Theorem 2.

11. Estimate for I_3. By integration by parts we obtain

$$\begin{aligned} I_3 &= -\sum_j \int FK_{0,n_2+1}w_{j0}^2 s(v)dy\bar{d}t - \sum_{j,k}\int 2FK_{0k}w_{j0}w_{jk}s(v)dy\bar{d}t \\ &\quad + \sum_{j,k}\int FK_{0k}w_{j0}^2 s'(v)w_{0k}dy\bar{d}t \\ &= -J_1 - J_2 + J_3 , \quad \text{say} . \end{aligned}$$

We first estimate J_1. Integration by parts gives

$$\begin{aligned} J_1 &= \sum_j \int Ft_2\Big(\frac{\partial}{\partial t_2}F\Big)w_{j0}^2 s(v)dy\bar{d}t \\ &= \sum_j \int [F^2(y, t_1, \varepsilon^{-1})w_{j0}^2(y, t_1, \varepsilon^{-1})s(v(y, t_1, \varepsilon^{-1})) \\ &\quad - F^2(y, t_1, \varepsilon)w_{j0}^2(y, t_1, \varepsilon)s(v(y, t_1, \varepsilon))]dy\frac{dt_1}{t_1} \\ &\quad - J_1 - \sum_j \int 2F^2 w_{j0}w_{j,n_2+1}s(v)dy\bar{d}t + \sum_j \int F^2 w_{j0}^2 s'(v)w_{0,n_2+1}dy\bar{d}t \\ &= L_1 - J_1 - L_2 + L_3 , \quad \text{say} . \end{aligned}$$

By Lemma 10 clearly we have

$$|L_1| \leqq c\sum_j \int \{w_{j0}^2(y, t_1, \varepsilon^{-1}) + w_{j0}^2(y, t_1, \varepsilon)\}dy\frac{dt_1}{t_1} .$$

Using the Plancherel theorem on the right hand side of the above inequality, we find $|L_1| \leqq c|\complement E|$.

Next we estimate L_2.

$$L_2 = 2\sum_j \int F^2 w_{j0}\Big(\sum_{k=1}^{n_2} T_{t_2}^{(2)^{-1}} \partial_k^{(2)} T_{t_2}^{(2)} w_{jk}\Big) s(v)\, dy\bar{d}t$$

$$= -\sum_{j,k} 4 \int F K_{0k} w_{j0} w_{jk} s(v)\, dy\bar{d}t - \sum_{j,k} 2 \int F^2 w_{jk}^2 s(v)\, dy\bar{d}t$$

$$+ \sum_{j,k} 2 \int F^2 w_{j0} w_{jk} w_{0k} s'(v)\, dy\bar{d}t$$

$$= -M_1 - M_2 + M_3\,, \quad \text{say}\,.$$

It is easy to see that

$$|M_1| \leqq \delta I_3 + c\sum_{j,k}\int w_{jk}^2\, dy\bar{d}t \leqq \delta I_3 + c|\complement E|$$

and

$$|M_2| \leqq c|\complement E|\,.$$

By Lemma 12, we have

$$|M_3| \leqq c\sum_{j,k}\Big(\int w_{j0}^2 w_{0k}^2\, dy\bar{d}t\Big)^{1/2}\Big(\int w_{jk}^2\, dy\bar{d}t\Big)^{1/2} \leqq c|\complement E|\,.$$

Thus

$$|L_2| \leqq |M_1| + |M_2| + |M_3| \leqq \delta I_3 + c|\complement E|\,.$$

In order to estimate L_3, note that

$$L_3 = \sum_j \int F^2 w_{j0}^2 s'(v)\Big(\sum_{k=1}^{n_2} T_{t_2}^{(2)^{-1}} \partial_k^{(2)} T_{t_2}^{(2)} w_{0k}\Big) dy\bar{d}t\,.$$

Thus by integration by parts we have

$$L_3 = -\sum_{j,k} 2\int F K_{0k} w_{j0}^2 s'(v) w_{0k}\, dy\bar{d}t - \sum_{j,k} 2\int F^2 w_{j0} w_{jk} s'(v) w_{0k}\, dy\bar{d}t$$

$$+ \sum_{j,k}\int F^2 w_{j0}^2 s''(v) w_{0k}^2\, dy\bar{d}t$$

$$= -M_4 - M_5 + M_6\,, \quad \text{say}\,.$$

We estimate M_4, M_5, M_6 as follows.

