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COMPACT OPERATORS IN TYPE III_λ AND TYPE III_0 FACTORS, II

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1. Introduction and notations. In this paper we continue the program started in [8] of studing notions of compact operators in type III_λ $(0 \leqq \lambda < 1)$ factors. Given a type III_λ factor $\boldsymbol{M}$ operating on a separable Hilbert space $\boldsymbol{H}$, we represent it as the crossed product of a type II_∞ algebra $\boldsymbol{N}$ (a factor for $0<\lambda<1$ or an algebra with diffuse center for $\lambda=0$) by an automorphism θ that λ-scales a trace τ (i.e., $\tau\circ\theta = \lambda\tau$ for $0<\lambda<1$ or $\tau\circ\theta \leqq \lambda_0\tau$ with $\lambda_0 < 1$ for $\lambda = 0$). We embed $\boldsymbol{N}$ in $\boldsymbol{M}$ and let E be the canonical normal conditional expectation $E: \boldsymbol{M}\to\boldsymbol{N}$, u be the unitary operator implementing θ (i.e., $\mathrm{Ad}\, u = \theta$) such that $\{\boldsymbol{N}, u\}'' = \boldsymbol{M}$ and $\varphi = \tau\circ E$ be the dual weight of τ. Then φ is a lacunary weight, i.e., 1 is an isolated point in $\mathrm{Sp}\,\sigma^\varphi$, $\lambda_0 = \sup\{\lambda \in \mathrm{Sp}\,\sigma^\varphi \mid \lambda < 1\}$, $\boldsymbol{N}$ is the centralizer of φ and $\boldsymbol{M}\cap\boldsymbol{N}' = \boldsymbol{N}\cap\boldsymbol{N}'$. For further references see [2, §4, 5] and [16, §30.4].

In [8] we denoted by $\boldsymbol{I}(\boldsymbol{N})$ the two sided ideal of $\boldsymbol{N}$ generated by the finite projections of $\boldsymbol{N}$, by $\boldsymbol{J}(\boldsymbol{N})$ the norm closure of $\boldsymbol{I}(\boldsymbol{N})$ and we defined

$$\boldsymbol{I} = \mathrm{span}\{x\in\boldsymbol{M}^+ \mid E(x)\in\boldsymbol{I}(\boldsymbol{N})\}\,,$$
$$\boldsymbol{M}_\varphi = \mathrm{span}\{x\in\boldsymbol{M}^+ \mid \varphi(x) < \infty\}\,,$$
$$\boldsymbol{J} = \bar{\boldsymbol{I}} \quad \text{where the bar denotes the norm closure}\,.$$

We then obtained the embeddings for $0<\lambda<1$ [8, Theorem 6.2]

$$\boldsymbol{I}\subset\boldsymbol{M}_\varphi\subset\boldsymbol{J}$$

analogous to the classical embeddings of finite rank, trace-class and compact operator ideals. For the case $\lambda = 0$ we obtained a similar embedding involving the center of $\boldsymbol{N}$ [8, Corollary 6.5]. We then proved the generalization of several of the classical properties of compact operators, (Riesz, Calkin, Rellich and Hilbert properties [8, Theorem 5.3, Proposition 5.6]). In [8, Remark 4.6] we noticed that $\boldsymbol{J}$ is minimal among the C^*-algebras $\boldsymbol{C}\supset\boldsymbol{E}(\boldsymbol{C}) = \boldsymbol{J}(\boldsymbol{N})$ which are two sided $\boldsymbol{N}$-modules, while the maximal one is the algebra $\boldsymbol{K}$ given by:

DEFINITION 1.1. $\boldsymbol{K} = \mathrm{span}\{x\in\boldsymbol{M}^+ \mid E(x)\in\boldsymbol{J}(\boldsymbol{N})\}$.

By [8, Proposition 3.3], we have that $\boldsymbol{K}$ is a hereditary C^*-algebra, a two sided $\boldsymbol{N}$-module (actually a two sided module over span $N(E)$, where the latter is the normalizer group of the expectation E [8, Remark 3.4]) and by [8, Remark 4.6], $\boldsymbol{K}\cap\boldsymbol{N} = E(\boldsymbol{K}) = \boldsymbol{J}(\boldsymbol{N})$ and $\boldsymbol{J}\subset\boldsymbol{K}$. The hereditary algebras $\boldsymbol{I}$, $\boldsymbol{M}_\varphi$, $\boldsymbol{J}$ and $\boldsymbol{K}$ depend on the choice of the crossed product decomposition of $\boldsymbol{M}$ (or equivalently, on the choice of the lacunary weight φ) only up to inner automorphisms of $\boldsymbol{M}$ (this holds for $0 < \lambda < 1$; for $\lambda = 0$ an analogous condition involving central projections of $\boldsymbol{N}$ is given in [8, Remark 4.7]).

In § 2 we characterize the algebra $\boldsymbol{K}$ in terms of the essential central range of its elements. In particular we prove that $x \in \boldsymbol{K}^+$ if and only if $\boldsymbol{N}'$ meets the σ-weak closure of the convex hull of $\{vxv^* \mid v \text{ unitary in } \boldsymbol{N}\}$ only in $\{0\}$, and we discuss analogous conditions involving the closure in the uniform topology.

In § 3 we study the notion of θ-wandering projections in $\boldsymbol{N}$ (i.e., projections p such that $p\theta^n(p) = 0$ for all $n\neq 0$) and we prove that every nonzero projection majorizes a nonzero θ-wandering projection.

Using this notion we introduce in § 4 an isomorphism ρ of $\boldsymbol{B}(l^2(\mathbb{Z}))$ onto $\boldsymbol{D}\subset\boldsymbol{M}$ such that $E(\rho(a)) = \sum_{n=-\infty}^{\infty} a_{nn}\pi(\theta^n(p))$. This enables us to fully characterize in § 5 the positive part of the intersection of $\boldsymbol{D}$ with all the algebras introduced earlier ($\boldsymbol{I}$, $\boldsymbol{M}_\varphi$, $\boldsymbol{J}$ and $\boldsymbol{K}$) in terms of the matricial form of the elements of $\boldsymbol{B}(l^2(\mathbb{Z}))$. In particular we show that $\rho(a) \in \boldsymbol{J}^+$ if and only if the "upper left corner" of the (bi-infinite) matrix of a is compact in the usual sense. We prove also that in contradistinction to $\boldsymbol{B}(\boldsymbol{H})$ or to semifinite factors, the above listed algebras have properly different sets of projections. In particular this shows that $\boldsymbol{J} \neq \boldsymbol{K}$.

By exploiting module properties of $\boldsymbol{J}$ and $\boldsymbol{K}$ relative to the algebra generated by u (i.e., the algebra of Laurent operators tensored with 1) and some subalgebras of it, and by using some results on Toeplitz operators, we show in § 6 that $\boldsymbol{J}$ is not an ideal of $\boldsymbol{K}$.

2. The essential central range. In this section we are going to study a generalization to $\boldsymbol{M}$ of the following characterization of $\boldsymbol{J}(\boldsymbol{N})$. For every $x \in \boldsymbol{B}(\boldsymbol{H})$ define

$$K(x) = \overline{\mathrm{co}}\{\mathrm{Ad}\, v(x) \mid v \in \boldsymbol{U}(\boldsymbol{N})\}$$

to be the norm closure of the convex hull of the unitary orbit of x, where $\mathrm{Ad}\, v(x) = vxv^*$ and $\boldsymbol{U}(\boldsymbol{N})$ is the group of unitary elements of $\boldsymbol{N}$. Let also $C(x)$ be the σ-weak closure of $K(x)$. Then for all $x \in \boldsymbol{N}$ we have by [6, Theorem 4.12, Corollary 4.17] that $K(x)\cap\boldsymbol{N}' = C(x)\cap\boldsymbol{N}' = \{\omega(x) \mid \omega \text{ center-valued state on } \boldsymbol{N},\ \omega(\boldsymbol{J}(\boldsymbol{N})) = \{0\}\}$. Here a center-valued state is a positive bounded $N \cap N'$-module homomorphism of N onto the center of N with $\omega(1) = 1$. If N is a factor, then this notion coincides with the usual notion of state.

Thus $C(x) \cap N'$ is the essential central range of $x \in N$, and $x \in J(N)$ if and only if $C(x) \cap N' = \{0\}$. For further information on the notion of essential central range (modulo the ideal $J(N)$) we refer the reader to [6] and [7].

In order to simplify notations, let us define F to be the set of all finite-support functions $f: U(N) \to [0, 1]$ such that $\sum \{f(v) \mid v \in U(N)\} = 1$. Define an action of F on M by setting

$$f \cdot x = \sum \{f(v) \mathrm{Ad}\, v(x) \mid v \in U(N)\} .$$

Then f is a positive contraction, i.e., $\|f \cdot x\| \leqq \|x\|$ for all $x \in M$, and $f \cdot x \geqq 0$ for all $x \in M^+$. The norm closure (resp. the σ-weak closure) of $\{f \cdot x \mid f \in F\}$ coincides with $K(x)$ (resp. with $C(x)$). Explicitly, $y \in K(x)$ (resp. $y \in C(x)$) if and only if there is a sequence $f_n \in F$ such that $f_n \cdot x \to y$ in norm (resp. σ-weakly, using the metrizability of the unit ball).

Notice that if $x \in M$ and $y \in K(x)$ then $K(y) \subset K(x) \subset M$; E and $f \in F$ commute, i.e., $E(f \cdot x) = f \cdot E(x)$; f leaves $N \cap N'$ pointwise invariant and leaves every two sided N-module globally invariant (in particular N, I, M_φ, J and K). Finally, F is closed under composition, i.e., for all $f, g \in F$, $f \circ g$ is in F and coincides with the usual convolution product.

Recall that Dixmier [4, Théorème 1, Ch. III, § 5] proved for all von Neumann algebras N that $K(x) \cap N' \neq \emptyset$ for all $x \in N$ (Dixmier property) and Schwartz [15] defined and studied the algebras $N \subset B(H)$ for which $C(x) \cap N' \neq \emptyset$ for all $x \in B(H)$ (P-property). We need to generalize both properties.

DEFINITION 2.1. An embedding $A \subset B$ has:

(a) the relative Dixmier property if $K(x) \cap A' \neq \emptyset$ for all $x \in B$;

(b) the relative P-property if $C(x) \cap A' \neq \emptyset$ for all $x \in B$.

It is usually difficult to analyze the relative Dixmier property: recall for instance that the long standing pure state extension problem for $B(H)$ is equivalent to the relative Dixmier property for the embedding of the algebra of diagonal operators in $B(H)$ ([1], [9]). In our case, we can however prove the relative P-property.

