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A FAMILY OF COMPACT SOLVABLE G_2-CALIBRATED MANIFOLDS

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1. Introduction. The G_2-calibrated manifolds are the G_2 analogous of the symplectic manifolds (see [HL 1], [HL 2]). Such a manifold M is 7-dimensional and has a twofold vector cross product P such that the associated 3-form ϕ is closed. In particular if ϕ is *closed and coclosed,* then P is a 2-fold *parallel* vector cross product, or equivalently M *has a subgroup of* G_2 *as the holonomy group* (see [GR]).

Bryant has shown that locally there are many 7-dimensional Riemannian manifolds with "Hol$\subset G_2$" (see [BR]) but a compact example is still conjectural.

Recently we reported in [FE] the existence of a compact G_2-calibrated manifold V^7 and, as far as we know, no more examples of this kind are known in the literature. V^7 is a nilmanifold and its first Betti number is equal to five.

In the present paper, we shall give a family of compact solvable nonnilpotent G_2-calibrated manifolds $M^7(k)$ and we shall prove that its first Betti number is equal to three.

We show that with respect to the natural metric $M^7(k)$ does not have a parallel vector cross product, so the holonomy group is not a subgroup of G_2. Thus $M^7(k)$ satisfies a natural weakening from "Hol$\subset G_2$" to "G_2-calibrated".

2. The manifolds $M^7(k)$. Let $G(k)$ be the group of matrices of complex numbers of the form

$$a = \begin{pmatrix} e^{kz_1} & 0 & 0 & z_2 \\ 0 & e^{-kz_1} & 0 & z_3 \\ 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $k \in \boldsymbol{R}$ is fixed such that $e^k + e^{-k}$ is an integer different from 2 and $z_1, z_2, z_3 \in \boldsymbol{C}$. $G(k)$ is a connected simply connected solvable nonnilpotent Lie group of (complex) dimension 3.

A global system of (complex) coordinates (z_1, z_2, z_3) on $G(k)$ is defined by

$$z_1(a) = z_1\,, \qquad z_2(a) = z_2\,, \qquad z_3(a) = z_3\,.$$

A basis for the right invariant (complex) 1-forms on $G(k)$ is given by

$$\omega_1 = dz_1\,, \qquad \omega_2 = dz_2 - kz_2dz_1\,, \qquad \omega_3 = dz_3 + kz_3dz_1\,.$$

Now, to get a compact quotient, we take $X(k) = G(k)/\Gamma(k)$, where $\Gamma(k)$ is a uniform subgroup of $G(k)$. (We refer to [NA, § 2] for a classification of 3-dimensional compact complex solvmanifolds, as well as for some explicit realizations of $X(k)$. See also [UE, Chapter VII, § 17].)

The 1-forms ω_1, ω_2 and ω_3 all descend to $X(k)$; denote the (complex) 1-forms induced on $X(k)$ by ϕ_1, ϕ_2 and ϕ_3 respectively. Moreover, (ϕ_1, ϕ_2, ϕ_3) is a basis for the holomorphic 1-forms on $X(k)$ such that

$$d\phi_1 = 0\,, \qquad d\phi_2 = k\phi_1 \wedge \phi_2\,, \qquad d\phi_3 = -k\phi_1 \wedge \phi_3\,.$$

THEOREM 1 ([NA]). *The first Betti number of $X(k)$ is equal to* 2.

Next we shall denote by $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ the (real) 1-forms $\mathrm{Re}(\phi_1)$, $\mathrm{Im}(\phi_1)$, $\mathrm{Re}(\phi_2)$, $\mathrm{Im}(\phi_2)$, $\mathrm{Re}(\phi_3)$, $\mathrm{Im}(\phi_3)$. Then we have

$$\begin{aligned} d\alpha_1 &= d\alpha_2 = 0\,,\\ d\beta_1 &= k(\alpha_1 \wedge \beta_1 - \alpha_2 \wedge \beta_2)\,,\\ d\beta_2 &= k(\alpha_2 \wedge \beta_1 + \alpha_1 \wedge \beta_2)\,,\\ d\gamma_1 &= -k(\alpha_1 \wedge \gamma_1 - \alpha_2 \wedge \gamma_2)\,,\\ d\gamma_2 &= -k(\alpha_2 \wedge \gamma_1 + \alpha_1 \wedge \gamma_2)\,. \end{aligned}$$

Let us now consider the compact manifold $M^7(k) = X(k) \times S^1$.

THEOREM 2. *There exists a (nonparallel) vector cross product on $M^7(k)$ such that the associated 3-form is closed.*

PROOF. Let ϕ be the 3-form on $M^7(k)$ defined by

$$\begin{aligned} (1)\qquad \phi = {}& \beta_1 \wedge \beta_2 \wedge \alpha_1 + \beta_2 \wedge \gamma_2 \wedge \eta + \gamma_2 \wedge \alpha_1 \wedge \gamma_1 + \alpha_1 \wedge \eta \wedge \alpha_2\\ &+ \eta \wedge \gamma_1 \wedge \beta_1 + \gamma_1 \wedge \alpha_2 \wedge \beta_2 + \alpha_2 \wedge \beta_1 \wedge \gamma_2\,, \end{aligned}$$

where η denotes the canonical 1-form on S^1. On the right hand side of (1), all of the terms except the second, fifth, sixth and seventh are closed. But $d(\beta_2 \wedge \gamma_2 - \beta_1 \wedge \gamma_1) = d(\beta_2 \wedge \gamma_1 + \beta_1 \wedge \gamma_2) = 0$, and so the sum of the second and fifth terms and sixth and seventh terms are each closed.

Define a metric on $M^7(k)$ by

$$\langle\ ,\ \rangle = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 + \gamma_1^2 + \gamma_2^2 + \eta^2\,.$$

Let $\{E_0 \cdots E_6\}$ be the basis dual to $\{\beta_1, \beta_2, \gamma_2, \alpha_1, \eta, \gamma_1, \alpha_2\}$. Then a twofold vector cross product P on $M^7(k)$ is given by $P(E_i, E_j) = -P(E_j, E_i)$, and $P(E_i, E_{i+1}) = E_{i+3}$, $P(E_{i+3}, E_i) = E_{i+1}$, $P(E_{i+1}, E_{i+3}) = E_i$ $(i \in Z_7)$. It is not hard to show that P satisfies the axioms for a twofold vector cross product (see [GR]) and ϕ is the associated 3-form. To show that P is not parallel we prove that $\delta\phi$ is nonzero, δ being the coderivative of $M^7(k)$ with respect to the metric given above. Indeed, a calculation shows that $\delta\phi = 4k(\gamma_1 \wedge \gamma_2 - \beta_1 \wedge \beta_2) \neq 0$.

THEOREM 3. *The first Betti number of $M^7(k)$ is equal to* 3.

PROOF. It is a direct consequence of Theorem 1.

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