Tôhoku Math. Journ.
39 (1987), 259–279.

HYPERCUSPIDALITY OF AUTOMORPHIC CUSPIDAL REPRESENTATIONS OF THE UNITARY GROUP $U(2, 2)$

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(Received April 22, 1986)

Introduction. In this paper, we study the hypercuspidality of automorphic cuspidal representations of the unitary group $U(2, 2)$.

The hypercuspidality in the case of the symplectic group was introduced by Piatetski-Shapiro [6]. For $G = GSp_4$, a cusp form f on G_A is called hypercuspidal if the Whittaker function corresponding to f vanishes (cf. [7]).

Analogously, we define the hypercuspidality in the case of $U(2, 2)$ by the vanishing of some Whittaker functions occuring in the Fourier expansion of the cusp form. More precisely, for a cusp form f on $U(2, 2)$, we consider the Fourier expansion of f with respect to the center of the unipotent radical of the Borel subgroup. Then we obtain two Whittaker functions W_f and V_f, where W_f is the ordinary Whittaker function and V_f is as defined in Section 1. We note that in the case of Sp_4, the function V_f did not appear in a similar Fourier expansion of a cusp form f. In terms of these functions, we say f is U-cuspidal (resp. N-cuspidal) if W_f (resp. V_f) vanishes. Further, if both of the functions W_f and V_f vanish, f is called hypercuspidal.

Next, using the notion of the dual reductive pair, we investigate cuspidal representations obtained from the Weil-lifting of cuspidal representations of $U(1, 1)$ or $U(2, 1)$. Symbolicically, $U(1, 1)$, $U(2, 1)$, $\cdots$, denote unitary groups over a global field of degree 2, 3, $\cdots$, with maximal index. Let τ be a cuspidal representation of $U(1, 1)$ or $U(2, 1)$ and $\Theta(\tau, \psi)$ a cuspidal representation of $U(2, 2)$ obtained from the Weil-lifting of τ. For $\varphi \in \tau$, let f_φ be an element in $\Theta(\tau, \psi)$ corresponding to φ. By an explicit computation of the Fourier coefficients of f_φ, we have relations between Whittaker functions of φ and f_φ (Lemma (3.2), Theorem (4.3) and Proposition (4.4)). Using these relations, we prove the non-vanishing of $\Theta(\tau, \psi)$. Further, under an additional assumption, we obtain some results about the hypercuspidality of $\Theta(\tau, \varphi)$ (Theorem (3.1) and Corollary to Proposition (4.4)).

The author would like to express his gratitude to Professor I. Satake for useful advice and encouragement and to Professor Y. Morita for many valuable comments.

1. Notation and preliminaries. Let F be a global field whose characteristic is different from 2. Let E be a quadratic extension of F, and denote its Galois involution by $x \to \bar{x}$. Let A_F (resp. A_E) be the adele ring of F (resp. E). We denote the trace and norm of E over F by $\mathrm{Tr}_{E/F}$ and $N_{E/F}$, respectively. We fix, once and for all, an element i in E^* such that $\mathrm{Tr}_{E/F}(i) = 0$ and a non-trivial character ψ of A_F/F.

Let $\mathfrak{G}$ be an algebraic group defined over F. Then we denote by $\mathfrak{G}_F$ (resp. $\mathfrak{G}_A$) the F-rational points (resp. A_F-rational points) in $\mathfrak{G}$. When $\mathfrak{G}$ is reductive, let $\mathscr{A}(\mathfrak{G}_A)$ (resp. $\mathscr{A}_0(\mathfrak{G}_A)$) denote the space consisting of automorphic forms (resp. cusp forms) on $\mathfrak{G}_A$. Also, when A is a locally compact group, let $\hat{A}$ be the group consisting of unitary characters on A.

Now, let V be a 4-dimensional vector space over E with a basis $\{e_1, e_2, e_3, e_4\}$, and $(\ ,\ )_V$ the skew-Hermitian form on V which is represented by the matrix

$$\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$$

with respect to $\{e_1, e_2, e_3, e_4\}$. Let

$$G_F = \left\{ g \in GL_4(E) \,\middle|\, g\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} {}^t\bar{g} = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \right\}$$

and

$$H_F = \left\{ h \in GL_2(E) \,\middle|\, h\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t\bar{h} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

First, we construct representatives for proper F-parabolic subgroups of G.

(1) Let B_F be a Borel subgroup with the Levi-factor

$$T_F = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & \bar{a}^{-1} & \\ & & & \bar{b}^{-1} \end{pmatrix} \,\middle|\, a, b \in E^* \right\}$$

and the unipotent radical

$$U_F = \left\{ \begin{pmatrix} 1 & a & x - \bar{a}b & b \\ 0 & 1 & \bar{b} - \bar{a}y & y \\ & & 1 & 0 \\ & 0 & -\bar{a} & 1 \end{pmatrix} \,\middle|\, a, b \in E,\ x, y \in F \right\}.$$

For simplicity, we put

$$u(a, b, x, y) = \begin{pmatrix} 1 & a & x - \bar{a}b & b \\ 0 & 1 & \bar{b} - \bar{a}y & y \\ & & 1 & 0 \\ & 0 & -\bar{a} & 1 \end{pmatrix},$$

$$u(a) = u(a, 0, 0, 0) \quad \text{and} \quad z(x) = u(0, 0, x, 0)$$

for $a, b \in \boldsymbol{A}_E$ and $x, y \in \boldsymbol{A}_F$.

(2) Let P_F be the parabolic subgroup stabilizing the isotropic line Ee_3, for which the Levi-factor and the unipotent radical are given by

$$L_F = \left\{ \begin{pmatrix} a' & & & \\ & a & & b \\ & & \bar{a}'^{-1} & \\ & c & & d \end{pmatrix} \middle| \, a' \in E^*, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_F \right\}$$

and $N_F = \{u(a, b, x, 0) \,|\, a, b \in E, x \in F\}$, respectively.

(3) Let Q_F be the parabolic subgroup stabilizing the isotropic subspace $Ee_3 + Ee_4$, for which the Levi-factor and the unipotent radical are given by

$$M_F = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t\bar{A}^{-1} \end{pmatrix} \middle| \, A \in GL_2(E) \right\}$$

and $S_F = \{u(0, b, x, y) \,|\, b \in E, x, y \in F\}$, respectively.

Let Z_F be the center of U_F, that is, $Z_F = \{z(x) \,|\, x \in F\}$. We identify the group L_F and $E^* \times H_F$ by

$$\ell : E^* \times H_F \xrightarrow{\sim} L_F \quad \left(a', \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \mapsto \begin{pmatrix} a' & & & \\ & a & & b \\ & & \bar{a}'^{-1} & \\ & c & & d \end{pmatrix}.$$

Further, for $a, b \in E$, we put $n(a, b) = u(a, b, 0, 0) \bmod Z_F$. We also use the same notation in the adelic case. Then, P_F/Z_F is isomorphic to $(E^* \times H_F) \ltimes (E \oplus E)$ by the correspondence

$$(a', h) \ltimes (a, b) \mapsto \ell(a', h) n(a, b) ,$$

where $(a', h) \in E^* \times H_F$ and $(a, b) \in E \oplus E$. Also, we have

$$\ell(a', h)^{-1} n(a, b) \ell(a', h) = n(a'^{-1}(a, b)h) . \tag{1.1}$$

Next, we determine groups $(U_F \backslash U_A)^\wedge$, $(N_F \backslash N_A)^\wedge$ and $(Z_F \backslash Z_A)^\wedge$ consisting of unitary characters of $U_F \backslash U_A$, $N_F \backslash N_A$ and Z_F / Z_A, respectively. For each $\xi, \zeta \in E$ and $t \in F$, we define characters $\psi_{(\xi, t)}$, $\psi_{(\xi, \zeta)}$ and ψ_t of $U_F \backslash U_A$, $N_F \backslash N_A$

and $Z_F\backslash Z_A$, respectively, by

$$\psi_{(\xi,t)}(u(a, b, x, y)) = \psi(\mathrm{Tr}_{E/F}(\xi a) + ty) ,$$
$$\psi_{(\xi,\zeta)}(u(a, b, x, 0)) = \psi(\mathrm{Tr}_{E/F}(\xi a + \zeta b))$$

and

$$\psi_t(z(x)) = \psi(tx) ,$$

where $a, b \in A_E/E$ and $x, y \in A_F/F$. Then we have

$$(U_F\backslash U_A)\hat{} = \{\psi_{(\xi,t)} \mid \xi \in E,\ t \in F\} , \qquad (N_F\backslash N_A)\hat{} = \{\psi_{(\xi,\zeta)} \mid \xi, \zeta \in E\} ,$$
$$(Z_F\backslash Z_A)\hat{} = \{\psi_t \mid t \in F\} .$$