THEOREM 2.2. *The embedding $N \subset M$ has the relative* P-*property.*

PROOF. Let A be a maximal abelian von Neumann subalgebra of N. Then by [2, 4.2.3], A is maximal abelian in M. Let $x \in M$ and let $C_A(x)$ be the σ-weak closure of the convex hull of $\{\mathrm{Ad}\, v(x) \mid v \in U(A)\}$. Then

$C_A(x)\subset C(x)\subset \boldsymbol{M}$ and $C_A(x)$ is bounded and hence σ-weakly compact. Therefore by the Markov-Kakutani fixed point theorem [17, Lemma A.1], $C_A(x)$ contains a point y fixed under all maps $\operatorname{Ad} v$, $v\in U(\boldsymbol{A})$ and hence belonging to $\boldsymbol{A}'$. But then, $y\in \boldsymbol{A}'\cap \boldsymbol{M}=\boldsymbol{A}\subset \boldsymbol{N}$ and hence because of the Dixmier property for $\boldsymbol{N}$, the set $K(y)\cap \boldsymbol{N}'$ is nonvoid. Since $y\in C(x)$, then $K(y)\subset C(x)$ and hence $\varnothing\neq K(y)\cap \boldsymbol{N}'\subset C(x)\cap \boldsymbol{N}'$. □

REMARK 2.3. In [12, Corollary 4.9] Longo has proved with different methods the same result for the case of the embedding of a separably operating *factor* $\boldsymbol{N}$ in its crossed product by a discrete group.

COROLLARY 2.4. *Let* $x\in \boldsymbol{M}$; *then*

$$C(x)\cap \boldsymbol{N}' = C(E(x))\cap \boldsymbol{N}' = K(E(x))\cap \boldsymbol{N}' .$$

PROOF. The second equality has been proven in [6, Corollary 4.17]. Let $z\in C(x)\cap \boldsymbol{N}'$. Then there is a sequence $f_n\in \boldsymbol{F}$ such that $f_n\cdot x\to z$ (σ-weakly). By the normality and hence σ-weak continuity of E and the fact that $z\in \boldsymbol{M}\cap \boldsymbol{N}'=\boldsymbol{N}\cap \boldsymbol{N}'$, we have that

$$f_n\cdot E(x) = E(f_n\cdot x)\to E(z) = z .$$

Thus $z\in C(E(x))\cap \boldsymbol{N}'$. Conversely, assume that $z\in C(E(x))\cap \boldsymbol{N}'$ and let $f_n\in \boldsymbol{F}$ be such that $f_n\cdot E(x)\to z$ (σ-weakly). Since $f_n\cdot x$ is bounded, we can assume, by passing to a subsequence if necessary, that $f_n\cdot x\to y$ (σ-weakly) for some $y\in C(x)$. Then again

$$f_n\cdot E(x)\to E(y) = z .$$

By Theorem 2.2, $C(y)\cap \boldsymbol{N}'\neq\varnothing$ and by the first part of this proof

$$C(y)\cap \boldsymbol{N}'\subset C(E(y))\cap \boldsymbol{N}' = C(z)\cap \boldsymbol{N}' = \{z\}$$

because the center of $\boldsymbol{N}$ is pointwise invariant under the action of $\boldsymbol{F}$. Thus $z\in C(y)\cap \boldsymbol{N}'\subset C(x)\cap \boldsymbol{N}'$. □

COROLLARY 2.5. *Let* $x\in \boldsymbol{M}$. *Then* $C(x)\cap \boldsymbol{N}' = \{\omega(x)\,|\,\omega$ *is an* $\boldsymbol{N}\cap \boldsymbol{N}'$-*valued positive module homomorphism on* $\boldsymbol{M}$, *with* $\omega(1)=1$, $\omega=\omega\circ E$ *and* $\omega(\boldsymbol{J})=\{0\}\}$.

PROOF. From Corollary 2.4 we have that $C(x)\cap \boldsymbol{N}' = \{\tilde{\omega}(E(x))\,|\,\tilde{\omega}$ is a center-valued state on $\boldsymbol{N}$, $\tilde{\omega}(\boldsymbol{J}(\boldsymbol{N}))=\{0\}\}$. Let $\tilde{\omega}$ be a center-valued state on $\boldsymbol{N}$ vanishing on $\boldsymbol{J}(\boldsymbol{N})$ and let $\omega=\tilde{\omega}\circ E$ be its extension to $\boldsymbol{M}$; then ω is an $\boldsymbol{N}\cap \boldsymbol{N}'$-valued positive module homomorphism on $\boldsymbol{M}$, with $\omega(1)=1$ and $\omega=\omega\circ E$. For every $x\in \boldsymbol{J}^+$, there is a $y\in \boldsymbol{J}(\boldsymbol{N})$ such that $x\leqq y$ [8, Theorem 4.3.(b)]; therefore,

$$0\leqq \omega(x)\leqq \omega(y) = \tilde{\omega}(y) = 0 .$$

As $J = \operatorname{span} J^+$, we thus have $\omega(J) = \{0\}$. Conversely, if ω is as in the statement of the Corollary, its restriction $\tilde{\omega}$ to N is a center-valued state on N, $\tilde{\omega}(J(N)) = \{0\}$ and

$$\omega(x) = \omega(E(x)) = \tilde{\omega}(E(x)) . \qquad \square$$

Thus $C(x) \cap N'$ *is an essential central range of* x. In particular for $0 < \lambda < 1$ the center of N is trivial, center-valued states are simply states and the essential central range is an essential numerical range. It is thus natural to investigate the class of elements x of M with $C(x) \cap N' = \{0\}$. As we have already mentioned, this condition for N characterizes the class of compact operators $J(N)$. In M^+ it characterizes K^+.

THEOREM 2.6. *Let $x \in M^+$. Then $x \in K^+$ if and only if $C(x) \cap N' = \{0\}$.*

PROOF. We have that $x \in K^+$ if and only if $E(x) \in J(N)$ if and only if $C(E(x)) \cap N' = \{0\}$ if and only if $C(x) \cap N' = \{0\}$ (by Corollary 2.4). $\square$

The proof actually shows that for all $x \in M$, $E(x) \in J(N)$ if and only if $C(x) \cap N' = \{0\}$. The class characterized by this condition is, however, much too large to be of interest as it includes all the elements x with $E(x) = 0$. Let us collect here for ease of reference some facts about K.

PROPOSITION 2.7.

(a) *K is a hereditary C^*-subalgebra of M and a two sided N-module.*

(b) $K = \operatorname{span} K^+ = \{x \in M \mid E(xx^* + x^*x) \in J(N)\}$.

(c) *K is globally invariant under the action of F.*

(d) *$I \subset K$, hence $J \subset K$.*

(e) $K \cap N = E(K) = J(N)$.

(f) *$N + K = \{x \in M \mid x - E(x) \in K\}$ is a C^*-algebra with two sided ideal K and $(N + K)/K$ is isomorphic to the generalized Calkin algebra $N/J(N)$.*

(g) *J is minimal and K is maximal among the hereditary C^*-algebras C such that $E(C) = J(N)$.*

PROOF. (a) and (b) follow from [8, Proposition 3.3], (c) is a consequence of (a), while (d) and (e) follow immediately from the definition. The proof of (f) is essentially identical to the proof of [8, Proposition 4.5] and (g) follows from [8, Proposition 4.5 and Remark 4.6]. $\square$

While for $x \in N$ we know that $K(x) \cap N' = C(x) \cap N'$, this is no longer obvious for $x \in M$ and therefore we have to investigate the set $K(x) \cap N'$ independently. Notice however that the above equality would hold also for every x in M if we knew that the embedding $N \subset M$ had the relative Dixmier property (see next lemma, part (a)).

LEMMA 2.8. *Let* $x \in \boldsymbol{M}$. *Then*
(a) *if* $K(f \cdot x) \cap \boldsymbol{N}' \neq \emptyset$ *for all* $f \in \boldsymbol{F}$, *then* $K(x) \cap \boldsymbol{N}' = C(x) \cap \boldsymbol{N}'$;
(b) *if* $0 \in K(f \cdot x)$ *for all* $f \in \boldsymbol{F}$, *then* $K(x) \cap \boldsymbol{N}' = \{0\}$.

PROOF. (a) Let $z \in C(x) \cap \boldsymbol{N}'$. Then $z \in K(E(x)) \cap \boldsymbol{N}'$ (Corollary 2.4) and thus for every $\varepsilon > 0$ there is an $f \in \boldsymbol{F}$ such that $\| f \cdot E(x) - z \| < \varepsilon$. By hypothesis there is a $z' \in K(f \cdot x) \cap \boldsymbol{N}'$ and hence a $g \in \boldsymbol{F}$ such that $\|(g \circ f) \cdot x - z'\| < \varepsilon$. Therefore we obtain

$$\begin{aligned}\|(g \circ f) \cdot x - z\| &\leqq \|(g \circ f) \cdot x - z'\| + \|z' - (g \circ f) \cdot E(x)\| + \|(g \circ f) \cdot E(x) - z\| \\ &< \varepsilon + \|E((g \circ f) \cdot x - z')\| + \|g \cdot (f \cdot E(x) - z)\| < 3\varepsilon\end{aligned}$$

by using the facts that E commutes with the action of $\boldsymbol{F}$, $E(z') = z'$, $g \cdot z = z$ and that both E and g are contractions. Thus $z \in K(x) \cap \boldsymbol{N}'$. The opposite inclusion follows from $K(x) \subset C(x)$.

(b) Let $z \in K(x) \cap \boldsymbol{N}'$, let $\varepsilon > 0$ and let $f \in \boldsymbol{F}$ be such that $\| f \cdot x - z \| < \varepsilon$. By hypothesis there is a $g \in \boldsymbol{F}$ such that $\|(g \circ f) \cdot x\| < \varepsilon$. Thus

$$\|z\| \leqq \|(g \circ f) \cdot x\| + \|g \circ (f \cdot x - z)\| < 2\varepsilon$$

by the same reasoning as in (a). Consequently, $z = 0$. Also, by (a) and by Theorem 2.2, $K(x) \cap \boldsymbol{N}' \neq \emptyset$. □

PROPOSITION 2.9. *Let* $x \in \boldsymbol{M}^+$; *then* $0 \in K(E(x))$ *if and only if* $0 \in K(x)$.