$$|M_4| \leqq \delta I_3 + \sum_{j,k} c\int w_{j0}^2 w_{0k}^2\, dy\bar{d}t \leqq \delta I_3 + c|\complement E|\,,$$

$$|M_5| \leqq \sum_{j,k} c\Big(\int w_{j0}^2 w_{0k}^2\, dy\bar{d}t\Big)^{1/2}\Big(\int w_{jk}^2\, dy\bar{d}t\Big)^{1/2} \leqq c|\complement E|\,,$$

$$|M_6| \leqq \sum_{j,k} c\int w_{j0}^2 w_{0k}^2\, dy\bar{d}t \leqq c|\complement E|\,.$$

This implies that

$$|L_3| \leqq |M_4| + |M_5| + |M_6| \leqq \delta I_3 + c|\complement E| .$$

Therefore

$$|J_1| \leqq \frac{1}{2}|L_1| + \frac{1}{2}|L_2| + \frac{1}{2}|L_3| \leqq \delta I_3 + c|\complement E| .$$

It is easy to obtain the following estimates for J_2 and J_3:

$$|J_2| \leqq \delta I_3 + \sum_{j,k} c \int w_{jk}^2 dy d\bar{t} \leqq \delta I_3 + c|\complement E| ,$$

$$|J_3| \leqq \delta I_3 + \sum_{j,k} c \int w_{j0}^2 w_{0k}^2 dy d\bar{t} \leqq \delta I_3 + c|\complement E| .$$

Consequently

$$|I_3| \leqq |J_1| + |J_2| + |J_3| \leqq 3\delta I_3 + c|\complement E| .$$

Thus

$$I_3 \leqq c|\complement E| ,$$

which proves (10.1).

12. Estimate for I_4. Set

$$I_4^{(1)} = \sum_k \int K_{0k}^2(y, \varepsilon, t_2) r(v(y, \varepsilon, t_2)) dy \frac{dt_2}{t_2} .$$

Then by integration by parts we have

$$\begin{aligned} I_4^{(1)} &= \sum_k \int (T_{t_2}^{(2)-1} \partial_k^{(2)} T_{t_2}^{(2)} F) K_{0k} r(v) dy \frac{dt_2}{t_2} \\ &= -\int F K_{0,n_2+1} r(v) dy \frac{dt_2}{t_2} + \sum_k \int F K_{0k} r'(v) w_{0k} dy \frac{dt_2}{t_2} \\ &= -J_4 + J_5 , \quad \text{say} . \end{aligned}$$

We estimate J_4. Integration by parts gives

$$\begin{aligned} J_4 &= \int F \frac{\partial}{\partial t_2} F r(v) dy dt_2 \\ &= \int [F^2(y, \varepsilon, \varepsilon^{-1}) r(v(y, \varepsilon, \varepsilon^{-1})) - F^2(y, \varepsilon, \varepsilon) r(v(y, \varepsilon, \varepsilon))] dy - J_4 \\ &\quad + \int F^2 r'(v) w_{0,n_2+1} dy \frac{dt_2}{t_2} \\ &= L_4 - J_4 + L_5 , \quad \text{say} . \end{aligned}$$

Clearly

$$|L_4| \leqq c \int \{N(F) \wedge 1\}^2 dy \ .$$

Next we estimate L_5.

$$L_5 = \int F^2 r'(v) \sum_{k=1}^{n_2} (T_{t_2}^{(2)-1} \partial_k^{(2)} T_{t_2}^{(2)} w_{0k}) dy \frac{dt_2}{t_2}$$

$$= -\sum_k 2 \int F K_{0k} w_{0k} r'(v) dy \frac{dt_2}{t_2} + \sum_k \int F^2 w_{0k}^2 r''(v) dy \frac{dt_2}{t_2}$$

$$= -M_7 + M_8 \ , \quad \text{say} \ .$$

It is easy to see that

$$|M_7| \leqq \delta I_4^{(1)} + c \sum_k \int w_{0k}^2 dy \frac{dt_2}{t_2} \leqq \delta I_4^{(1)} + c |\complement E| \ ,$$

$$|M_8| \leqq c \sum_k \int w_{0k}^2 dy \frac{dt_2}{t_2} \leqq c |\complement E| \ .$$