Finally, for a given automorphic form f on G_A, we define three Whittaker functions corresponding to f by

$$W_f^{\psi_{(\xi,t)}}(g) = \int_{U_F\backslash U_A} \overline{\psi_{(\xi,t)}(u)} f(ug)du ,$$
$$V_f^{\psi_{(\xi,\zeta)}}(g) = \int_{N_F\backslash N_A} \overline{\psi_{(\xi,\zeta)}(n)} f(ng)dn$$

and

$$J_f^{\psi_t}(g) = \int_{Z_F\backslash Z_A} \overline{\psi_t(z)} f(zg)dz .$$

2. Fourier expansions and the hypercuspidality. In this section, we define the hypercuspidality for cusp forms on $U(2, 2)$.

Let $E^1 = \{a \in E^* \mid N_{E/F}(a) = 1\}$. Let $[F^*]$ (resp. $[E^*]$) be a complete set of representatives of $F^*/N_{E/F}(E^*)$ (resp. E^*/E^1). For a cusp form f on G_A, we consider the Fourier expansion of f along $Z_F\backslash Z_A$. Fix g in G_A. As a function on the compact abelian group $Z_F\backslash Z_A$, $f(zg)$ has a Fourier expansion of the form

$$(2.1) \qquad f(g) = \int_{Z_F\backslash Z_A} f(zg)dz + \sum_{t \in F^*} J_f^{\psi_t}(g)$$
$$= \int_{Z_F\backslash Z_A} f(zg)dz + \sum_{t \in [F^*]} \sum_{a \in [E^*]} J_f^{\psi_t}\left(\begin{pmatrix} a & & & \\ & 1 & & \\ & & \bar{a}^{-1} & \\ & & & 1 \end{pmatrix} g\right) .$$

We put

$$f_0(g) = \int_{Z_F\backslash Z_A} f(zg)dz .$$

We shall express this function f_0 by Whittaker functions W_f and V_f. In order to do so, we first describe the L_F-orbit decomposition of $(N_F\backslash N_A)\hat{}$.

L_F acts on $(N_F\backslash N_A)^\wedge$ by

$$\psi^{\ell}_{(\xi,\zeta)}(n) = \psi_{(\xi,\zeta)}(\ell^{-1}n\ell)\ ,$$

where $\psi_{(\xi,\zeta)} \in (N_F\backslash N_A)^\wedge$ and $\ell \in L_F$. Noting that Z_A is the derived group of N_A, we can deduce from (1, 1) that

$$\psi^{\ell(a',h)}_{(\xi,\zeta)} = \psi_{a'^{-1}(\xi,\zeta)h}$$

for $(a', h) \in E^* \times H_F$. Thus we obtain the following L_F-orbit decomposition:

$$(N_F\backslash N_A)^\wedge = \{\psi_{(0,0)}\} \cup \psi^{L_F}_{(1,0)} \cup \Big(\bigcup_{t\in[F^*]} \psi^{L_F}_{(1,ti)}\Big)\ . \tag{2.2}$$

We denote the stabilizers of $\psi_{(1,0)}$ and $\psi_{(1,ti)}$ in L_F by $L(1, 0)$ and $L(1, ti)$, respectively, and put

$$R_F = \left\{ \begin{pmatrix} c & & & \\ & c & & cy \\ & & c & \\ & & & c \end{pmatrix} \middle|\ c \in E^1,\ y \in F \right\}.$$

LEMMA 2.1. *For any cusp form f on G_A, we have*

$$f_0(g) = \sum_{t\in[F^*]} \Big\{ \sum_{\gamma\in R_F\backslash L_F} W_f^{\psi_{(1,t)}}(\gamma g) + \sum_{\gamma\in L(1,ti)\backslash L_F} V_f^{\psi_{(1,ti)}}(\gamma g)\Big\}\ . \tag{2.3}$$

PROOF. First we put

$$\lambda(p) = \int_{Z_F\backslash Z_A} f(zp)dz \quad (p \in P_A)\ .$$

Then this is a function on $P_F Z_A\backslash P_A$. Note that this group is isomorphic to $(E^*\backslash A_E^* \times H_F\backslash H_A) \ltimes (A_E/E)^2$. Fix p in P_A. As a function on $(A_E/E)^2$, $\lambda(n(a, b)p)$ has the Fourier expansion of the form

$$\lambda(p) = \sum_{(\xi,\zeta)\in E^2} \int_{(A_E/E)^2} \overline{\psi_{(\xi,\zeta)}(n(a, b))}\lambda(n(a, b)p)dadb\ .$$

From (2.2), we have

$$\lambda(p) = \int_{(A_E/E)^2} \lambda(n(a, b)p)dadb + \sum_{\gamma\in L(1,0)\backslash L_F} \int_{(A_E/E)^2} \overline{\psi^{\gamma^{-1}}_{(1,0)}(n(a, b))}\lambda(n(a, b)p)dadb$$
$$+ \sum_{t\in[F^*]} \sum_{\gamma\in L(1,ti)\backslash L_F} \int_{(A_E/E)^2} \overline{\psi^{\gamma^{-1}}_{(1,ti)}(n(a, b))}\lambda(n(a, b)p)dadb\ .$$

Since f is a cusp form, the first integral vanishes. Further,

$$\int_{(A_E/E)^2} \overline{\psi^{\gamma^{-1}}_{(1,ti)}(n(a, b))}\lambda(n(a, b)p)dadb = \int_{N_F\backslash N_A} \overline{\psi_{(1,ti)}(\gamma n\gamma^{-1})}f(np)dn$$
$$= V_f^{\psi_{(1,ti)}}(\gamma p)\ .$$

Similary,

$$\int_{(A_E/E)^2} \overline{\psi^{\gamma^{-1}}_{(1,0)}(n(a,b))}\lambda(n(a,b)p)dadb = \int_{N_F\backslash N_A} \overline{\psi_{(1,0)}(n)}f(n\gamma p)dn .$$

Put

$$\lambda_1(p) = \int_{N_F\backslash N_A} \overline{\psi_{(1,0)}(n)}f(np)dn .$$

Therefore we have

$$\lambda(p) = \sum_{\gamma\in L(1,0)\backslash L_F} \lambda_1(\gamma p) + \sum_{t\in[F^*]} \sum_{\gamma\in L(1,ti)\backslash L_F} V_f^{\psi_{(1,ti)}}(\gamma p) .$$

Next let

$$P_{1,F} = \left\{\ell\left(1, \begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)\middle| y\in F\right\}$$

and

$$D_F = \left\{\ell\left(a, \begin{pmatrix}a & 0\\ 0 & \bar{a}^{-1}\end{pmatrix}\right)\middle| a\in E^*\right\} .$$

Note that $L(1, 0) = D_F \ltimes P_{1,F}$. For $t\in F$, we define a character ${}_t\psi$ of $P_{1,F}\backslash P_{1,A}$ by

$${}_t\psi\left(\ell\left(1, \begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)\right) = \psi(ty) .$$

Then, as before, D_F acts on $(P_{1,F}\backslash P_{1,A})^\wedge$ and we have the D_F-orbit decomposition

$$(P_{1,F}\backslash P_{1,A})^\wedge = \{{}_0\psi\}\cup(\bigcup_{t\in[F^*]} {}_t\psi^{D_F}) .$$

Let $D_{0,F}$ be a common stabilizer of ${}_t\psi$ for $t\in[F^*]$ in D_F, that is, $D_{0,F} = \{c1_4 \mid c\in E^1\}$. Now, viewing $p_1 \mapsto \lambda_1(p_1p)$ as a function on $P_{1,F}\backslash P_{1,A}$, we can express its Fourier expansion in the form

$$\lambda_1(p) = \int_{P_{1,F}\backslash P_{1,A}} \lambda_1(p_1p)dp_1 + \sum_{t\in[F^*]} \sum_{\delta\in D_{0,F}\backslash D_F} \int_{P_{1,F}\backslash P_{1,A}} \overline{{}_t\psi^{\delta^{-1}}(p_1)}\lambda_1(p_1p)dp_1 .$$