PROOF. The condition is sufficient, even for a nonpositive x, by Corollary 2.4 and the inclusion $K(x) \subset C(x)$. Assume now that $0 \in K(E(x))$. Then 0 is in the central convex hull of the essential central spectrum of $E(x)$ [6, Theorem 4.4]. Since $E(x) \geqq 0$, 0 belongs also to the essential central spectrum of $E(x)$ [6, Proposition 3.12]. Hence we can apply [7, Theorem 2.10] to the case of the (central) ideal $\boldsymbol{J}(\boldsymbol{N})$ of $\boldsymbol{N}$ and thus we can find a sequence of mutually orthogonal equivalent projections $p_n \in \boldsymbol{N}$ with central support 1, such that $\| p_n E(x) p_n \| < 2^{-n}$. By passing if necessary to subprojections, we can assume that $\tau(p_n) < \infty$. Let $p = \sum_{n=1}^{\infty} p_n$. Then p is properly infinite, $p \sim 1$ and

$$\varphi(pxp) = \tau(pE(x)p) = \sum_{n=1}^{\infty} \tau(p_n E(x) p_n) \leqq \sum_{n=1}^{\infty} 2^{-n} \tau(p_n) = \tau(p_1) < \infty .$$

Therefore $pxp \in \boldsymbol{M}_\varphi \subset \boldsymbol{J}$ and hence there is a $y \in \boldsymbol{J}(\boldsymbol{N})$ such that $pxp \leqq y$ by [8, Theorems 6.2 and 4.3(b)]. Thus

$$x \leqq 2(pxp + (1 - p)x(1 - p)) \leqq 2y + 2\|x\|(1 - p) .$$

Let $\varepsilon > 0$ and let $1/n < \varepsilon$. Because $p \sim 1$ and $\boldsymbol{N}$ is properly infinite, we can find as in the proof of [5, Proposition 5] n unitary operators $u_i \in \boldsymbol{N}$ such that $\{u_i(1 - p)u_i^* \mid i = 1, \cdots, n\}$ are mutually orthogonal. Let $f \in \boldsymbol{F}$

be such that $f\cdot z = (1/n)\sum_{i=1}^{n} u_i z u_i^*$ for all $z \in \boldsymbol{M}$. Then

$$\|f\cdot(1-p)\| = (1/n)\left\|\sum_{i=1}^{n} u_i(1-p)u_i^*\right\| \leqq (1/n)\sup\|u_i(1-p)u_i^*\| \leqq 1/n < \varepsilon .$$

As $f\cdot y \in \boldsymbol{J}(\boldsymbol{N})$ we have that $0 \in K(f\cdot y)$ and hence there is a $g \in \boldsymbol{F}$ such that $\|(g\circ f)\cdot y\| < \varepsilon$. By the order preserving property of the action of $g\circ f$, we have that

$$\|(g\circ f)\cdot x\| \leqq 2\|(g\circ f)\cdot y\| + 2\|x\|\,\|(g\circ f)\cdot(1-p)\| \leqq 2(1+\|x\|)\varepsilon .$$

Therefore $0 \in K(x)$. □

THEOREM 2.10. *Let* $x \in \boldsymbol{M}^+$. *Then the following conditions are equivalent*:

(a) $x \in \boldsymbol{K}^+$,

(b) $0 \in K(f\cdot x)$ *for all* $f \in \boldsymbol{F}$,

(c) $K(y) \cap \boldsymbol{N}' = \{0\}$ *for all* $y \in K(x)$.

PROOF. (a)$\Rightarrow$(b) Assume that $x \in \boldsymbol{K}^+$ and let $f \in \boldsymbol{F}$. Then $f\cdot x \in K^+$ by Proposition 2.7 (c); but then $0 \in K(E(f\cdot x))$ and hence $0 \in K(f\cdot x)$ by Proposition 2.9.

(b)$\Rightarrow$(c) Let $y \in K(x)$ and let $\varepsilon > 0$. Then there is an $f \in \boldsymbol{F}$ such that $\|f\cdot x - y\| < \varepsilon$. Choose any $g \in \boldsymbol{F}$; then by hypothesis $0 \in K((g\circ f)\cdot x)$. Hence, there is an $h \in \boldsymbol{F}$ such that $\|(h\circ g\circ f)\cdot x\| < \varepsilon$. Thus, we obtain

$$\|h\cdot(g\cdot y)\| \leqq \|(h\circ g)\cdot(f\cdot x - y)\| + \|(h\circ g\circ f)\cdot x\| < 2\varepsilon ,$$

whence $0 \in K(g\cdot y)$. From Lemma 2.8 (b) it follows that $K(y) \cap \boldsymbol{N}' = \{0\}$.

Clearly (c)$\Rightarrow$(b).

(b)$\Rightarrow$(a) By Lemma 2.8 (a) and (b), we have $\{0\} = K(x) \subset \boldsymbol{N}' = C(x) \cap \boldsymbol{N}'$. Therefore $x \in \boldsymbol{K}^+$ by Theorem 2.6. □

Notice that the equivalence of (b) and (c) holds also for a nonpositive x. If $\boldsymbol{N} \subset \boldsymbol{M}$ had the relative Dixmier property, then (b) and (c) would also be equivalent to the condition $K(x) \cap \boldsymbol{N}' = C(x) \cap \boldsymbol{N}' = \{0\}$, which by Corollary 2.4, is equivalent to $C(E(x)) \cap \boldsymbol{N}' = \{0\}$ and hence to $E(x) \in \boldsymbol{J}(\boldsymbol{N})$. This leads us to study the class:

DEFINITION 2.11. $\boldsymbol{K}^\sim = \{x \in \boldsymbol{M} \mid K(y) \cap \boldsymbol{N}' = \{0\}$ for all $y \in K(x)\}$.

As noted above, $\boldsymbol{K}^\sim = \{x \in \boldsymbol{M} \mid 0 \in K(f\cdot x)$ for all $f \in \boldsymbol{F}\}$. In the next proposition we shall see that $\boldsymbol{K}^\sim$ satisfies a form of the Weyl Perturbation Theorem [10, Theorem 3.3].

PROPOSITION 2.12. (a) $\boldsymbol{K}^\sim$ *is a selfadjoint Banach space containing* $\boldsymbol{K}$,

(b) $K(x+y) \cap \boldsymbol{N}' = K(x) \cap \boldsymbol{N}'$ *for all* $x \in \boldsymbol{M}$ *and* $y \in \boldsymbol{K}^\sim$.

PROOF. (a) Let $x_1, x_2 \in \boldsymbol{K}^\sim$, $f \in \boldsymbol{F}$ and $\varepsilon > 0$. Then there are $g_1, g_2 \in \boldsymbol{F}$ such that $\|(g_1 \circ f)\cdot x_1\| < \varepsilon$, $\|(g_2 \circ g_1 \circ f)\cdot x_2\| < \varepsilon$ and hence $\|(g_2 \circ g_1 \circ f)\cdot(x_1 + x_2)\| < 2\varepsilon$. Therefore $0 \in K(f\cdot(x_1 + x_2))$ and thus $x_1 + x_2 \in \boldsymbol{K}^\sim$. Clearly $\alpha x \in \boldsymbol{K}^\sim$ and $x^* \in \boldsymbol{K}^\sim$ for all $\alpha \in \mathbb{C}$ and $x \in \boldsymbol{K}^\sim$, so that $\boldsymbol{K}^\sim$ is a linear, selfadjoint space. As $(\boldsymbol{K}^\sim)^+ = \boldsymbol{K}^+$ by Theorem 2.10, $\boldsymbol{K}^\sim \supset \boldsymbol{K}$ by Proposition 2.7 (b). Let now x be in the norm closure of $\boldsymbol{K}^\sim$ and choose any $f \in \boldsymbol{F}$. For every $\varepsilon > 0$ there is a $y \in \boldsymbol{K}^\sim$ such that $\|x - y\| < \varepsilon$, and since $0 \in K(f\cdot y)$, there is a $g \in \boldsymbol{F}$ such that $\|(g \circ f)\cdot y\| < \varepsilon$. But then

$$\|g\cdot(f\cdot x)\| \leqq \|(g \circ f)\cdot y\| + \|(g \circ f)\cdot(x - y)\| < 2\varepsilon .$$

Therefore $0 \in K(f\cdot x)$ and hence $x \in \boldsymbol{K}^\sim$.

(b) Since $-y \in \boldsymbol{K}^\sim$, it is enough to prove that $K(x) \cap \boldsymbol{N}' \subset K(x + y) \cap \boldsymbol{N}'$. Let $z \in K(x) \cap \boldsymbol{N}'$ and let $\varepsilon > 0$; then there are $f, g \in \boldsymbol{F}$ such that $\|f\cdot x - z\| < \varepsilon$ and $\|(g \circ f)\cdot y\| < \varepsilon$. Thus, we obtain

$$\|(g \circ f)\cdot(x + y) - z\| \leqq \|g\cdot(f\cdot x - z)\| + \|(g \circ f)\cdot y\| < 2\varepsilon ,$$

whence $z \in K(x + y)$. □

We have shown in [9, Theorem 3.5], that if a unitary $v \in \boldsymbol{M}$ implements a properly outer automorphism of $\boldsymbol{N}$, then v belongs to $\boldsymbol{K}^\sim$. Thus in particular, we have that $u \in \boldsymbol{K}^\sim$.

3. Wandering projections.

In this section we let $\boldsymbol{N}$ be any countably decomposable von Neumann algebra with a given faithful semifinite normal trace τ (f.s.n. for short) and scaling automorphism θ (i.e., $\tau \circ \theta \leqq \lambda_0 \tau$ for some fixed $0 < \lambda_0 < 1$). In particular the results of this section will apply to the algebra $\boldsymbol{N}$ of the rest of this paper.

DEFINITION 3.1. (a) A nonzero projection $p \in \boldsymbol{N}$ is called a *θ-wandering projection* (or simply a wandering projection) if $p\theta^n(p) = 0$ for all nonzero integers n.

(b) Let $q \in \boldsymbol{N}$ be a projection. Then we call *θ-span of q* the projection $q_\theta = \sup\{\theta^n(q) \mid n \in \mathbb{Z}\}$.

Let us collect in the following lemma some simple facts about wandering projections and θ-spans.