This implies that

$$|L_5| \leqq |M_7| + |M_8| \leqq \delta I_4^{(1)} + c |\complement E| \ .$$

Consequently

$$|J_4| \leqq \frac{1}{2} |L_4| + \frac{1}{2} |L_5| \leqq \delta I_4^{(1)} + c \int \{N(F) \wedge 1\}^2 dy \ .$$

Since

$$|J_5| \leqq \delta I_4^{(1)} + c \sum_k \int w_{0k}^2 dy \frac{dt_2}{t_2} \leqq \delta I_4^{(1)} + c |\complement E| \ ,$$

we have

$$I_4^{(1)} \leqq |J_4| + |J_5| \leqq 2\delta I_4^{(1)} + c \int \{N(F) \wedge 1\}^2 dy \ .$$

Thus

$$I_4^{(1)} \leqq c \int \{N(F) \wedge 1\}^2 dy \ .$$

If we set

$$I_4^{(2)} = \sum_k \int K_{0k}^2(y, \varepsilon^{-1}, t_2) r(v(y, \varepsilon^{-1}, t_2)) dy \frac{dt_2}{t_2} \ ,$$

then in the same way as above, we obtain

$$I_4^{(2)} \leqq c \int \{N(F) \wedge 1\}^2 dy \ .$$

Thus

$$|I_4| \leqq I_4^{(1)} + I_4^{(2)} \leqq c \int \{N(F) \wedge 1\}^2 dy ,$$

which proves (10.2).

13. Proof of Theorem 3. Let $K^{(i)}$ be the same as in Theorem 3 and suppose that P_i is diagonal. Let $\zeta_0 \in C^\infty(\boldsymbol{R}^1)$ be such that

$$\zeta_0 \geqq 0 , \qquad \zeta_0(u) = \begin{cases} 1 & \text{if} \quad u \leqq 1 \\ 0 & \text{if} \quad u \geqq 2 . \end{cases}$$

Set

$$\begin{aligned} \bar{K}^{(i)}(x^{(i)}) &= K^{(i)}(x^{(i)})\zeta_0(a\rho^{(i)}(x^{(i)})) \quad (a > 0) , \\ \bar{K} &= \bar{K}^{(1)} \times \bar{K}^{(2)} , \\ \bar{K}^{(i),\delta}(x^{(i)}) &= \bar{K}^{(i)}(x^{(i)})(1 - \zeta_0(\delta^{-1}\rho^{(i)}(x^{(i)}))) \quad (\delta > 0) , \\ \bar{K}^{(\delta)} &= \bar{K}^{(1),\delta} \times \bar{K}^{(2),\delta} \\ \bar{K}^{(i)}_{\varepsilon_i}(x^{(i)}) &= \bar{K}^{(i)}(x^{(i)})(1 - \chi_{[0,1]}(\varepsilon_i^{-1}\rho^{(i)}(x^{(i)}))) , \\ \bar{K}_\varepsilon &= \bar{K}^{(1)}_{\varepsilon_1} \times \bar{K}^{(2)}_{\varepsilon_2} \quad (\varepsilon = (\varepsilon_1, \varepsilon_2),\ \varepsilon_i > 0) . \end{aligned}$$

On account of a theorem of Stein [15], which generalizes an indirect method of Kolmogoroff [12], Theorem 3 follows from the next lemma. (The constant a in the definition of $\bar{K}^{(i)}$ will be determined depending on the sets A and B. See [9, p. 138].)