The first integral equals

$$\int_{(A_E/E)^2} \overline{\psi_{(1,0)}(n(a,b))}\left\{\int_{P_{1,F}Z_F\backslash P_{1,A}Z_A} f(zn(a,b)p_1p)dzdp_1\right\}dadb$$
$$= \int_{A_F/E} \overline{\psi(\mathrm{Tr}_{E/F}(a))}\left\{\int_{S_F\backslash S_A} f(su(a)p)ds\right\}da = 0 .$$

Furthermore,

$$\int_{P_{1,F}\backslash P_{1,A}} \overline{{}_t\psi^{\delta^{-1}}(p_1)}\lambda_1(p_1p)\,dp_1 = \int_{P_{1,F}\backslash P_{1,A}} \overline{{}_t\psi(\delta p_1\delta^{-1})}\left\{\int_{N_F\backslash N_A} \overline{\psi_{(1,0)}(n)}f(np_1p)\,dn\right\}dp_1 .$$

Here if we put

$$p_1 = \ell\left(1, \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) \quad \text{and} \quad n = u(a, b, x, 0)\,,$$

this integral equals

$$\int_{A_F/F} \overline{\psi(ty)} \left\{ \int_{N_F \backslash N_A} \overline{\psi(\mathrm{Tr}_{E/F}(a))} f((\delta n \delta^{-1}) p_1 \delta p) dn \right\} dy$$
$$= \int_{A_F/F} \int_{N_F \backslash N_A} \overline{\psi(\mathrm{Tr}_{E/F}(a) + ty)} f(n p_1 \delta p) dn dy = W_f^{\psi(1,t)}(\delta p)\,.$$

Thus

$$\lambda_1(p) = \sum_{t \in [F^*]} \sum_{\delta \in D_{0,F} \backslash D_F} W_f^{\psi(1,t)}(\delta p)\,.$$

Hence

$$\begin{aligned} f_0(p) &= \lambda(p) \\ &= \sum_{\gamma \in L(1,0) \backslash L_F} \lambda_1(\gamma p) + \sum_{t \in [F^*]} \sum_{\gamma \in L(1,ti) \backslash L_F} V_f^{\psi(1,ti)}(\gamma p) \\ &= \sum_{t \in [F^*]} \Big\{ \sum_{\gamma \in D_F P_{1,F} \backslash L_F} \sum_{\delta \in D_{0,F} \backslash D_F} W_f^{\psi(1,t)}(\delta \gamma p) + \sum_{\gamma \in L(1,ti) \backslash L_F} V_f^{\psi(1,ti)}(\gamma p) \Big\} \\ &= \sum_{t \in [F^*]} \Big\{ \sum_{\gamma \in D_{0,F} P_{1,F} \backslash L_F} W_f^{\psi(1,t)}(\gamma p) + \sum_{\gamma \in L(1,ti) \backslash L_F} V_f^{\psi(1,ti)}(\gamma p) \Big\}\,, \end{aligned}$$

where

$$D_{0,F} P_{1,F} = \left\{ \ell\left(c, \begin{pmatrix} c & cy \\ 0 & c \end{pmatrix}\right) \middle| \, c \in E^1,\ y \in F \right\} = R_F\,.$$

We have thus proved the relation (2.3) for $g \in P_A$.

On the other hand, since there exists a compact subgroup K in G_A such that $G_A = P_A K$, we obtain the assertion for all $g \in G_A$ by the right translation with respect to elements of K. q.e.d.

From this Lemma and (2.1), we have

$$f(g) = \sum_{t \in [F^*]} \left\{ \sum_{\gamma \in R_F \backslash L_F} W_f^{\psi(1,t)}(\gamma g) + \sum_{\gamma \in L(1,ti) \backslash L_F} V_f^{\psi(1,ti)}(\gamma g) \right.$$
$$\left. + \sum_{a \in [E^*]} J_f^{\psi_t}\left(\begin{pmatrix} a & & & \\ & 1 & & \\ & & \bar{a}^{-1} & \\ & & & 1 \end{pmatrix} g \right) \right\}.$$

In view of this expansion, we put

$$W(\psi) = \{(W_f^{\psi(1,t)})_{t\in[F^*]} \mid f \in \mathscr{A}_0(G_A)\}$$

and

$$V(\psi) = \{(V_f^{\psi(1,ti)})_{t\in[F^*]} \mid f \in \mathscr{A}_0(G_A)\} .$$

We define a linear map $\mathscr{D}$ from $\mathscr{A}_0(G_A)$ to $W(\psi) \oplus V(\psi)$ by $\mathscr{D}(f) = ((W_f^{\psi(1,t)})_t, (V_f^{\psi(1,ti)})_t)$. Noting that the mapping $f \mapsto (J_f^{\psi t})_{t\in[F^*]}$ is injective on $\mathscr{A}_0(G_A)$, we give the following definition.

DEFINITION. Let f be a cusp form on G_A. We say f is *N-cuspidal* (resp. *U-cuspidal*) if f is contained in $\mathscr{D}^{-1}(W(\psi))$ (resp. $\mathscr{D}^{-1}(V(\psi))$). Further, we say f is *hypercuspidal* if f is contained in $\mathrm{Ker}(\mathscr{D})$.

Clearly, these subspaces are invariant under the action of the Hecke algebra of G_A, and are independent of a choice of a character ψ and a set of representatives $[F^*]$.

EXAMPLE. Let F be an algebraic number field. We assume that F has a real place v which does not split in E. Let G_v be the group consisting of F_v-rational points in G. It is known that holomorphic discrete series representations of G_v do not have an ordinary Whittaker model for any non-degenerate characters of U_v (Hashizume [2]). Thus, if π is a cuspidal representation of G_A whose G_v-component is a holomorphic discrete series representation, then π is U-cuspidal. But, we do not know whether π is hypercuspidal or not.

In Sections 3 and 4, we will construct U-cusp forms and N-cusp forms by the Weil-lifting.

3. Lifting from $U(1, 1)$ to $U(2, 2)$. In this section, we consider the Weil-lifting $\Theta(\tau, \psi)$ of an irreducible automorphic cuspidal representation τ of H_A to G_A, and investigate the cuspidality of $\Theta(\tau, \psi)$.

Let W be a 2-dimensional vector space over E, $(\ ,\ )_W$ the skew-Hermitian form on W which is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with respect to a basis $\{w_1, w_2\}$. Let $X_F = (V \otimes W)_F$ be a vector space over F. We consider the symplectic space X_F obtained by taking the imaginary part of the Hermitian form $(\ ,\ )_W \cdot (\ ,\ )_V$. We have a dual reductive pair $(H, G) \subset Sp_{16}$. Let $Sp_{16}(A_F)^\sim$ be the two-fold covering group of $Sp_{16}(A_F)$. Let ω_ψ be the Weil-representation of $Sp_{16}(A_F)^\sim$ associated to ψ. Then, in the same manner as in [1, Sections 6 and 8], ω_ψ gives an ordinary representation of $G_A H_A$. Let $X_F = X_1 \oplus X_2$ be a complete polarization of X_F, and $\mathscr{S}(X_{1,A})$ the Schwarz-Bruhat space on $X_{1,A}$. Now, let (τ, V_τ) be an irreducible automorphic cuspidal representation of H_A in $\mathscr{A}_0(H_A)$. For each $\varphi \in V_\tau$ and $\Phi \in \mathscr{S}(X_{1,A})$, we put

$$f_\varphi^\Phi(g) = \int_{H_F \backslash H_A} \{\sum_{v \in X_{1,F}} \omega_\psi(g \cdot h)\Phi(v)\}\varphi(h)dh \quad (g \in G_A)$$

and

$$\Theta(\tau, \psi) = \{f_\varphi^\Phi \mid \varphi \in V_\tau,\ \Phi \in \mathscr{S}(X_{1,A})\} .$$