LEMMA 3.2. (a) *A nonzero projection p is wandering if and only if $p\theta^n(p) = 0$ for all positive integers.*

(b) *For every projection q in $\boldsymbol{N}$, $q_\theta \in \boldsymbol{N}^\theta = \{x \in \boldsymbol{N} \mid x = \theta(x)\}$ and q is wandering if and only if it is nonzero and $q_\theta = \sum_{n=-\infty}^{\infty} \theta^n(q)$.*

(c) *Nonzero subprojections of wandering projections are wandering.*

(d) *For all projections p, q in $\boldsymbol{N}$, $pq_\theta = 0$ if and only if $p_\theta q_\theta = 0$.*

(e) *The sum of wandering projections is wandering if and only if* *their θ-spans are mutually orthogonal.*

THEOREM 3.3. *Every nonzero projection of N majorizes a wandering projection.*

PROOF. Let q be a nonzero projection. By the semifiniteness of τ, we can assume without loss of generality that $\tau(q) < \infty$. Let k be a positive integer such that $\lambda_0^{k+1} \leqq (1 - \lambda_0)/2$. Let us denote by $l(x)$ the left support of x, i.e., the range projection of x. Define $p_0 = q$,

$$p_j = p_{j-1} - l(p_{j-1}\theta^j(p_{j-1})) \quad \text{for} \quad j = 1, 2, \cdots, k$$
$$p = p_k - l(p_k \sup\{\theta^j(p_k) \mid j \geqq k + 1\}) .$$

By construction $p \leqq p_k \leqq p_{k-1} \leqq \cdots \leqq p_0 = q$. Since $p(p_k \sup\{\theta^j(p_k) \mid j \geqq k+1\}) = 0$, we have that $p\theta^j(p_k) = 0$, hence $p\theta^j(p) = 0$ for $j \geqq k+1$. Similarly, for $j = 1, 2, \cdots, k$ we have that $p_j\theta^j(p_j) = 0$, hence $p\theta^j(p_j) = 0$, and thus $p\theta^j(p) = 0$. Therefore $p\theta^j(p) = 0$ for all $j > 0$ and hence for all $j \neq 0$. We have to prove now that $p \neq 0$. Recall that $l(x) \sim l(x^*)$ for all $x \in N$. Then

$$\tau(l(p_{j-1}\theta^j(p_{j-1}))) = \tau(l(\theta^j(p_{j-1})p_{j-1})) \leqq \tau(\theta^j(p_{j-1})) \leqq \lambda_0^j\tau(p_{j-1}) .$$

Therefore,

$$\tau(p_j) \geqq \tau(p_{j-1}) - \lambda_0^j\tau(p_{j-1}) = (1 - \lambda_0^j)\tau(p_{j-1})$$

and hence $\tau(p_k) \geqq \alpha\tau(q)$ where $\alpha = (1 - \lambda_0)(1 - \lambda_0^2) \cdots (1 - \lambda_0^k)$. Similarly,

$$\begin{aligned}
&\tau(l(p_k \sup\{\theta^j(p_k) \mid j \geqq k + 1\})) \\
&\qquad \leqq \tau(\sup\{\theta^j(p_k) \mid j \geqq k + 1\}) \leqq \sum \{\tau(\theta^j(p_k)) \mid j \geqq k + 1\} \\
&\qquad \leqq \sum \{\lambda_0^j\tau(p_k) \mid j \geqq k + 1\} \leqq \tau(p_k)/2 .
\end{aligned}$$

Thus

$$\tau(p) = \tau(p_k) - \tau(l(p_k \sup\{\theta^j(p_k) \mid j \geqq k + 1\})) \geqq \tau(p_k)/2 \geqq \alpha\tau(q)/2 > 0 ,$$

whence $p \neq 0$. □

COROLLARY 3.4. *Every nonzero projection $q \in N$ with finite trace majorizes a wandering projection $p \in \hat{N} = \{\theta^n(q) \mid n \in \mathbb{Z}\}'' \subset q_\theta N q_\theta$.*

PROOF. It is easy to see that $\hat{N}$ is θ-invariant and contained in $q_\theta N q_\theta$; thus the restriction of θ to $\hat{N}$ is an automorphism. Since the generators of $\hat{N}$ have all finite trace, as $\tau(\theta^n(q)) \leqq \lambda_0^n\tau(q) < \infty$, the restriction of τ to $\hat{N}$ is semifinite. Clearly it is also faithful, normal and scaled by θ. Thus Theorem 3.3 applied to $\hat{N}$ guarantees that the wandering projection p is in $\hat{N}$. □

Notice that if $\tau(q) = \infty$, the restriction of τ to $\hat{N}$ may not be semifinite and then $\hat{N}$ may fail to contain any wandering projections. As an example, consider a projection $0 \neq q \in N$ such that $q\theta(q) = 0$ and $\theta^2(q) = q$; then $\tau(q) = \infty$ and $\hat{N} = \mathbb{C}q \oplus \mathbb{C}\theta(q)$ does not contain wandering projections.

The following proposition will be used in the next section.

PROPOSITION 3.5. *Let q be a nonzero projection of N^θ (i.e., $\theta(q) = q$). Then there is a wandering projection p with finite trace such that $q = p_\theta$.*

PROOF. Let $\{p_i \mid i = 1, 2, \cdots, n \leqq \infty\}$ be a maximal family (at most countable since H is separable) of wandering projections majorized by q and having mutually orthogonal θ-spans and finite traces. Since $p_i \leqq q$, we have $(p_i)_\theta \leqq q_\theta = q$. Let $q_0 = q - \sum_{i=1}^n (p_i)_\theta$. If $q_0 \neq 0$, then by Theorem 3.3 and Lemma 3.2 (c) there is a wandering projection $p_0 \leqq q_0$ with finite trace. Since $q_0 \in N^\theta$, it follows that $(p_0)_\theta \leqq q_0$ and hence $(p_0)_\theta$ is orthogonal to $\sum_{i=1}^n (p_i)_\theta$, contradicting the maximality of the family (see Lemma 3.2 (e)). Thus $q = \sum_{i=1}^n (p_i)_\theta$. Choose now for each i an integer $m(i)$ such that $\tau(\theta^{m(i)}(p_i)) \leqq 2^{-i}$ and let $p = \sum_{i=1}^n \theta^{m(i)}(p_i)$. Then p has finite trace, $p \leqq q$ and p is wandering (Lemma 3.2 (e)). Finally, we have

$$p_\theta = \sum_{j=-\infty}^{\infty} \theta^j\Big(\sum_{i=1}^{n} \theta^{m(i)}(p_i)\Big) = \sum_{i=1}^{n} \sum_{j=-\infty}^{\infty} \theta^{j+m(i)}(p_i) = \sum_{i=1}^{n} \sum_{j=-\infty}^{\infty} \theta^j(p_i) = \sum_{i=1}^{n} (p_i)_\theta$$
$$= q . \qquad \square$$

REMARK 3.6. (a) Assume that N is a continuous algebra. Then the wandering projection p such that $p_\theta = q$ can be chosen to have infinite trace.

Indeed, by decomposing if necessary the wandering projection p_i in the proof of Proposition 3.5 into infinitely many subprojections, and by using Lemma 3.2 (c) and (e), we can assume that the maximal family $\{p_i\}$ constructed in the above proof is infinite. Since $\tau \circ \theta^{-1} \geqq \lambda^{-1}\tau$, we can choose integers $m(i)$ so that $\tau(\theta^{m(i)}(p_i)) \geqq 1$ and define $p = \sum_{i=1}^\infty \theta^{m(i)}(p_i)$. Then $\tau(p) = \infty$ and, as in the above proof, we see that p is wandering and $p_\theta = q$.

(b) Assume furthermore that $\tau \circ \theta = \lambda\tau$. Then, for any preassigned number $\alpha > 0$, the wandering projection p such that $p_\theta = q$ can be chosen to have trace $\tau(p) = \alpha$.

Indeed, by (a) we can first find a wandering projection r with infinite trace, such that $r_\theta = q$. We then decompose r into an infinite sum of mutually orthogonal projections p_i, $i = 0, 1, \cdots$ with trace $\alpha(1 - \lambda)$ and we define $p = \sum_{i=0}^\infty \theta^i(p_i)$. Then $\tau(p) = \alpha$ and the same argument as above shows that $p_\theta = q$.

(c) If $\boldsymbol{N}$ is not a continuous algebra, then the properties in Remarks 3.6(a) and (b) may be false.

Indeed, consider $\boldsymbol{N} = l^\infty(\mathbb{Z})$, with the canonical basis $\{m_n \mid n \in \mathbb{Z}\}$ of rank one projections. Let the automorphism θ and the trace τ be defined by $\theta(m_n) = m_{n+1}$ and $\tau(m_n) = \lambda^n$ for all $n \in \mathbb{Z}$. Then $\tau \circ \theta = \lambda\tau$, but the set of wandering projections of $\boldsymbol{N}$ is $\{m_n \mid n \in \mathbb{Z}\}$ and hence neither (a) nor (b) is true.

Another way of generating wandering projections is the following generalization of a technique used by Dye for abelian algebras [18, Lemma 8.8].

PROPOSITION 3.7. *Let q be a projection of $\boldsymbol{N}$ with finite trace. Then there is a wandering projection p with finite trace such that $q \leqq \sum_{n=0}^\infty \theta^n(p)$.*

PROOF. Let $r = \sup\{\theta^n(q) \mid n \geqq 0\}$. Then

$$\tau(r) \leqq \sum_{n=1}^\infty \tau(\theta^n(q)) \leqq \sum_{n=0}^\infty \lambda_0^n \tau(q) = (1-\lambda_0)^{-1}\tau(q) < \infty .$$

Clearly $\theta(r) \leqq r$ and thus $\{\theta^n(r) \mid n \geqq 0\}$ is monotone decreasing, whence it is easy to verify that $p = r - \theta(r)$ is wandering. Now $r \geqq \theta^n(r) \geqq \theta^n(p)$ for $n \geqq 0$, hence $r \geqq \sum_{n=0}^\infty \theta^n(p)$. But $\sum_{k=0}^{n-1} \theta^k(p) = r - \theta^n(r)$, and hence

$$\tau\Big(\sum_{n=0}^\infty \theta^n(p)\Big) = \lim \tau(r - \theta^n(r)) \geqq \lim(1-\lambda_0^n)\tau(r) = \tau(r) ,$$

whence $\tau(r - \sum_{n=0}^\infty \theta^n(p)) = 0$. Therefore $q \leqq r = \sum_{n=0}^\infty \theta^n(p)$. □