LEMMA 13. *If f is a function on $\boldsymbol{R}^n$ with compact support and satisfies*

$$\int |f|\log(2 + |f|)dx < \infty ,$$

$$\int f(x^{(1)}, x^{(2)})dx^{(2)} = 0 \quad \textit{for all} \quad x^{(1)} \in \boldsymbol{R}^{n_1} ,$$

$$\int f(x^{(1)}, x^{(2)})dx^{(1)} = 0 \quad \textit{for all} \quad x^{(2)} \in \boldsymbol{R}^{n_2} ,$$

then we have $\sup_\varepsilon |f * \bar{K}_\varepsilon(x)| < \infty$ *for almost every x.*

We begin the proof of Lemma 13. Let $\phi^{(i)} \in \mathscr{S}_1(\boldsymbol{R}^{n_i})$ be such that $\phi^{(i)} \geqq 0$, $\operatorname{supp} \phi^{(i)} \subset \{x^{(i)}: \rho^{(i)}(x^{(i)}) \leqq 1\}$ and set $\phi = \phi^{(1)} \times \phi^{(2)}$. To estimate $f * \bar{K}_\varepsilon$, we use the following lemmas.

LEMMA 14. *There is $\sigma_i > 0$ such that*

$$|\bar{K}^{(i)}_{\varepsilon_i}(x^{(i)}) - \bar{K}^{(i),\delta} * \phi^{(i)}_{\varepsilon_i}(x^{(i)})| \leqq c\varepsilon_i^{-\gamma_i}(1 + \varepsilon_i^{-1}\rho^{(i)}(x^{(i)}))^{-\gamma_i - \sigma_i} \quad \textit{if} \quad 2\delta < \varepsilon_i .$$

LEMMA 15. *Let $\eta^{(i)} \in \mathscr{S}(\boldsymbol{R}^{n_i})$ and* $\operatorname{supp} \eta^{(i)} \subset \{x^{(i)}: \rho^{(i)}(x^{(i)}) \leqq 1\}$. *Given*

$L>0$, there exists $M>0$ such that if

$$\int \eta^{(i)}(x^{(i)})x^{(i)\alpha}dx^{(i)} = 0 \tag{13.1}$$

for all multi-indices α satisfying $|\alpha| \leqq M$, then we have

$$\left|\int t_i^{-\gamma_i}\bar{K}^{(i),\delta}(A_{t_i}^{(i)-1}(x^{(i)} - y^{(i)}))\eta^{(i)}(y^{(i)})dy^{(i)}\right| \leqq c(1 + \rho^{(i)}(x^{(i)}))^{-L},$$

where c is independent of t_i and δ.

When g is a function on $\boldsymbol{R}^n$, set

$$M^{(1)}g(x^{(1)}, x^{(2)}) = \sup_{t_1>0} t_1^{-\gamma_1}\int_{B_1(x^{(1)},t_1)} |g(y^{(1)}, x^{(2)})|dy^{(1)},$$

$$M^{(2)}g(x^{(1)}, x^{(2)}) = \sup_{t_2>0} t_2^{-\gamma_2}\int_{B_2(x^{(2)},t_2)} |g(x^{(1)}, y^{(2)})|dy^{(2)},$$

where $B_i(x^{(i)}, t_i) = \{y^{(i)}: \rho^{(i)}(x^{(i)} - y^{(i)}) < t_i\}$. Then if f satisfies the assumptions of Lemma 13, by arguing as in [9] and by using Lemma 14 we have

$$\begin{aligned}\sup_\varepsilon |f * \bar{K}_\varepsilon| \leqq\ & cM^{(2)}(\sup_{\varepsilon_1}|f *_{(1)} \bar{K}_{\varepsilon_1}^{(1)}|) + cM^{(1)}(\sup_{\varepsilon_2}|f *_{(2)} \bar{K}_{\varepsilon_2}^{(2)}|) \\ & + cM^{(1)}M^{(2)}f + \liminf_{\delta\to 0}(\sup_\varepsilon|f * \bar{K}^{(\delta)} * \phi_\varepsilon|),\end{aligned}$$

where the symbol $*_{(i)}$ denotes the operation of convolution in $\boldsymbol{R}^{n_i}$. It is clear that $M^{(1)}M^{(2)}f < \infty$ a.e. since $\int |f|\log(2 + |f|)dx < \infty$.