It is well known that $\Theta(\tau, \psi)$ gives an automorphic representation of G_A in $\mathscr{A}(G_A)$. We call it the Weil-lifting of τ. The aim of this section is to prove the following:

THEOREM 3.1. *Let (τ, V_τ) be an irreducible cuspidal representation of H_A in $\mathscr{A}_0(H_A)$. If τ is non-trivial, then $\Theta(\tau, \psi)$ is also non-trivial. Further, if $\Theta(\tau, \psi)$ is cuspidal, then it is U-cuspidal but not hypercuspidal.*

We need a few lemmas for the proof. We give a complete polarization of X_F by $X_1 = e_1 \otimes W + V_1 \otimes w_1$ and $X_2 = e_3 \otimes W + V_1 \otimes w_2$, where $V_1 = Ee_2 + Ee_4$. As a basis of X_1 we take $\{e_1 \otimes w_1, ie_1 \otimes w_1, e_1 \otimes w_2, ie_1 \otimes w_2, e_2 \otimes w_1, ie_2 \otimes w_1, e_4 \otimes w_1, ie_4 \otimes w_1\}$ and choose a basis of X_2 in such a way that the symplectic form $\mathrm{Im}(\ ,\ )_V \cdot (\ ,\ )_W$ is represented by the matrix

$$\begin{pmatrix} 0 & 1_8 \\ -1_8 & 0 \end{pmatrix} .$$

Then we can use the Schroedinger realization of ω_ψ on $\mathscr{S}(X_{1,A})$ (cf. [7], [9]). We identifiy X_1 with $W \oplus V_1 = \{a_1w_1 + a_2w_2 + a_3e_2 + a_4e_4 \mid a_j \in E,\ 1 \leqq j \leqq 4\}$, and we write a Schwarz-Bruhat function $\Phi(X)$ on $X_{1,A}$ as $\Phi(w, v)$ or $\Phi(a_1, a_2, a_3, a_4)$. Then for $\Phi \in \mathscr{S}(X_{1,A})$ and $u = u(a, b, x, y) \in U_A$, the action of $\omega_\psi(u)$ on Φ is given by

$$(3.1) \qquad \omega_\psi(u)\Phi(a_1, a_2, a_3, a_4) = \psi(\mathrm{Im}(a_1\bar{a}_2(x - \bar{a}b) + \bar{a}_2a_3(\bar{b} - \bar{a}y) + a_2\bar{a}_4a) \times \Phi(a_1, a_2, aa_1 + a_3, ba_1 + ya_3 + a_4) .$$

Also, when we put

$$h(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in H_A ,$$

the action of $\omega_\psi(h(y))$ on Φ is given by

$$(3.2) \qquad \omega_\psi(h(y))\Phi(a_1, a_2, a_3, a_4) = \psi(y\,\mathrm{Im}(a_3\bar{a}_4))\Phi(a_1, ya_1 + a_2, a_3, a_4) .$$

By a calculation analogous to that in [7], for any $f = f_\varphi^\Phi \in \Theta(\tau, \psi)$, we obtain a formula

$$(3.3)\qquad f(g)=\int_{H_F\backslash H_A}\{\sum_{v\in V_{1,F}}\omega_\psi(g\cdot h)\Phi(0,v)\}\varphi(h)dh + \sum_{y\in F}\int_{H_{y,F}\backslash H_A}\{\sum_{v\in V_{1,F}}\omega_\psi(g\cdot h)\Phi(yiw_1+w_2,v)\}\varphi(h)dh\ ,$$

where $H_{y,F}$ is the stabilizer of yiw_1+w_2 in H_F. For $y=0$, we have $H_{0,F}=\{h(y)\mid y\in F\}$. Let U' be the derived group of U. For each automorphic form f on G_A, we put

$$f_{00}(g)=\int_{U'_F\backslash U'_A} f(ug)du\ .$$

Then, by (3.1), (3.2) and (3.3), after a simple calculation, we obtain the following formulas for $f=f_\varphi^\Phi$ in $\Theta(\tau,\psi)$.

$$(3.4)\qquad f_0(g)=\int_{H_F\backslash H_A}\{\sum_{v\in V_{1,F}}\omega_\psi(g\cdot h)\Phi(0,v)\}\varphi(h)dh + \int_{H_{0,F}\backslash H_A}\{\sum_{v\in V_{1,F}}\omega_\psi(g\cdot h)\Phi(w_2,v)\}\varphi(h)dh$$

and

$$(3.5)\qquad f_{00}(g)=\int_{H_F\backslash H_A}\{\sum_{v\in V_{1,F}}\omega_\psi(g\cdot h)\Phi(0,v)\}\varphi(h)dh\ .$$

Using these formulas, we compute the Fourier coefficients W_f^ψ and V_f^ψ of f.

LEMMA 3.2. *For any* $f=f_\varphi^\Phi\in\Theta(\tau,\psi)$ *and a non-trivial character* $\psi_{(\xi,\zeta)}\in(N_F\backslash N_A)^\wedge$, *we have*:

(1) *If* $\mathrm{Im}(\xi\bar\zeta)=0$, *then* $V_f^{\psi_{(\xi,\zeta)}}\equiv 0$,

(2) *If* $\mathrm{Im}(\xi\bar\zeta)\neq 0$, *then* $V_f^{\psi_{(\xi,\zeta)}}$ *is equal to the integral*

$$\int_{H_{0,A}\backslash H_A}\omega_\psi(g\cdot h)\Phi(0,1,-\bar\zeta,\bar\xi)\,W_\varphi^{\psi_{\mathrm{Im}(\xi\bar\zeta)}}(h)dh\ ,$$

where

$$W_\varphi^{\psi_{\mathrm{Im}(\xi\bar\zeta)}}(h)=\int_{A_F/F}\overline{\psi(y\,\mathrm{Im}(\xi\bar\zeta))}\varphi(h(y)\cdot h)dy\ .$$

PROOF. Clearly we have

$$V_f^{\psi_{(\xi,\zeta)}}(g)=\int_{N_FZ_A\backslash N_A}\overline{\psi_{(\xi,\zeta)}(n)}f_0(ng)dn\ .$$

By (3.4), the right hand side equals

$$\int_{N_FZ_A\backslash N_A}\overline{\psi_{(\xi,\zeta)}(n)}\left[\int_{H_F\backslash H_A}\{\sum_{v\in V_{1,F}}\omega_\psi(ng\cdot h)\Phi(0,v)\}\varphi(h)dh\right]dn + \int_{N_FZ_A\backslash N_A}\overline{\psi_{(\xi,\zeta)}(n)}\left[\int_{H_{0,F}\backslash H_A}\{\sum_{v\in V_{1,F}}\omega_\psi(ng\cdot h)\Phi(w_2,v)\}\varphi(h)\,dh\right]dn\ .$$

By (3.1), the first term equals

$$\left[\int_{N_F Z_A \backslash N_A} \overline{\psi_{(\xi,\zeta)}(n)}\, dn\right]\left[\int_{H_F\backslash H_A} \{\sum_{v\in V_{1,F}} \omega_\psi(g\cdot h)\Phi(0, v)\}\varphi(h)dh\right] = 0 .$$

The second term equals

$$\begin{aligned}
&\int_{(A_E/E)^2} \overline{\psi_{(\xi,\zeta)}(n(a, b))}\left[\int_{H_{0,F}\backslash H_A} \{\sum_{v\in V_{1,F}} \omega_\psi(n(a, b)g\cdot h)\Phi(w_2, v)\}\varphi(h)dh\right]dadb \\
&\quad = \int_{H_{0,F}\backslash H_A}\left[\sum_{(a_3,a_4)\in E^2}\left\{\int_{A_E/E}\psi(\mathrm{Im}(\bar{b}(a_3+\bar{\zeta})))db\right\}\right. \\
&\qquad \left.\times\left\{\int_{A_E/E}\psi(\mathrm{Im}(a(\bar{a}_4-\xi)))da\right\}\omega_\psi(g\cdot h)\Phi(0, 1, a_3, a_4)\right]\varphi(h)dh \\
&\quad = \int_{H_{0,F}\backslash H_A} \omega_\psi(g\cdot h)\Phi(0, 1, -\bar{\zeta}, \bar{\xi})\varphi(h)dh \\
&\quad = \int_{H_{0,A}\backslash H_A} \omega_\psi(g\cdot h)\Phi(0, 1, -\bar{\zeta}, \bar{\xi})\left\{\int_{A_F/F}\overline{\psi(y\,\mathrm{Im}(\xi\bar{\zeta}))}\varphi(h(y)\cdot h)\, dy\right\}dh .
\end{aligned}$$

Since φ is a cusp form, if $\mathrm{Im}(\xi\bar{\zeta}) = 0$, the inner integral equals zero. This implies the assertion (1). On the other hand, if $\mathrm{Im}(\xi\bar{\zeta}) \neq 0$, the last integral is no more than the one in the assertion (2). q.e.d.