Notice also that for abelian algebras, the wandering projection p constructed in Proposition 3.7 also satisfies $p \leqq q$ since then

$$r = \sup\{q, \theta(r)\} = q + \theta(r) - q\theta(r)$$

implies

$$p = r - \theta(r) = q(1 - \theta(r)) \leqq q .$$

4. Type I subfactors of $\boldsymbol{M}$. For the rest of this paper, we use explicitly the discrete crossed product decomposition of $\boldsymbol{M} = \boldsymbol{N} \otimes_\theta \mathbb{Z}$ where θ is a (properly outer) automorphism that scales the trace τ of $\boldsymbol{N}$. If $\boldsymbol{N}$ acts on the separable Hilbert space $\boldsymbol{H}$, then $\boldsymbol{M}$ acts on $\boldsymbol{H} \otimes l^2(\mathbb{Z})$ which we identify with $l^2(\boldsymbol{H}, \mathbb{Z})$ via the correspondence $(\zeta \otimes \eta)(n) = \eta(n)\zeta$ for $\zeta \in \boldsymbol{H}$, $\eta \in l^2(\mathbb{Z})$ and $n \in \mathbb{Z}$. We shall henceforth distinguish between $\boldsymbol{N}$ and its isomorphic image $\pi(\boldsymbol{N}) \subset \boldsymbol{M}$, where for all $x \in \boldsymbol{N}$, $\pi(x)$ is defined by:

$$(\pi(x)\xi)(n) = \theta^{-n}(x)\xi(n) \quad \text{for all} \quad \xi \in l^2(\boldsymbol{H}, \mathbb{Z}) \text{ and } n \in \mathbb{Z} .$$

Recall that the unitary operator u which, together with $\pi(\boldsymbol{N})$, generates $\boldsymbol{M}$ is given by $u = 1 \otimes w$, where w is the bilateral shift on $l^2(\mathbb{Z})$, i.e., $(u\xi)(n) = \xi(n-1)$ for all $\xi \in l^2(\boldsymbol{H}, \mathbb{Z})$ and $n \in \mathbb{Z}$. Recall also the covariance formula

$$\operatorname{Ad} u(\pi(x)) = \pi(\theta(x)) \quad \text{for all} \quad x \in \boldsymbol{N}$$

and the characterization of $\boldsymbol{M}$ as

$$\boldsymbol{M} = \{x \in \boldsymbol{N} \otimes \boldsymbol{B}(l^2(\mathbb{Z})) \,|\, (\theta \otimes \operatorname{Ad} w^{-1})(x) = x\} .$$

For these and further properties of crossed products, see [3], [17].

For the remainder of this section, let $p \in \boldsymbol{N}$ be a wandering projection with finite trace such that $p_\theta = \sum_{n=-\infty}^{\infty} \theta^n(p) = 1$ (see Proposition 3.5). Define $p_i = \pi(\theta^i(p))$ for all $i \in \mathbb{Z}$. A useful tool for studying $\boldsymbol{M}$ is given by the following embedding of type I factors in $\boldsymbol{M}$.

DEFINITION 4.1. Let $\rho\colon \boldsymbol{B}(l^2(\mathbb{Z})) \to \boldsymbol{N} \otimes \boldsymbol{B}(l^2(\mathbb{Z}))$ be defined by

$$\rho(a) = \sum_{n=-\infty}^{\infty} \theta^{-n}(p) \otimes \operatorname{Ad} w^n(a) \quad \text{for every} \quad a \in \boldsymbol{B}(l^2(\mathbb{Z})) .$$

REMARK 4.2. Since the projections $\theta^{-n}(p)$ are mutually orthogonal and $\|\operatorname{Ad} w^n(a)\| = \|a\|$, we see that the series converges in the strong topology and thus $\rho(a)$ belongs to $\boldsymbol{N} \otimes \boldsymbol{B}(l^2(\mathbb{Z}))$. We actually have more: the convergence is unconditional, in the sense that the net of the finite partial sums converges strongly to $\rho(a)$. Notice in particular that if $\zeta \in \boldsymbol{H}$, $\eta \in l^2(\mathbb{Z})$ and $k \in \mathbb{Z}$ then

$$(\rho(a)\zeta \otimes \eta)(k) = \sum_{n=-\infty}^{\infty} (\operatorname{Ad} w^n(a)\eta)(k)\theta^{-n}(p)\zeta$$

where the convergence is unconditional in the strong topology of $\boldsymbol{H}$.

For every $a \in \boldsymbol{B}(l^2(\mathbb{Z}))$ let $[a_{ij}]$ be the matrix representation of a with respect to the canonical basis $\{\mu_i \,|\, i \in \mathbb{Z}\}$ of $l^2(\mathbb{Z})$ and let $\{m_i \,|\, i \in \mathbb{Z}\}$ be the corresponding canonical decomposition of the identity in rank one diagonal projections. Then we have:

THEOREM 4.3. (a) *ρ is a normal isomorphism of $\boldsymbol{B}(l^2(\mathbb{Z}))$ into $\boldsymbol{M}$.*
(b) *$\rho(w) = u$ and $\rho(m_i) = p_i$ for all $i \in \mathbb{Z}$.*
(c) *$E(\rho(a)) = \sum_{n=-\infty}^{\infty} a_{nn} p_n$ for all $a \in \boldsymbol{B}(l^2(\mathbb{Z}))$.*

PROOF. (a) Given the unconditional strong convergence of the series, it is easy to verify that ρ is indeed a *-isomorphism and hence an isometry. Let $a, a_\gamma \in \boldsymbol{B}(l^2(\mathbb{Z}))$ and assume that a_γ is increasing to a. Then for every $k \in \mathbb{Z}$, $\zeta_k \in \theta^{-k}(p)\boldsymbol{H}$ and $\eta \in l^2(\mathbb{Z})$ we have:

$$\begin{aligned}(\rho(a) - \rho(a_\gamma))(\zeta_k \otimes \eta) &= \sum_{n=-\infty}^{\infty} \theta^{-n}(p)\zeta_k \otimes \operatorname{Ad} w^n(a - a_\gamma)\eta \\ &= \zeta_k \otimes \operatorname{Ad} w^k(a - a_\gamma)\eta \to 0\end{aligned}$$

in the strong topology. Since the span of the vectors $\zeta_k \otimes \eta$ is dense (by definition) in $\boldsymbol{H} \otimes l^2(\mathbb{Z})$ and since $\rho(a) - \rho(a_\gamma)$ is bounded by $2\|a\|$, we see that $\rho(a_\gamma) \to \rho(a)$, which proves the normality of ρ. Moreover, for all $a \in \boldsymbol{B}(l^2(\mathbb{Z}))$, we have by the normality of $\theta \otimes \operatorname{Ad} w^{-1}$ that

$$(\theta \otimes \operatorname{Ad} w^{-1})(\rho(a)) = \sum_{n=-\infty}^{\infty} \theta(\theta^{-n}(p)) \otimes \operatorname{Ad} w^{-1}(\operatorname{Ad} w^n(a)) = \rho(a) ,$$

whence by the above mentioned characterization of $\boldsymbol{M}$, we see that $\rho(a) \in \boldsymbol{M}$.

(b) We have that

$$\rho(w) = \sum_{n=-\infty}^{\infty} \theta^{-n}(p) \otimes \operatorname{Ad} w^n(w) = \sum_{n=-\infty}^{\infty} \theta^{-n}(p) \otimes w = 1 \otimes w = u .$$

Let $\zeta \in \boldsymbol{H}$ and $i, j, k \in \mathbb{Z}$. Then by Remark 4.2 we have

$$\begin{aligned}(\rho(m_j)\zeta \otimes \mu_i)(k) &= \sum_{n=-\infty}^{\infty} (\operatorname{Ad} w^n(m_j)\mu_i)(k)\theta^{-n}(p)\zeta = \sum_{n=-\infty}^{\infty} (m_{j+n}\mu_i)(k)\theta^{-n}(p)\zeta \\ &= \delta_{k,i}\theta^{j-k}(p)\zeta = (\theta^{-k}(\theta^j(p))(\zeta \otimes \mu_i))(k) = (\pi(\theta^j(p))\zeta \otimes \mu_i)(k) \\ &= (p_j(\zeta \otimes \mu_i))(k) .\end{aligned}$$

Since the span of the vectors $\zeta \otimes \mu_i$ is dense in $\boldsymbol{H} \otimes l^2(\mathbb{Z})$, we have that $\rho(m_j) = p_j$ for all j.

(c) Let R be the map from $l^2(\boldsymbol{H}, \mathbb{Z})$ onto $\boldsymbol{H}$ given by $R\xi = \xi(0)$ for all $\xi \in l^2(\boldsymbol{H}, \mathbb{Z})$. Then $R^*\zeta = \zeta \otimes \mu_0$ for all $\zeta \in \boldsymbol{H}$. Moreover, $E(x) = \pi(RxR^*)$ for all $x \in \boldsymbol{M}$ ([19, Ch. V, §7] or [14, Ch. 7, §11]). Therefore, for every $a \in \boldsymbol{B}(l^2(\mathbb{Z}))$ and every $\zeta \in \boldsymbol{H}$, we have

$$\begin{aligned}(R\rho(a)R^*)\zeta &= (R\rho(a))\zeta \otimes \mu_0 = R\Big(\sum_{n=-\infty}^{\infty} \theta^{-n}(p)\zeta \otimes \operatorname{Ad} w^n(a)\mu_0\Big) \\ &= \sum_{n=-\infty}^{\infty} R(\theta^{-n}(p)\zeta \otimes \operatorname{Ad} w^n(a)\mu_0) = \sum_{n=-\infty}^{\infty} (\operatorname{Ad} w^n(a)\mu_0)(0)\theta^{-n}(p)\zeta \\ &= \sum_{n=-\infty}^{\infty} (a\mu_n)(n)\theta^n(p)\zeta = \Big(\sum_{n=-\infty}^{\infty} a_{nn}\theta^n(p)\Big)\zeta .\end{aligned}$$

Therefore

$$E(\rho(a)) = \pi(R\rho(a)R^*) = \pi\Big(\sum_{n=-\infty}^{\infty} a_{nn}\theta^n(p)\Big) = \sum_{n=-\infty}^{\infty} a_{nn}p_n . \qquad \square$$

Recall that every $x \in \boldsymbol{M}$ has a generalized Fourier series $x = \sum_{n=-\infty}^{\infty} \pi(x_n)u^n$ where the series converges in the $\boldsymbol{N}$-Bures topology and $\pi(x_n) = E(xu^{-n})$ for all $n \in \mathbb{Z}$ [13]. Then we easily obtain the following corollary:

COROLLARY 4.4. (a) *For every* $a \in \boldsymbol{B}(l^2(\mathbb{Z}))$ *the generalized Fourier series of* $\rho(a)$ *is given by* $\rho(a) = \sum_{n=-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} a_{k,k-n}p_k)u^n$.

(b) $\boldsymbol{D} = \rho(\boldsymbol{B}(l^2(\mathbb{Z})))$ *is a type* I *factor with matrix units* $\{u^i p_0 u^{-j} \mid i, j \in \mathbb{Z}\}$.