We next note that if $F(y, t) = f * \phi_t(y)$, then $\int N^{p_0}(F)dx < \infty$ for some p_0 with $0 < p_0 < 1$. (This follows from a direct estimate for $N(F)$.) Thus to prove $\liminf_{\delta\to 0}(\sup_\varepsilon|f * \bar{K}^{(\delta)} * \phi_\varepsilon|) < \infty$ a.e., it is sufficient to show that

$$\sup_\delta \int \sup_\varepsilon|f * \bar{K}^{(\delta)} * \phi_\varepsilon|^{p_0}dx \leqq c\int N^{p_0}(F)dx. \tag{13.2}$$

Now we prove (13.2). Let $\eta^{(i)} \in \mathscr{S}(\boldsymbol{R}^{n_i})$ and $\operatorname{supp}\eta^{(i)} \subset \{x^{(i)}: \rho^{(i)}(x^{(i)}) \leqq 1\}$. Suppose also that $\eta^{(i)}$ satisfies the condition (13.1) for sufficiently large M and the condition:

$$\sup_{t_i>0}|\hat{\eta}^{(i)}(A_{t_i}^{(i)*}\xi^{(i)})| > 0 \quad (\xi^{(i)} \neq 0).$$

Set $H^{(\delta)}(y, t) = f * \bar{K}^{(\delta)} * \eta_t * \eta_t(y)$ ($\eta = \eta^{(1)} \times \eta^{(2)}$), $J(y, t) = f * \eta_t(y)$. Then by Theorem 1 we have

$$\int \sup_\varepsilon|f * \bar{K}^{(\delta)} * \phi_\varepsilon|^{p_0}dx \leqq c\int S^{p_0}(H^{(\delta)})dx.$$

Using Lemma 6, Lemma 8 and Lemma 15 (for sufficiently large L), we obtain

$$\int S^{p_0}(H^{(\delta)})dx \leqq c \int S^{p_0}(J)dx ,$$

where c is independent of δ. By the corollary to Theorem 2, we have

$$\int S^{p_0}(J)dx \leqq c \int N^{p_0}(F)dx .$$

Combining the above inequalities, we obtain (13.2).

To prove $M^{(2)}(\sup_{\varepsilon_1}|f *_{(1)} \bar{K}_{\varepsilon_1}^{(1)}|) < \infty$ a.e., note that

$$M^{(2)}(\sup_{\varepsilon_1}|f *_{(1)} \bar{K}_{\varepsilon_1}^{(1)}|) \leqq cM^{(2)}M^{(1)}f + cM^{(2)}(\Omega) ,$$

where $\Omega = \liminf_{\delta\to 0}(\sup_{\varepsilon_1}|f *_{(1)} \bar{K}^{(1),\delta} *_{(1)} \phi_{\varepsilon_1}^{(1)}|)$ (this follows from Lemma 14). Since $M^{(2)}M^{(1)}f$ is finite almost everywhere, we only have to prove $M^{(2)}(\Omega) < \infty$ a.e. Let V be a compact set in $\boldsymbol{R}^n$. Then since $M^{(2)}$ is of weak type (1, 1), we have

$$\int_V \{M^{(2)}(\Omega)\}^q dx \leqq c + c\int_W \Omega dx \quad (0 < q < 1) \tag{13.3}$$

for some compact set W in $\boldsymbol{R}^n$. If $F^{(1)}(y^{(1)}, t_1) = (\phi_{t_1}^{(1)} *_{(1)} f(\cdot, x^{(2)}))(y^{(1)})$ for fixed $x^{(2)}$, then we can prove directly

$$\int N^{(1)}(F^{(1)})dx^{(1)} \leqq c + c\int |f(x^{(1)}, x^{(2)})|\log(2 + |f(x^{(1)}, x^{(2)})|)dx^{(1)} .$$

Thus the finiteness of the integral on the right hand side of (13.3) follows from the equivalence of $S^{(1)}$ and $N^{(1)}$ if we argue as in the proof of (13.2). This proves the almost everywhere finiteness of $M^{(2)}(\Omega)$, which completes the proof of the fact that $M^{(2)}(\sup_{\varepsilon_1}|f *_{(1)} \bar{K}_{\varepsilon_1}^{(1)}|) < \infty$ a.e. The almost everywhere finiteness of $M^{(1)}(\sup_{\varepsilon_2}|f *_{(2)} \bar{K}_{\varepsilon_2}^{(2)}|)$ is proved similarly. This completes the proof of Lemma 13.

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