LEMMA 3.3. *For any* $f = f_\varphi^\Phi \in \Theta(\tau, \psi)$, $\xi \in E^*$ *and* $t \in F$, *we have* $W_f^{\psi_{(\xi,t)}} \equiv 0$.

PROOF. From (3.5), we have

$$\begin{aligned}
W_f^{\psi_{(\xi,t)}}(g) &= \int_{U_F U'_A\backslash U_A} \overline{\psi_{(\xi,t)}(u)} f_{00}(ug)du \\
&= \int_{U_F U'_A\backslash U_A} \overline{\psi_{(\xi,t)}(u)}\left[\int_{H_F\backslash H_A} \{\sum_{v\in V_{1,F}} \omega_\psi(ug\cdot h)\Phi(0, v)\}\varphi(h)dh\right]du \\
&= \int_{A_E/E+A_F/F} \overline{\psi_{(\xi,t)}(u(a, 0, 0, y))} \\
&\quad \times\left[\int_{H_F\backslash H_A} \{\sum_{v\in V_{1,F}} \omega_\psi(u(a, 0, 0, y)g\cdot h)\Phi(0, v)\}\varphi(h)dh\right]dady \\
&= \left\{\int_{A_E/E} \overline{\psi(\mathrm{Tr}_{E/F}(\xi a))}da\right\}\left\{\int_{A_F/F}\overline{\psi(ty)}\right. \\
&\quad \left.\times\left[\int_{H_F\backslash H_A} \{\sum_{(a_3,a_4)\in E^2} \omega_\psi(g\cdot h)\Phi(0, 0, a_3, ya_3+a_4)\}\varphi(h)dh\right]dy\right\} = 0 .
\end{aligned}$$

q.e.d.

Note that Lemmas 3.2 and 3.3 remain true without the assumption of the cuspidality of $\Theta(\tau, \psi)$.

PROOF OF THEOREM 3.1. Let (τ, V_τ) be a non-trivial irreducible cuspidal representation of H_A. For any $\alpha \in F$, we define a character ψ_α of

$H_{0,F}\backslash H_{0,A}$ by $\psi_\alpha(h(y)) = \psi(\alpha y)$. Then for each $\varphi \in V_\tau$, we have a Fourier expansion of the form

$$\varphi(h) = \sum_{t \in [F^*]} \sum_{a \in [E^*]} W_\varphi^{\psi_t}\left(\begin{pmatrix} a & \\ & \bar{a}^{-1} \end{pmatrix} h\right).$$

Thus, if we put $W(\tau, \psi_t) = \{W_\varphi^{\psi_t} \mid \varphi \in V_\tau\}$, then there exists at least one $t' \in [F^*]$ such that $W(\tau, \psi_{t'}) \neq \{0\}$. We choose elements ξ, $\zeta \in E^*$ such that $\mathrm{Im}(\xi\bar{\zeta}) = t'$. Then, from Lemma 3.2, for any $f = f_\varphi^\Phi \in \Theta(\tau, \psi)$, we have

$$V_f^{\psi(\xi,\zeta)}(1) = \int_{H_{0,A}\backslash H_A} \omega_\psi(h)\Phi(0, 1, -\bar{\zeta}, \bar{\xi}) W_\varphi^{\psi_{t'}}(h) dh .$$

Since $W_\varphi^{\psi_{t'}} \neq 0$, this integral does not vanish at least for one $\Phi \in \mathscr{S}(X_{1,A})$. Hence $\Theta(\tau, \psi)$ is non-trivial. The last assertion is obvious by Lemma 3.3. q.e.d.

Finally, we state a result on the cuspidality of $\Theta(\tau, \psi)$. We define a theta-series of H_A with respect to ω_ψ by

$$\Theta_\Phi(h) = \sum_{v \in V_{1,F}} \omega_\psi(h)\Phi(0, v)$$

for $\Phi \in \mathscr{S}(X_{1,A})$. Let χ be the central character of τ. We denote by $\Theta(\psi, \chi^{-1})$ the space consisting of the theta-series of H_A which are transformed according to χ^{-1} under the center of H_A. We can easily show that $f \in \Theta(\tau, \psi)$ is cuspidal if and only if $f_{00} \equiv 0$. Therefore, by (3.5), $\Theta(\tau, \psi)$ is cuspidal if and only if V_τ is orthogonal to $\Theta(\psi, \chi^{-1})$.

4. **Lifting from $U(2, 1)$ to $U(2, 2)$.** We use an argument similar to that in Section 3.

Let W be a 3-dimensional vector spaces over E with a basis $\{w_{-1}, w_0, w_1\}$, and $(\ ,\ )_W$ the Hermitian form which is represented by the matrix

$$\begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix}$$

with respect to $\{w_{-1}, w_0, w_1\}$. Let H° be the corresponding unitary group and N° the unipotent subgroup of H°:

$$N_F^\circ = \left\{ \begin{pmatrix} 1 & a & z \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a,\ z \in E,\ \mathrm{Tr}_{E/F}(z) = -N_{E/F}(a) \right\}.$$

Let Z° be the center of N°:

$$Z_F^\circ = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| z \in E,\ \mathrm{Tr}_{E/F}(z) = 0 \right\} .$$

For the general theory of cusp forms on H_A°, we refer the reader to [1]. We define a character ψ° of $N_F^\circ \backslash N_A^\circ$ by

$$\psi^\circ\left(\begin{pmatrix} 1 & a & z \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{pmatrix}\right) = \psi(\mathrm{Tr}_{E/F}(a)) .$$

We denote by $L_0^2(H_A^\circ)$ the space consisting of the square-integrable cusp forms on H_A°. For each $\varphi \in L_0^2(H_A^\circ)$, we put

$$W_\varphi^{\psi^\circ}(h) = \int_{N_F^\circ \backslash N_A^\circ} \overline{\psi^\circ(n)} \varphi(nh) dn$$

and

$$\varphi_0(h) = \int_{Z_F^\circ \backslash Z_A^\circ} \varphi(zh) dz .$$

Then we have

$$\varphi_0(h) = \sum_{a \in E^*} W_\varphi^{\psi^\circ}\left(\begin{pmatrix} a & & \\ & 1 & \\ & & \bar{a}^{-1} \end{pmatrix} h\right) .$$

In particular, φ_0 vanishes if and only if so does $W_\varphi^{\psi^\circ}$. Let

$$L_{0,0}^2(H_A^\circ) = \{\varphi \in L_0^2(H_A^\circ) \mid W_\varphi^{\psi^\circ} \equiv 0\}$$

and let $L_{0,1}^2(H_A^\circ)$ be the orthogonal complement of $L_{0,0}^2$ in L_0^2. These spaces are invariant under H_A° and independent of ψ. Clearly, we have an orthogonal decomposition $L_0^2(H_A^\circ) = L_{0,0}^2(H_A^\circ) \oplus L_{0,1}^2(H_A^\circ)$. We know from [1] that the multiplicity one theorem holds for $L_{0,1}^2(H_A^\circ)$.