The following construction will help shed more light on the pair $\{\rho, \boldsymbol{D}\}$. Let us define the von Neumann subalgebras of $\boldsymbol{N}$:

$$\boldsymbol{L}_0 = \mathbb{C}1 ,$$

$$\boldsymbol{D}_0 = \sum_{n=-\infty}^{\infty} \oplus \mathbb{C}\theta^n(p) = \left\{ \sum_{n=-\infty}^{\infty} \alpha_n \theta^n(p) \mid \alpha_n \in \mathbb{C},\ \sup|\alpha_n| < \infty \right\} ,$$

$$\boldsymbol{N}_0 = \sum_{n=-\infty}^{\infty} \oplus \boldsymbol{N}_{\theta^n(p)} \text{ where } \boldsymbol{N}_{\theta^n(p)} \text{ is the restriction of } \theta^n(p)\boldsymbol{N}\theta^n(p) \text{ to } \theta^n(p)\boldsymbol{H} .$$

Clearly $\boldsymbol{L}_0 \subset \boldsymbol{D}_0 \subset \boldsymbol{N}_0 \subset \boldsymbol{N}$ are globally θ-invariant algebras and thus we can form the crossed products

$$\boldsymbol{L} = \boldsymbol{L}_0 \otimes_\theta \mathbb{Z} , \quad \boldsymbol{D}^\sim = \boldsymbol{D}_0 \otimes_\theta \mathbb{Z} \quad \text{and} \quad \boldsymbol{M}_0 = \boldsymbol{N}_0 \otimes_\theta \mathbb{Z} .$$

Therefore we have

$$\boldsymbol{L} \subset \boldsymbol{D}^\sim \subset \boldsymbol{M}_0 \subset \boldsymbol{M} .$$

Notice the $\boldsymbol{L}_0$ and $\boldsymbol{N}$ are independent of the wandering projection p, hence $\boldsymbol{L}$ and $\boldsymbol{M}$ do not depend on p, while the other algebras do.

Since the action of θ on $\boldsymbol{L}_0$ is trivial, $\boldsymbol{L}$ is the von Neumann algebra generated by u, hence $\boldsymbol{L} = 1 \otimes \mathscr{L}$ where $\mathscr{L}$ is the *algebra of Laurent operators*, i.e., the algebra generated by the bilateral shift w.

Notice that by the definition of the isomorphism ρ we easily obtain that $\rho(a) = 1 \otimes a$ for all $a \in \mathscr{L}$. The expression $1 \otimes a$ is independent of the wandering projection p. In Proposition 6.2, we shall use this fact to study the module and ideal structure of $\boldsymbol{J}$.

As $\boldsymbol{D}_0$ is generated by $\{\theta^n(p) \mid n \in \mathbb{Z}\}$, $\boldsymbol{D}^\sim$ is generated by u and $\{p_n \mid n \in \mathbb{Z}\}$, hence has the same generators as $\boldsymbol{D}$ (see Corollary 4.4(b)) and therefore $\boldsymbol{D}^\sim = \boldsymbol{D}$.

REMARK 4.5. There is an isomorphism of $l^\infty(\mathbb{Z})$ (realized as an algebra of operators acting on $l^2(\mathbb{Z})$) onto $\boldsymbol{D}_0$ under which $\operatorname{Ad} w$ corresponds to θ and thus by [3, Proposition 2.13] there is an isomorphism between the crossed products, namely $l^\infty(\mathbb{Z}) \otimes_{\operatorname{Ad} w} \mathbb{Z}$ and $\boldsymbol{D}$. It is then easy to verify that ρ is the composite of this isomorphism with the isomorphism of $\boldsymbol{B}(l^2(\mathbb{Z}))$ onto $l^\infty(\mathbb{Z}) \otimes_{\operatorname{Ad} w} \mathbb{Z}$ mapping the matrix units $\{w^i m_0 w^{-j} \mid i, j \in \mathbb{Z}\}$ onto $\{(1 \otimes w^i)\pi_{(\operatorname{Ad} w)}(m_0)(1 \otimes w^{-j}) | i, j \in \mathbb{Z}\}$. Notice that this last isomorphism maps the algebra $\boldsymbol{A} = \{m_j \mid j \in \mathbb{Z}\}''$ of the diagonal operators of $\boldsymbol{B}(l^2(\mathbb{Z}))$ onto the image in $l^\infty(\mathbb{Z}) \otimes_{\operatorname{Ad} w} \mathbb{Z}$ of $l^\infty(\mathbb{Z})$ and intertwines the corresponding conditional expectations. Thus if $\widetilde{E}: \boldsymbol{B}(l^2(\mathbb{Z})) \to \boldsymbol{A}$ is the conditional expectation given by $\widetilde{E}(a) = \sum_{n=-\infty}^{\infty} a_{nn} m_n$ (i.e., $\widetilde{E}(a)$ "is the main diagonal of the matrix a"), then ρ intertwines $\widetilde{E}$ and E. This is actually part (c) of Theorem 4.3.

5. Classes of finite rank, finite trace and compact projections. For the remainder of this paper we shall assume that $0 < \lambda < 1$. Thus $\boldsymbol{N}$ is a factor and $\tau \circ \theta = \lambda\tau$. In this section we shall use the embedding of $\boldsymbol{B}(l^2(\mathbb{Z}))$ in $\boldsymbol{M}$ introduced in § 4 in order to separate the classes of the projections of $\boldsymbol{I}$, $\boldsymbol{M}_\varphi$, $\boldsymbol{K}$ and $\boldsymbol{J}$. In particular this will show that $\boldsymbol{K} \neq \boldsymbol{J}$.

Let us choose a wandering projection p with finite trace such that $p_\theta = 1$ and let ρ be the corresponding isomorphism of $\boldsymbol{B}(l^2(\mathbb{Z}))$ onto $\boldsymbol{D} \subset \boldsymbol{M}$.

THEOREM 5.1. *Let* $a \in \boldsymbol{B}(l^2(\mathbb{Z}))^+$. *Then*

(a) $\rho(a) \in \boldsymbol{I}$ *if and only if* $\{n \in \mathbb{Z} \mid a_{nn} \neq 0\}$ *is bounded below;*

(b) $\rho(a) \in \boldsymbol{M}_\varphi$ *if and only if* $\sum_{n=-\infty}^{\infty} \lambda^n a_{nn} < \infty$;

(c) $\rho(a) \in \boldsymbol{K}$ *if and only if* $a_{nn} \to 0$ *for* $n \to -\infty$.

PROOF. (a) $\rho(a) \in \boldsymbol{I}$ if and only if

$$E(\rho(a)) = \sum_{n=-\infty}^{\infty} a_{nn} p_n = \pi\Big(\sum_{n=-\infty}^{\infty} a_{nn}\theta^n(p)\Big) \in \pi(\boldsymbol{I}(\boldsymbol{N}))$$

(by Theorem 4.3 (c) and the definition of $\boldsymbol{I}^+$), if and only if the range projection $\sum \{\theta^n(p) \mid a_{nn} \neq 0\}$ of $\sum_{n=-\infty}^{\infty} a_{nn}\theta^n(p)$ is finite, if and only if (using the fact that $\boldsymbol{N}$ is a factor)

$$\tau(\textstyle\sum \{\theta^n(p) \mid a_{nn} \neq 0\}) = \sum \{\lambda^n \mid a_{nn} \neq 0\}\tau(p) < \infty ,$$

if and only if $a_{nn} \neq 0$ for only finitely many negative integers n.

(b) $\rho(a) \in \boldsymbol{M}_\varphi$ if and only if

$$\varphi(E(\rho(a))) = \varphi\Big(\sum_{n=-\infty}^{\infty} a_{nn}p_n\Big) = \sum_{n=-\infty}^{\infty} a_{nn}\lambda^n\tau(p) < \infty .$$

(c) $\rho(a) \in \boldsymbol{K}$ if and only if $E(\rho(a)) = \sum_{n=-\infty}^{\infty} a_{nn}p_n \in \pi(\boldsymbol{J}(\boldsymbol{N}))$ if and only if the spectral projection

$$\textstyle\sum \{p_n \mid a_{nn} > \varepsilon\} = \pi(\sum \{\theta^n(p) \mid a_{nn} > \varepsilon\})$$

of $E(\rho(a))$ corresponding to the interval (ε, ∞) is finite for every $\varepsilon > 0$ [10, Propositions 3.8, 3.9], if and only if (again using the fact that $\boldsymbol{N}$ is a factor)

$$\tau(\textstyle\sum \{\theta^n(p) \mid a_{nn} > \varepsilon\}) = \sum \{\lambda^n \mid a_{nn} > \varepsilon\}\tau(p) < \infty$$

for every $\varepsilon > 0$, if and only if $a_{nn} \to 0$ for $n \to -\infty$. □

Notice that since $\boldsymbol{I}$, $\boldsymbol{M}_\varphi$ and $\boldsymbol{K}$ are the span of their positive parts, the conditions in Theorem 5.1 are necessary also for nonpositive operators in $\boldsymbol{B}(l^2(\mathbb{Z}))$. Clearly, they are not sufficient, as the example of the bilateral shift $w \in \boldsymbol{B}(l^2(\mathbb{Z}))$ shows. Indeed $w_{nn} = 0$ for all $n \in \mathbb{Z}$, however $\boldsymbol{K}$ (and hence $\boldsymbol{I}$ and $\boldsymbol{M}_\varphi$) is a *-algebra that does not contain the identity and hence does not contain any unitary operator.

The following characterization of $\boldsymbol{D} \cap \boldsymbol{J}$ will establish a further link between the class $\boldsymbol{J}$ of $\boldsymbol{M}$ and the ideal of compact opertors $\boldsymbol{K}(l^2(\mathbb{Z}))$ of $\boldsymbol{B}(l^2(\mathbb{Z}))$.

The notion of relative weak (RW for short) vector convergence, introduced by the second named author in [11], plays a role in the theory of compact operators in von Neumann algebras similar to the role that the weak vector convergence plays in $\boldsymbol{B}(\boldsymbol{H})$. A net $\xi_\lambda \in \boldsymbol{H}$ converges to 0 weakly relatively to a semifinite algebra $\boldsymbol{N}$ ($\xi_\lambda \to 0$ ($\boldsymbol{N}$RW)) if it is norm bounded and if for every finite projection q in $\boldsymbol{N}$, $\|q\xi_\lambda\| \to 0$. A generalized Hilbert condition holds for semifinite algebras [11, Theorem 7]. For the case of a type III_λ $(0 < \lambda < 1)$ factor, we also have that $x \in \boldsymbol{J}^+$ if and only if $\|x\xi_\lambda\| \to 0$ for every $\xi_\lambda \to 0$ ($\pi(\boldsymbol{N})$RW), [8, Proposition 5.6]. This property is used in the following theorem in order to characterize $\boldsymbol{D} \cap \boldsymbol{J}^+$.

THEOREM 5.2. *Let* $r_- = \sum_{i=0}^\infty m_{-i}$ *and let* $a \in \boldsymbol{B}(l^2(\mathbb{Z}))^+$. *Then* $\rho(a) \in \boldsymbol{J}$ *if and only if* $r_- a r_- \in \boldsymbol{K}(l^2(\mathbb{Z}))$.

PROOF. For every positive integer n, let $q_n = \sum \{m_{-i} \mid i \geqq n\}$. Then $q_n \leqq r_-$ and q_n decreases to zero. Notice that by Theorem 5.1 (a), $1 - \rho(q_n) \in \boldsymbol{I}$ for all n (actually, $1 - \rho(q_n) \in \pi(\boldsymbol{I}(\boldsymbol{N}))$ by Theorem 4.3 (b)). Assume that $r_- a r_-$ is compact in $\boldsymbol{B}(l^2(\mathbb{Z}))$. Then $q_n a q_n = q_n r_- a r_- q_n$ converges in norm to zero and hence

$$\|a - (1 - q_n)a(1 - q_n)\| = \|\rho(a) - (1 - \rho(q_n))\rho(a)(1 - \rho(q_n))\| \to 0 .$$

As

$$(1 - \rho(q_n))\rho(a)(1 - \rho(q_n)) \leqq \|a\|(1 - \rho(q_n)) \in \boldsymbol{I} ,$$

we conclude that $\rho(a) \in \boldsymbol{J}$.

Conversely, suppose that $r_- a r_-$ is not compact in $\boldsymbol{B}(l^2(\mathbb{Z}))$. Now by a routine argument, we can find an $\alpha > 0$ and a strictly increasing sequence $\{n_j\}$ of positive integers such that

$$\|(q_{n_j} - q_{n_{j+1}})a(q_{n_j} - q_{n_{j+1}})\| > \alpha$$

for each j. Let ν_j be a unit vector in the range of $q_{n_j} - q_{n_{j+1}}$ such that

$$\omega_{\nu_j}(a) = (a\nu_j, \nu_j) > \alpha .$$

Let $0 \neq \zeta_0 \in p\boldsymbol{H}$ be such that $\omega_{\zeta_0} \leqq \tau(p \cdot p)$. Since $\boldsymbol{N}$ is a factor and

$$\tau(p) < \tau(\theta^{-j}(p)) = \lambda^{-j}\tau(p) \quad \text{for} \quad j = 1, 2, \cdots ,$$

we have that $p \prec \theta^{-j}(p)$. Thus there is a partial isometry $u_j \in \boldsymbol{N}$ such that $p = u_j^* u_j$ and $u_j u_j^* < \theta^{-j}(p)$. Setting $\zeta_j = u_j \zeta_0$, we see that $\zeta_j \in \theta^{-j}(p)\boldsymbol{H}$ and that for every $x \in \boldsymbol{N}^+$ and $j = 1, 2, \cdots$, we have

$$\omega_{\zeta_j}(x) = \omega_{\zeta_0}(u_j^* x u_j) \leqq \tau(p u_j^* x u_j p) = \tau(u_j u_j^* x u_j u_j^*) \leqq \tau(\theta^{-j}(p) x \theta^{-j}(p)) .$$

In other words,

$$\omega_{\zeta_j} \leqq \tau(\theta^{-j}(p) \cdot \theta^{-j}(p)) \quad \text{for all} \quad j .$$

Define $\xi_j = \zeta_j \otimes w^j \nu_j$ for $j = 1, 2, \cdots$. Then by using the strong convergence of the series giving $\rho(a)$, we obtain

$$\begin{aligned}(\rho(a)\xi_j, \xi_j) &= \sum_{n=-\infty}^{\infty} ((\theta^{-n}(p) \otimes \operatorname{Ad} w^n(a))\zeta_j \otimes w^j \nu_j, \zeta_j \otimes w^j \nu_j) \\ &= \sum_{n=-\infty}^{\infty} ((\theta^{-n}(p)\zeta_j, \zeta_j)(\operatorname{Ad} w^{n-j}(a)\nu_j, \nu_j) = \|\zeta_j\|^2 (a\nu_j, \nu_j) > \alpha \|\zeta_0\|^2 .\end{aligned}$$

Thus, in view of [8, Proposition 5.6], in order to obtain that $\rho(a)$ is not in $\boldsymbol{J}$, it is enough to show that $\xi_j \to 0$ ($\pi(\boldsymbol{N})$RW). Notice that ξ_j is bounded since $\|\xi_j\| = \|\zeta_0\|$ for all j. Let s be any finite projection in $\boldsymbol{N}$. Then we have:

$$\begin{aligned}\|\pi(s)\xi_j\|^2 &= \sum_{n=-\infty}^{\infty} \|(\pi(s)\xi_j)(n)\|^2 = \sum_{n=-\infty}^{\infty} \|\theta^{-n}(s)\xi_j(n)\|^2 \\ &= \sum_{n=-\infty}^{\infty} \|\theta^{-n}(s)\zeta_j\|^2 |\nu_j(n-j)|^2 \leqq \sum_{n=0}^{\infty} \|\theta^n(s)\zeta_j\|^2 ,\end{aligned}$$

from the fact that $|\nu_j(k)| \leqq \|\nu_j\| = 1$ for all k and from the fact that $\nu_j(k) = 0$ for $k > -j$, because $\nu_j \in q_{n_j}\boldsymbol{H} \subset q_j \boldsymbol{H}$. Summing over j, we obtain:

$$\begin{aligned}\sum_{j=1}^{\infty} \|\pi(s)\xi_j\|^2 &\leqq \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \|\theta^n(s)\zeta_j\|^2 = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \omega_{\zeta_j}(\theta^n(s)) \leqq \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \tau(\theta^{-j}(p)\theta^n(s)\theta^{-j}(p)) \\ &\leqq \sum_{n=0}^{\infty} \tau(\theta^n(s)) = \left(\sum_{n=0}^{\infty} \lambda^n\right)\tau(s) < \infty ,\end{aligned}$$

from the fact that the finite projection s in the factor $\boldsymbol{N}$ has finite trace. Thus $\|\pi(s)\xi_j\| \to 0$ and hence $\xi_j \to 0$ ($\pi(\boldsymbol{N})$RW). □

As a consequence of Theorems 5.1 and 5.2 we obtain the following corollary:

COROLLARY 5.3. *The set of the projections in the classes $\boldsymbol{I} \subset \boldsymbol{M}_\varphi \subset \boldsymbol{J} \subset \boldsymbol{K}$ are all distinct; hence, the inclusions are proper.*

PROOF. Let ζ be any unit vector in $l^2(\mathbb{Z})$; the one-dimensional projection s on span ζ has matrix representation $s_{ij} = \zeta(i)\overline{\zeta(j)}$. Choose $\zeta(n) = \lambda^{|n|}\beta$, $(n \in \mathbb{Z})$ with $\beta = (1 + 2\sum_{n=1}^{\infty} \lambda^{2n})^{-1/2}$; then $\sum_{n=-\infty}^{\infty} \lambda^n s_{nn} < \infty$ but $s_{nn} \neq 0$ for all n. Thus by Theorem 5.1 (a) and (b), $\rho(s) \in \boldsymbol{M}_\varphi$ but $\rho(s) \notin \boldsymbol{I}$. Choose now $\zeta(n) = \lambda^{|n|/2}\nu$, $(n \in \mathbb{Z})$ with $\nu = (1 + 2\sum_{n=1}^{\infty} \lambda^n)^{-1/2}$; then $\sum_{n=-\infty}^{\infty} \lambda^n s_{nn} = \infty$.

Hence by Theorems 5.1(b) and 5.2 $\rho(s) \in \boldsymbol{J}$ but $\rho(s) \notin \boldsymbol{M}_\varphi$. For any infinite projection $s \leqq r_-$ such that $s_{nn} \to 0$ for $n \to -\infty$, we have $\rho(s) \in \boldsymbol{K}$ but $\rho(s) \notin \boldsymbol{J}$. Choose for example

$$s_{i,j} = \begin{cases} 2^{-k}, & \text{for } i, j = -2^k - 1, \cdots, -2^{k+1} \text{ and } k = 0, 1, 2, \cdots \\ 0, & \text{otherwise}. \end{cases}$$

Then s is the direct sum of $2^k \times 2^k$ blocks each of whose entries is equal to 2^{-k} and thus each block is a rank one projection and s is an infinite dimensional projection. We see that $s_{nn} \to 0$ as $n \to -\infty$; hence $\rho(s) \in \boldsymbol{K}$ but $\rho(s) \notin \boldsymbol{J}$. □

We conclude this section with an example of two projections in $\boldsymbol{M}_\varphi$ whose supremum is the identity of $\boldsymbol{M}$; this shows that unlike their analogue in a semifinite algebra and unlike $\boldsymbol{I}$, the classes of the projections in $\boldsymbol{M}_\varphi$, $\boldsymbol{J}$ and $\boldsymbol{K}$ are not closed under supremum, (see also [8, Example 7.4]).

EXAMPLE 5.4. Consider for $k \in \mathbb{N}$ the rank one projection s_k on the unit vector $\alpha_k \mu_{-k} + \beta_k \mu_k$ where $\{\mu_k \mid k \in \mathbb{Z}\}$ is the canonical basis of $l^2(\mathbb{Z})$ and choose $0 \neq \alpha_k$ small enough so that $\sum_{k=1}^\infty \lambda^{-k} |\alpha_k|^2 < \infty$. Let $s = \sum_{k=1}^\infty s_k$ and let $m = \sum_{k=0}^\infty m_k$; since $\sup\{s_k, m_k\} = m_{-k} + m_k$ for all k, we have that $\sup\{s, m\} = 1$ and thus $\sup\{\rho(s), \rho(m)\} = 1$. On the other hand we already know that $\rho(m) \in \pi(\boldsymbol{I}(\boldsymbol{N}))$ and we see that $\rho(s)$ satisfies by construction the condition of Theorem 5.1 (b). Thus, both projections are in $\boldsymbol{M}_\varphi$. □

6. Multipliers of the hereditary algebra *J*. In this section we investigate module and ideal structures for $\boldsymbol{J}$. We have already considered in §4 the algebra $\mathscr{L} \subset \boldsymbol{B}(l^2(\mathbb{Z}))$ of Laurent operators generated by the bilateral shift w. There is an isomorphism $L: L^\infty(\mathbb{T}) \to \mathscr{L}$ given by $L_f = \sum_{n=-\infty}^\infty \hat{f}(n) w^n$ where $\{\hat{f}(n) \mid n \in \mathbb{Z}\}$ are the Fourier coefficients of $f \in L^\infty(\mathbb{T})$ and the series is the generalized Fourier expansion of L_f. The matrix representation of L_f relative to the standard basis of $l^2(\mathbb{Z})$ is $(L_f)_{ij} = \hat{f}(i - j)$ for $i, j \in \mathbb{Z}$.