In the same manner as in Section 3, let $X_F = (V \otimes W)_F$ be a vector space over F with the symplectic form $\langle\ ,\ \rangle = \mathrm{Re}(\ ,\ )_W \cdot (\ ,\ )_V$. We have a dual reductive pair $(H^\circ, G) \subset Sp_{24}$. Let (τ, V_τ) be an irreducible cuspidal representation of H_A°. We denote by $\Theta(\tau, \psi)$ the Weil-lifting of τ with respect to the Weil-representation ω_ψ of $Sp_{24}(A_F)^\sim$. We give a complete polarization of X_F by $X_F = X_1 \oplus X_2$, where $X_1 = e_1 \otimes W + e_2 \otimes W$ and $X_2 = e_3 \otimes W + e_4 \otimes W$. Further, as a basis of X_1 we take $\{e_1 \otimes w_{-1}, ie_1 \otimes w_{-1}, e_1 \otimes w_0, ie_1 \otimes w_0, \cdots, e_2 \otimes w_1, ie_2 \otimes w_1\}$ and choose a basis of X_2 in such a way that the symplectic form $\langle\ ,\ \rangle$ is represented by the matrix

$$\begin{pmatrix} 0 & 1_{12} \\ -1_{12} & 0 \end{pmatrix} .$$

As before, for each $\varphi \in V_\tau$ and each Schwarz-Bruhat function Φ on $X_{1,A}$, we put

$$f_\varphi^\Phi(g) = \int_{H_F^\circ \backslash H_A^\circ} \{\sum_{v \in X_{1,F}} \omega_\psi(g\cdot h)\Phi(v)\}\varphi(h)dh .$$

We identify X_1 with $W \oplus W$. Then, for $\Phi \in \mathscr{S}(X_{1,A})$ and $u = u(a, b, x, y) \in U_A$, the action of u on Φ is given by

$$\begin{aligned}(4.1)\qquad &\omega_\psi(u)\Phi(X, Y)\\ &= \psi(1/2\{x(X, X)_W + 2\,\mathrm{Re}(b(X, Y)_W) + y(Y, Y)_W\})\Phi(X, aX + Y) ,\end{aligned}$$

where $X, Y \in W_A$. Also for $h \in H_A^\circ$, we have

$$(4.2)\qquad \omega_\psi(h)\Phi(X, Y) = \Phi(X\cdot h, Y\cdot h) .$$

First we consider the cuspidality of $\Theta(\tau, \psi)$. We define the action of H_F° on $W \oplus W$ by $(X, Y)\cdot h = (X\cdot h, Y\cdot h)$ for $(X, Y) \in W \oplus W$ and $h \in H_F^\circ$. Let $\mathrm{Gr}(X, Y)$ be the Gram matrix of (X, Y), that is,

$$\mathrm{Gr}(X, Y) = \begin{pmatrix} (X, X)_W & (X, Y)_W \\ (Y, X)_W & (Y, Y)_W \end{pmatrix} .$$

For $\alpha, t \in F$, we put

$$\mathrm{Gr}(\alpha, t) = \left\{(X, Y) \in W \oplus W \mid \mathrm{Gr}(X, Y) = \begin{pmatrix} 0 & ti \\ -ti & \alpha \end{pmatrix}\right\} .$$

and $\mathrm{Gr}(\alpha) = \mathrm{Gr}(\alpha, 0)$. Applying Witt's theorem, we can easily show the following:

LEMMA 4.1. *$\mathrm{Gr}(\alpha, t)$ has the following H_F°-orbit decomposition.*

(1) $\mathrm{Gr}(0) = \{(0, 0)\} \cup (w_1, 0)\cdot H_F^\circ \cup (\cup_{a \in E}(aw_1, w_1)\cdot H_F^\circ)$.

(2) *If $\alpha \in N_{E/F}(E^*)$, we write $\alpha = \alpha'\bar{\alpha}'$. Then*

$$\mathrm{Gr}(\alpha) = (0, \alpha' w_0)\cdot H_F^\circ \cup (w_1, \alpha' w_0)\cdot H_F^\circ .$$

(3) *If $\alpha \notin N_{E/F}(E^*)$ and $\alpha \neq 0$, then*

$$\mathrm{Gr}(\alpha) = (0, 1/2w_{-1} + \alpha w_1)\cdot H_F^\circ .$$

(4) *If $t \in F^*$, then for any $\alpha \in F$,*

$$\mathrm{Gr}(\alpha, t) = (tiw_1, w_{-1} + 1/2\alpha w_1)\cdot H_F^\circ .$$

For $X \in W$, let $H^\circ(X)$ be the stabilizer of X in H_F°. In particular, we put $H_{\alpha,F}^\circ = H^\circ(1/2w_{-1} + \alpha w_1)$ for $\alpha \in F^*$.

THEOREM 4.2. *Let (τ, V_τ) be an irreducible cuspidal representation of H_A° in $\mathscr{A}_0(H_A^\circ)$. Then $\Theta(\tau, \psi)$ is cuspidal if and only if*

$$\int_{H^\circ_{\alpha,F}\backslash H^\circ_{\alpha,A}} \varphi(kh)dk = 0$$

for all $\varphi \in V_\tau$, $h \in H^\circ_A$ *and* $\alpha \in F^*$.

PROOF. By definition, for a given automorphic form f on G_A, f is cuspidal if and only if

$$\int_{S_F\backslash S_A} f(sg)ds = 0 \qquad \text{and} \qquad \int_{N_F\backslash N_A} f(ng)dn = 0$$

for all $g \in G_A$. Thus, for $f = f^\Phi_\varphi \in \Theta(\tau, \psi)$, we compute these integrals. First

$$\int_{S_F\backslash S_A} f(sg)ds = \int_{(A_F/F)^2}\int_{A_E/E} f(u(0, b, x, y)g)dbdxdy$$
$$= \int_{(A_F/F)^2}\int_{A_E/E}\Big[\int_{H^\circ_F\backslash H^\circ_A}\{\sum_{(X,Y)\in X_{1,F}} \omega_\psi(u(0, b, x, y)g\cdot h)\Phi(X, Y)\}$$
$$\times\ \varphi(h)dh\Big]dbdxdy\ .$$

By (4.1), this equals

$$= \int_{H^\circ_F\backslash H^\circ_A}\Big[\sum_{(X,Y)\in X_{1,F}}\Big\{\int_{A_F/F}\psi(1/2x(X, X)_W)dx\Big\}\Big\{\int_{A_E/E}\psi(\mathrm{Re}(b(X, Y)_W))db\Big\}$$
$$\times\ \Big\{\int_{A_F/E}\psi(1/2y(Y, Y)_W)dy\Big\}\omega_\psi(g\cdot h)\Phi(X, Y)\Big]\varphi(h)dh$$
$$= \int_{H^\circ_F\backslash H^\circ_A}\{\sum_{(X,Y)\in \mathrm{Gr}(0)}\omega_\psi(g\cdot h)\Phi(X, Y)\}\varphi(h)dh\ .$$

By Lemma 4.1, this equals

$$\int_{H^\circ_F\backslash H^\circ_A}\omega_\psi(g\cdot h)\Phi(0, 0)\varphi(h)dh + \int_{H^\circ_F\backslash H^\circ_A}\{\sum_{\gamma\in H^\circ(w_1)\backslash H^\circ_F}\omega_\psi(g\cdot h)\Phi((w_1, 0)\cdot\gamma)\}\varphi(h)dh$$
$$+ \sum_{a\in E}\int_{H^\circ_F\backslash H^\circ_A}\{\sum_{\gamma\in H^\circ(w_1)\backslash H^\circ_F}\omega_\psi(g\cdot h)\Phi((aw_1, w_1)\cdot\gamma)\}\varphi(h)dh\ .$$

Since φ is a cusp form, it follows from (4.2) that the first integral is equal to zero. Also, since $H^\circ(w_1)$ contains N°_F, the second integral is equal to

$$\int_{H^\circ(w_1)\backslash H^\circ_A}\omega_\psi(g\cdot h)\Phi(w_1, 0)\varphi(h)dh$$
$$= \int_{H^\circ(w_1)N^\circ_A\backslash H^\circ_A}\omega_\psi(g\cdot h)\Phi(w_1, 0)\Big\{\int_{N^\circ_F\backslash N^\circ_A}\varphi(nh)dn\Big\}dh = 0\ .$$

For the same reason, the third term is equal to zero. Hence we have

$$\int_{S_F\backslash S_A} f(sg)ds = 0$$

for all $f \in \Theta(\tau, \psi)$.

Secondly, for $f = f_\varphi^\Phi \in \Theta(\tau, \psi)$, put

$$f_{00}(g) = \int_{U'_F \backslash U'_A} f(ug)du ,$$

where U' is the derived group of U. Then using the formula (4.1) and Lemma 4.1, and making a calculation similar to that above, we have

$$\begin{aligned} f_{00}(g) = \sum_{\alpha \in N_{E/F}(E^*)} &\left\{\int_{H^\circ(w_0)\backslash H^\circ_A} \omega_\psi(g\cdot h)\Phi(0, \alpha' w_0)\varphi(h)dh \right.\\ &\left. + \int_{Z^\circ_F\backslash H^\circ_A} \omega_\psi(g\cdot h)\Phi(w_1, \alpha' w_0)\varphi(h)dh\right\} \\ &+ \sum_{\alpha \in F^* - N_{E,F}(E^*)} \int_{H^\circ_{\alpha,F}\backslash H^\circ_{\alpha,A}} \omega_\psi(g\cdot h)\Phi(0, 1/2w_{-1} + \alpha w_1)\varphi(h)dh ,\end{aligned}$$

where, for $\alpha \in N_{E/F}(E^*)$, α' denotes an element of E^* such that $\alpha = N_{E/F}(\alpha')$. Moreover, we have

$$\begin{aligned} \int_{N_F\backslash N_A} f(ng)dn &= \int_{A_F/F} f_{00}(u(a)\cdot g)da \\ &= \sum_{\alpha \in N_{E/F}(E^*)} \left\{\int_{H^\circ(w_0)\backslash H^\circ_A} \omega_\psi(g\cdot h)\Phi(0, \alpha' w_0)\varphi(h)dh \right.\\ &\quad \left. + \int_{A_E/E}\int_{Z^\circ_F\backslash H^\circ_A} \omega_\psi(g\cdot h)\Phi(w_1, \alpha' w_0 + aw_1)\varphi(h)dhda\right\} \\ &\quad + \sum_{\alpha \in F^* - N_{E/F}(E^*)} \int_{H^\circ_{\alpha,F}\backslash H^\circ_A} \omega_\psi(g\cdot h)\Phi(0, 1/2w_{-1} + \alpha w_1)\varphi(h)dh .\end{aligned}$$

For any $a \in A_E/E$, we put

$$m(a) = \begin{pmatrix} 1 & a & -1/2a\bar{a} \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{pmatrix} \in N_F^\circ\backslash N_A^\circ .$$

Then

$$\begin{aligned} &\int_{A_E/E}\int_{Z^\circ_F\backslash H^\circ_A} \omega_\psi(g\cdot h)\Phi(w_1, \alpha' w_0 + aw_1)\varphi(h)dhda \\ &= \int_{A_E/E}\int_{Z^\circ_F\backslash H^\circ_A} \omega_\psi(g\cdot m(-\bar{\alpha}'^{-1}\bar{a})h)\Phi(w_1, \alpha' w_0)\varphi(h)dhda \\ &= \int_{A_E/E}\int_{Z^\circ_F\backslash H^\circ_A} \omega_\psi(g\cdot h)\Phi(w_1, \alpha' w_0)\varphi(m(a)h)dhda \\ &= \int_{Z^\circ_A\backslash H^\circ_A} \omega_\psi(g\cdot h)\Phi(w_1, \alpha' w_0)\left\{\int_{A_E/E}\int_{Z^\circ_F\backslash Z^\circ_A} \varphi(m(a)zh)dzda\right\}dh \\ &= \int_{Z^\circ_A\backslash H^\circ_A} \omega_\psi(g\cdot h)\Phi(w_1, \alpha' w_0)\left\{\int_{N^\circ_F\backslash N^\circ_A} \varphi(nh)dn\right\}dh \\ &= 0 .\end{aligned}$$

Note that for $\alpha \in N_{E/F}(E^*)$, we have

$$(0, \alpha' w_0) \cdot H_F^\circ = (0, 1/2w_{-1} + \alpha w_1) \cdot H_F^\circ .$$

Consequently, we obtain

$$\int_{N_F \backslash N_A} f(ng)dn = \sum_{\alpha \in F^*} \int_{H_{\alpha,F}^\circ \backslash H_A^\circ} \omega_\psi(g \cdot h)\Phi(0, 1/2w_{-1} + \alpha w_1)\varphi(h)dh$$

for each $f = f_\varphi^\Phi \in \Theta(\tau, \psi)$. Hence $\Theta(\tau, \psi)$ is cuspidal if and only if

$$\sum_{\alpha \in F^*} \int_{H_{\alpha,F}^\circ \backslash H_A^\circ} \omega_\psi(g \cdot h)\Phi(0, 1/2w_{-1} + \alpha w_1)\varphi(h)dh = 0$$

for all $\Phi \in \mathscr{S}(X_{1,A})$ and $\varphi \in V_\tau$.

For $\Phi \in \mathscr{S}(X_{1,A})$, we put $\Phi_1(Y) = \Phi(0, Y)$. The correspondence $\Phi \mapsto \Phi_1 \in \mathscr{S}(W_A)$ is surjective. Since we have

$$\omega_\psi(g \cdot h)\Phi(0, 1/2w_{-1} + \alpha w_1) = (\omega_\psi(g)\Phi)_1((1/2w_{-1} + \alpha w_1) \cdot h) ,$$

$\Theta(\tau, \psi)$ is cuspidal if and only if

$$\sum_{\alpha \in F^*} \int_{H_{\alpha,F}^\circ \backslash H_A^\circ} \Phi_1((1/2w_{-1} + \alpha w_1) \cdot h)\varphi(h)dh = 0$$

for all $\Phi_1 \in \mathscr{S}(W_A)$ and $\varphi \in V_\tau$. For $\alpha \in F^*$, we put $W_A(\alpha) = \{w \in W_A \mid (w, w)_W = \alpha\}$. Since $W_A(\alpha)$ is a closed subset in W_A, when we choose an element $w' \in W_A(\alpha)$, there exists a function $\Phi_{\alpha,w'} \in \mathscr{S}(W_A)$ such that $\Phi_{\alpha,w'}(w') = 1$ and that $\Phi_{\alpha,w'}|_{W_A(\beta)} = 0$ if $\beta \neq \alpha$. Thus for a fixed $\varphi \in V_\tau$,

$$\begin{aligned} &\sum_{\alpha \in F^*} \int_{H_{\alpha,F}^\circ \backslash H_A^\circ} \Phi_1((1/2w_{-1} + \alpha w_1) \cdot h)\varphi(h)dh \\ &\quad = \sum_{\alpha \in F^*} \int_{H_{\alpha,A}^\circ \backslash H_A^\circ} \Phi_1((1/2w_{-1} + \alpha w_1) \cdot h)\left\{\int_{H_{\alpha,F}^\circ \backslash H_{\alpha,A}^\circ} \varphi(kh)dk\right\}dh \\ &\quad = 0 \end{aligned}$$

for all $\Phi_1 \in \mathscr{S}(W_A)$ if and only if

$$\int_{H_{\alpha,F}^\circ \backslash H_{\alpha,A}^\circ} \varphi(kh)dk = 0$$

for all $\alpha \in F^*$ and $h \in H_A^\circ$. q.e.d.

In particular, if we put

$$T_F^\circ = \left\{ \begin{pmatrix} 1 & & 0 \\ & c & \\ 0 & & 1 \end{pmatrix} \middle| c \in E^1 \right\} ,$$

then T_F° is contained in $H_{\alpha,F}^\circ$ for any $\alpha \in F^*$. Therefore we obtain the following:

COROLLARY. *If V_τ satisfies the condition*

$$(\#) \qquad \int_{T_F^\circ \backslash T_A^\circ} \varphi(th)dt = 0$$

for all $\varphi \in V_\tau$ and $h \in H_A^\circ$, then $\Theta(\tau, \psi)$ is cuspidal.

Unfortunately, we do not know yet any example of the cuspidal representations satisfying the condition (#).

Next, we compute the Fourier coefficient W_f^ψ. We choose a complete set of representatives $[F^*]$ of $F^*/N_{E/F}(E^*)$ which contains 1. For the sake of convenience, we use $\{t/2 \mid t \in [F^*]\}$ instead of $[F^*]$. It is enough to compute $W_f^{\psi(1,t/2)}$ for $t \in [F^*]$.