If we let $r_+ = \sum_{n=0}^\infty m_n$, $r_- = \sum_{n=0}^\infty m_{-n}$, then the compression of L_f to $r_+ l^2(\mathbb{Z})$ is the Toeplitz matrix $T_f = r_+ L_f r_+$ with symbol f. Since we have to consider (because of Theorem 5.2) compressions to $r_- l^2(\mathbb{Z})$, let us define $S \in \boldsymbol{B}(l^2(\mathbb{Z}))$ to be the (unitary) reflection operator, i.e.,

$$(S\mu)(n) = \mu(-n) \quad \text{for all} \quad \mu \in l^2(\mathbb{Z}) \text{ and } n \in \mathbb{Z}.$$

Let f^* be the reflexion of $f \in L^\infty(\mathbb{T})$, i.e.,

$$f^*(t) = f(\bar{t}) \quad \text{for} \quad t \in \mathbb{T}.$$

Then it is easy to verify that $\operatorname{Ad} S(r_+) = r_-$ and that for all $f \in L^\infty(\mathbb{T})$

we have $\operatorname{Ad} S(L_{f*}) = L_f$ and thus $r_- L_f r_- = \operatorname{Ad} S(T_{f*})$. Let us finally recall that by [20, Theorems A and 1) if $f, g \in L^\infty(\mathbb{T})$ then

$$T_{fg} - T_f T_g \in \boldsymbol{K}(l^2(\mathbb{Z}))$$

if and only if

$$H[\bar{f}] \cap H[g] \subset H^\infty(\mathbb{T}) + C(\mathbb{T}) ,$$

where $H^\infty(\mathbb{T})$ is the Hardy space of the functions $f \in L^\infty(\mathbb{T})$ with $\hat{f}(n) = 0$ for $n < 0$, $C(\mathbb{T})$ is the space of continuous complex-valued functions on $\mathbb{T}$ and $H[\bar{f}]$ (resp. $H[g]$) is the subalgebra of $L^\infty(\mathbb{T})$ generated by $H^\infty(\mathbb{T})$ and $\bar{f}$ (resp. g).

PROPOSITION 6.1. *Let $f \in L^\infty(\mathbb{T})$, let $p \in \boldsymbol{N}$ be any wandering projection with finite trace and θ-span $p_\theta = \sum_{n=-\infty}^{\infty} \theta^n(p) = 1$, let ρ be the corresponding isomorphism from $\boldsymbol{B}(l^2(\mathbb{Z}))$ onto $\boldsymbol{D} \subset \boldsymbol{M}$ and let $x = \rho(L_f r_+ (L_f)^*)$. Then*

(a) $x \in \boldsymbol{K}$,

(b) $x \in \boldsymbol{J}$ *if and only if* $f \in H^\infty(\mathbb{T}) + C(\mathbb{T})$.

PROOF. (a) The (n, n)-entry of the matrix representation of $L_f r_+ (L_f)^*$ is

$$\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left\{ \hat{f}(n-i) \Big(\sum_{k=0}^{\infty} m_k \Big)_{ij} \hat{f}(n-j)^- \right\} = \sum_{i=0}^{\infty} |\hat{f}(n-i)|^2 \to 0 \quad \text{for} \quad n \to -\infty$$

as $\hat{f} \in l^2(\mathbb{Z})$. Thus $x \in \boldsymbol{K}$ by Theorem 5.1 (c).

(b) $x \in \boldsymbol{J}$ if and only if $r_- L_f r_+ (L_f)^* r_- \in \boldsymbol{K}(l^2(\mathbb{Z}))$ (Theorem 5.2). But

$$\begin{aligned} r_- L_f r_+ (L_f)^* r_- &= r_- L_f (1 - r_- + m_0) L_{\bar{f}} r_- = r_- (L_{|f|^2} - L_f r_- L_{\bar{f}}) r_- + r_- L_f m_0 L_{\bar{f}} r_- \\ &= \operatorname{Ad} S(T_{|f*|^2} - T_{f*} T_{\bar{f}*}) + r_- L_f m_0 L_{\bar{f}} r_- \in \boldsymbol{K}(l^2(\mathbb{Z})) , \end{aligned}$$

if and only if $T_{|f*|^2} - T_{f*} T_{\overline{(f*)}} \in \boldsymbol{K}(l^2(\mathbb{Z}))$ (using $\overline{(f^*)} = \bar{f}^*$), if and only if $H[\bar{f}^*] \subset H^\infty(\mathbb{T}) + C(\mathbb{T})$ [20, Theorems A and 1], if and only if $f \in H^\infty(\mathbb{T}) + C(\mathbb{T})$. □

Recall that $\rho(L_f) = 1 \otimes L_f \in \boldsymbol{L}$ for all $f \in L^\infty(\mathbb{T})$. The set $H^\infty(\mathbb{T}) + C(\mathbb{T})$ is a closed subalgebra of $L^\infty(\mathbb{T})$, thus its image $1 \otimes L(H^\infty(\mathbb{T}) + C(\mathbb{T}))$ under $\rho \circ L$ is a closed subalgebra of $\boldsymbol{L}$. Likewise $\rho \circ L(C(\mathbb{T})) = 1 \otimes L(C(\mathbb{T}))$ is a C^*-subalgebra of $\boldsymbol{L}$.

PROPOSITION 6.2. *$\boldsymbol{J}$ is a left module over $1 \otimes L(H^\infty(\mathbb{T}) + C(\mathbb{T}))$ and a two sided module over $1 \otimes L(C(\mathbb{T}))$.*

PROOF. Let $a = 1 \otimes L_f$ for some $f \in H^\infty(\mathbb{T}) + C(\mathbb{T})$ and let $x \in \boldsymbol{J}$. Then we have, by [8, Proposition 4.1 (b)], that $ax \in \boldsymbol{J}$ if and only if both $x^* a^* a x$ and $a x x^* a^*$ are in $\boldsymbol{J}^+$. But $x^* a^* a x \in \boldsymbol{J}^+$, since

$$x^*a^*ax \leqq \|a\|^2 x^*x \in \boldsymbol{J}^+ .$$

As $xx^* \in \boldsymbol{J}^+$, we can find, by [8, Theorem 4.3 (b)], some $z \in \pi(\boldsymbol{J}(\boldsymbol{N})^+)$ such that $xx^* \leqq z$ and hence $axx^*a^* \leqq aza^*$. Let $\varepsilon > 0$ and let q be the spectral projection of z corresponding to the interval $[\varepsilon, \infty)$. We shall prove that $aqa^* \in \boldsymbol{J}^+$. Notice first that q is finite in $\pi(\boldsymbol{N})$ [10, Propositions 3.8 and 3.9] and hence has finite trace. By Proposition 3.7, there is a wandering projection p' with finite trace such that $q \leqq \pi(\sum_{n=0}^{\infty} \theta^n(p'))$. There is also a second wandering projection p'' with finite trace such that $(p'')_\theta = 1 - (p')_\theta$ (Proposition 3.5). Thus $p = p' + p''$ is also wandering projection with finite trace (Lemma 3.2 (e)), $p_\theta = 1$ and $q \leqq \pi(\sum_{n=0}^{\infty} \theta^n(p))$. Let ρ be the isomorphism corresponding to p. Then $a = 1 \otimes L_f = \rho(L_f)$ and $\pi(\sum_{n=0}^{\infty} \theta^n(p)) = \rho(r_+)$. Therefore

$$azqa^* \leqq \|z\| aqa^* \leqq \|z\| \rho(L_f r_+ (L_f)^*) \in \boldsymbol{J}^+$$

by Proposition 6.1 (b). Hence we have $azqa^* \in \boldsymbol{J}^+$. Since

$$\|aza^* - azqa^*\| \leqq \varepsilon \|a\|^2$$

and ε is arbitrary, we obtain that aza^* and hence axx^*a^* are in $\boldsymbol{J}^+$. Thus $ax \in \boldsymbol{J}$ and consequently $\boldsymbol{J}$ is a left module over $1 \otimes L(H^\infty(\mathbb{T}) + C(\mathbb{T}))$ and in particular over $1 \otimes L(C(\mathbb{T}))$.

Since both $\boldsymbol{J}$ and $1 \otimes L(C(\mathbb{T}))$ are selfadjoint, $\boldsymbol{J}$ is also a two sided module over $1 \otimes L(C(\mathbb{T}))$. □

COROLLARY 6.3. *The C^*-subalgebra $\boldsymbol{J}$ of $\boldsymbol{K}$ is not an ideal of $\boldsymbol{K}$.*

PROOF. Choose a wandering projection p with finite trace and $p_\theta = 1$, and let ρ be the corresponding isomorphism of $\boldsymbol{B}(l^2(\mathbb{Z}))$ onto $\boldsymbol{D} \subset \boldsymbol{M}$. Let $q = \rho(r_- - m_0)$; recall that $1 - q$ is finite in $\pi(\boldsymbol{N})$. Let f be a function in the complement of $H^\infty(\mathbb{T}) + C(\mathbb{T})$ in $L^\infty(\mathbb{T})$ and let $a = 1 \otimes L_f = \rho(L_f)$ and $y = qa(1-q)$. Then

$$(1-q)a(1-q)a^*(1-q) \leqq \|a\|^2(1-q) \in \boldsymbol{J}^+ ,$$

and

$$a(1-q)a^* \leqq 2(qa(1-q)a^*q + (1-q)a(1-q)a^*(1-q)) .$$

Since $a(1-q)a^*$ is not in $\boldsymbol{J}$ (Proposition 6.1 (b)), we conclude that also $yy^* = qa(1-q)a^*q$ is not in $\boldsymbol{J}$. Thus y is not in $\boldsymbol{J}$.

On the other hand, $yy^* \in \boldsymbol{K}^+$ as $a(1-q)a^* \in \boldsymbol{K}$ (Proposition 6.1 (a)) and $\boldsymbol{K}$ is a $\pi(\boldsymbol{N})$-module. Moreover,

$$y^*y = (1-q)a^*qa(1-q) \leqq \|a\|^2(1-q) \in \boldsymbol{K}^+$$

hence $y^*y \in \boldsymbol{K}^+$ and thus $y \in \boldsymbol{K}$. Therefore $y = y(1-q)$ is the product of an element in K and an element in J (actually a finite projection in $\pi(N)$) and does not belong to J. □

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