THEOREM 4.3. *Let (τ, V_τ) be an irreducible cuspidal representation of H_A°. For $f = f_\varphi^\Phi \in \Theta(\tau, \psi)$, we have the following:*

(1) *If $V_\tau \subset L_{0,0}^2(H_A^\circ)$, then $W_f^{\psi(1,t/2)} \equiv 0$ for all $t \in [F^*]$.*

(2) *If $V_\tau \subset L_{0,1}^2(H_A^\circ)$, then for $t \in [F^*]$ we have*

$$W_f^{\psi(1,t/2)}(g) = \begin{cases} 0 & \text{if } t \neq 1 \\ \int_{Z_A^\circ \backslash H_A^\circ} \omega_\psi(g \cdot h)\Phi(w_1, w_0) W_\varphi^{\psi^\circ}(h)dh & \text{if } t = 1 \end{cases}$$

for all $g \in G_A$.

PROOF. For $t \in [F^*]$, we compute the integral

$$U_f^{\psi(1,t/2)}(g) = \int_{S_F \backslash S_A} \overline{\psi_{(1,t/2)}(s)} f(sg)ds .$$

By (4.1), this equals

$$\begin{aligned}
&= \int_{(A_F/F)^2} \int_{A_E/E} \overline{\psi_{(1,t/2)}(u(0, b, x, y))} f(u(0, b, x, y)g)dbdxdy \\
&= \int_{H_F^\circ \backslash H_A^\circ} \Big[\sum_{(X,Y) \in X_{1,F}} \Big\{ \int_{A_F/F} \psi((1/2)x(X, X)_W)dx \Big\} \\
&\quad \times \Big\{ \int_{A_E/E} \psi(\mathrm{Re}(b(X, Y)_W))db \Big\} \Big\{ \int_{A_F/F} \overline{\psi((1/2)y(t - (Y, Y)_W))}dy \Big\} \\
&\quad \times \omega_\psi(g \cdot h)\Phi(X, Y) \Big] \varphi(h)dh \\
&= \int_{H_F^\circ \backslash H_A^\circ} \Big\{ \sum_{(X,Y) \in \mathrm{Gr}(t)} \omega_\psi(g \cdot h)\Phi(X, Y) \Big\} \varphi(h)dh \\
&= \begin{cases} \int_{H_{t,F}^\circ \backslash H_A^\circ} \omega_\psi(g \cdot h)\Phi(0, (1/2)w_{-1} + tw_1)\varphi(h)dh & \text{if } t \neq 1 \\ \int_{H^\circ(w_0) \backslash H_A^\circ} \omega_\psi(g \cdot h)\Phi(0, w_0)\varphi(h)dh + \int_{Z_F^\circ \backslash H_A^\circ} \omega_\psi(g \cdot h)\Phi(w_1, w_0)\varphi(h)dh & \text{if } t = 1 . \end{cases}
\end{aligned}$$

If $t \neq 1$, we have

$$W_f^{\psi(1,t/2)}(g) = \int_{A_E/E} \overline{\psi(\mathrm{Tr}_{E/F}(a))}\, U_f^{\psi(1,t/2)}(u(a)g)da$$
$$= \left\{\int_{A_E/E} \overline{\psi(\mathrm{Tr}_{E/F}(a))}da\right\} U_f^{\psi(1,t/2)}(g) = 0 .$$

On the other hand, if $t = 1$, we have

$$\begin{aligned} W_f^{\psi(1,1/2)}(g) &= \int_{A_E/E} \overline{\psi(\mathrm{Tr}_{E/F}(a))}\, U_f^{\psi(1,1/2)}(u(a)g)da \\ &= \int_{A_E/E} \overline{\psi(\mathrm{Tr}_{E/F}(a))}\left\{\int_{H^\circ(w_0)\backslash H_A^\circ} \omega_\psi(u(a)g\cdot h)\Phi(0, w_0)\varphi(h)dh \right.\\ &\quad \left. + \int_{Z_F^\circ\backslash H_A^\circ} \omega_\psi(u(a)g\cdot h)\Phi(w_1, w_0)\varphi(h)dh\right\}da \\ &= \left\{\int_{A_E/E} \overline{\psi(\mathrm{Tr}_{E/F}(a))}da\right\}\left\{\int_{H^\circ(w_0)\backslash H_A^\circ} \omega_\psi(g\cdot h)\Phi(0, w_0)\varphi(h)dh\right\} \\ &\quad + \int_{A_E/E} \overline{\psi(\mathrm{Tr}_{E/F}(a))}\left\{\int_{Z_F^\circ\backslash H_A^\circ} \omega_\psi(g\cdot h)\Phi(w_1, aw_1 + w_0)\varphi(h)dh\right\}da \\ &= \int_{A_E/E} \overline{\psi(\mathrm{Tr}_{E/F}(a))}\left\{\int_{Z_A^\circ\backslash H_A^\circ} \omega_\psi(g\cdot h)\Phi(w_1, aw_1 + w_0)\varphi_0(h)dh\right\}da . \end{aligned}$$

If $\varphi \in L^2_{0,0}(H_A^\circ)$, then $\varphi_0 \equiv 0$. Thus $W_f^{\psi(1,1/2)} \equiv 0$. This proves the assertion (1).

On the other hand, if $\varphi \in L^2_{0,1}(H_A^\circ)$. then we have

$$\begin{aligned} &W_f^{\psi(1,1/2)}(g) \\ &= \int_{A_E/E} \overline{\psi(\mathrm{Tr}_{E/F}(a))}\left\{\int_{Z_A^\circ\backslash H_A^\circ} \omega_\psi(g\cdot m(-\bar{a})h)\Phi(w_1, w_0)\varphi_0(h)dh\right\}da \\ &= \int_{Z_A^\circ\backslash H_A^\circ} \omega_\psi(g\cdot h)\Phi(w_1, w_0)\left\{\int_{A_E/E} \overline{\psi(\mathrm{Tr}_{E/F}(a))}\varphi_0(m(a)h)da\right\}dh \\ &= \int_{Z_A^\circ\backslash H_A^\circ} w_\psi(g\cdot h)\Phi(w_1, w_0)\, W_\varphi^{\psi^\bullet}(h)dh . \end{aligned}$$

This proves the assertion (2). q.e.d.

Note that this theorem remains true without the assumption of the cuspidality of $\Theta(\tau, \psi)$.

By the verification similar to that for Theorem 3.1, we can show the following:

COROLLARY. *Suppose $V_\tau \subset L^2_{0,1}(H_A^\circ)$. If τ is non-trivial, then $\Theta(\tau, \psi)$ is also non-trivial.*

Finally, we compute the Fourier coefficient $V_f^{\psi(1,ti/2)}$. For each $\alpha \in F$ and $\varphi \in \mathscr{A}_0(H_A^\circ)$, we put

$$J_{\varphi}^{\psi_{\alpha}}(h) = \int_{A_F/F} \overline{\psi(\alpha x)}\varphi\left(\begin{pmatrix} 1 & 0 & xi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} h\right) dx \,.$$

Then, by Lemma 4.1. (4) and a simple calculation we can deduce the following:

PROPOSITION 4.4. *Let (τ, V_τ) be an irreducible cuspidal representation of H_A°. For any $f = f_\varphi^\Phi \in \Theta(\tau, \psi)$ and $t \in [F^*]$ we have*

$$V_f^{\psi(1, ti/2)}(g) = \int_{A_F} \int_{T_F^\circ \backslash H_A^\circ} \omega_\psi(g \cdot h)\Phi(tiw_1, w_{-1} + xw_1) J_\varphi^{\psi_{-2t^{-1}}}(h) dh dx \,.$$

Further, if V_τ satisfies the condition (♯), then $V_f^{\psi(1, ti/2)}$ vanishes for all $t \in [F^*]$.

Combining this proposition with Theorem 4.3, we obtain the following:

COROLLARY. *We assume that there exists a non-trivial irreducible cuspidal representation (τ, V_τ) of H_A° satisfying the condition (♯) in Corollary to Theorem 4.2. Then we have:*

(1) *If $V_\tau \subset L_{0,1}^2(H_A^\circ)$, then $\Theta(\tau, \psi)$ is N-cuspidal but not hypercuspidal.*

(2) *If $V_\tau \subset L_{0,0}^2(H_A^\circ)$, then $\Theta(\tau, \psi)$ is hypercuspidal.*

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