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HORIZONTAL LIFTS OF SPACELIKE CURVES WITH NON-DIFFERENTIABLE ENDPOINTS

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Let $P(M, G)$ be a principal fiber bundle with structure group G over a manifold M; let $\sigma: [0, L] \to M$ be a continuous curve in M which is differentiable on the half-open interval $[0, L)$. For a given connection on P, does σ admit a horizontal lift into P defined over the entire closed interval $[0, L]$? If the connection is flat, it surely does. Here is an example where it does not: $M = \boldsymbol{R}^2$, $G = GL(2)$, $P =$ bundle of linear frames in $\boldsymbol{R}^2$, $L = 1$, $\sigma(t) = (1-t)(\cos(1-t)^{-2}, \sin(1-t)^{-2})$, and the connection is the Levi-Civita connection associated with the metric $\exp(-y^2)\cdot(dx^2 + dy^2)$; a linear frame, parallel translated, in this metric, from $\sigma(0)$ to $\sigma(t)$, is rotated through an angle of $(1/4)\theta^{-1}\sin 2\theta - (1/2)\ln\theta$, where $\theta = (1-t)^{-2}$, so it has no limit as $t \to 1$.

The purpose of this paper is to show that if M admits a Lorentz metric for which σ is a finite-length spacelike curve with timelike acceleration (when parametrized by arc-length), then σ does, indeed, admit a horizontal lift over the entire closed interval, i.e., the lift over the differentiable part has a limit as $t \to L$. This is done by first showing that the horizontal lift over $[0, L]$ exists in the case that for some Riemannian metric on M, σ has finite length; since σ is compact, if this is the case for one Riemannian metric, so must it be for all Riemannian metrics. Next, it is shown that if σ has infinite Riemannian-length, then any scalar function F on M which, in the given Lorentz metric, has a timelike gradient which is (say) opposite-directed to $\nabla_{\dot\sigma}\dot\sigma$ with respect to future and past, must have $H_F(\dot\sigma, \dot\sigma)$ unbounded below, where H_F is the Hessian of F. Finally, it is shown how to construct, in a neighborhood of any point in any Lorentz manifold, a function with a timelike gradient (either past- or future-directed) and a positive-definite Hessian. Since it is only the behavior of σ and the connection in a neighborhood of $\sigma(L)$ that is significant, this is sufficient for the problem at hand.

THEOREM 1. *Let M be a manifold with a Lorentz metric g, let P be a principal fiber bundle over M with structure group G, and let ω be a connection form on P. Let $\sigma: [0, L] \to M$ be a continuous curve in M which is differentiable on $[0, L)$. If σ, on $[0, L)$, is spacelike, is para-*

metrized by arc-length, and has timelike acceleration—or, more generally, for some (continuous) unit-timelike vector-valued function N_t defined over σ and some $\kappa(t) \geqq 0$, $\nabla_{\dot\sigma}\dot\sigma = \kappa N$—then a horizontal lift $v: [0, L) \to P$ of σ has a limit as $t \to L$.

PROOF. In the course of this proof, the following elementary result from analysis will be used: For any differentiable function $x(t)$ on a finite interval $[0, L)$, if $\int_0^L x(t)dt$ is finite but $\int_0^L |x(t)|\,dt = \infty$, then both x and x' are unbounded both above and below on $[0, L)$.

Let $\mathscr{U}$ be a neighborhood of $\sigma(L)$ over which P is trivial; it does no harm to assume that σ is contained in $\mathscr{U}$. Let $u: \mathscr{U} \to P$ be a cross-section; then a lift $v_t = u_{\sigma(t)}a_t$ of σ, with $a: [0, L) \to G$, is horizontal if and only if $\dot a_t a_t^{-1} = -\omega[(d/dt)u_{\sigma(t)}]$ ($\dot v = \dot u a + u\dot a$, $\dot v a^{-1} = \dot u + u\dot a a^{-1}$, $\omega(\dot v a^{-1}) = \mathrm{ad}(a)\omega(\dot v) = \omega(\dot u) + \omega(u\dot a a^{-1}) = \omega(\dot u) + \dot a a^{-1}$; therefore, $\omega(\dot v) = 0$ iff $\dot a a^{-1} = -\omega(\dot u)$; see, e.g., [4], p. 69). Define $\alpha = -u^*\omega$. Let M have an arbitrary Riemannian metric, and let G have an arbitrary right-invariant Riemannian metric, both denoted by $\|-\|$; then at each x in $\mathscr{U}$, $\alpha_x: T_xM \to \mathfrak{g}$ has a norm $\|\alpha_x\|$ as a linear transformation, and $\|\alpha\|$ is bounded in a (possibly smaller) neighborhood of $\sigma(L)$. The equation $\dot a_t a_t^{-1} = \alpha(\dot\sigma_t)$ has a solution for $0 \leqq t < L$. As a curve in G, its length $L(a) = \int_0^L \|\dot a_t\| = \int_0^L \|\dot a_t a_t^{-1}\| = \int_0^L \|\alpha(\dot\sigma_t)\| \leqq \int_0^L \|\alpha\|\,\|\dot\sigma_t\|$. Therefore, if $\int_0^L \|\dot\sigma\|$ is finite, so is $L(a)$. Being homogeneous, G is complete, so if $L(a)$ is finite, a_t has a limit as $t \to L$. Therefore, if σ has finite Riemannian-length, the horizontal lift $u_{\sigma(t)}a_t$ has a limit $u_{\sigma(L)}a_L$.

Let U be any (non-vanishing) timelike vector field on M; let $U^\perp$ be its perpendicular space; and let $P_U: T_xM \to U_x^\perp$ be projection. Then $(X, Y) \mapsto \langle P_U(X), P_U(Y)\rangle + \langle X, U\rangle\langle Y, U\rangle$ is a Riemannian metric on M ($\langle -, -\rangle$ denotes g, as will $|-|$). Thus, if σ has infinite Riemannian-length, $\int_0^L (|P_U(\dot\sigma)|^2 + \langle\dot\sigma, U\rangle^2)^{1/2} = \infty$. Since σ is of unit-speed and spacelike,

$$\langle\dot\sigma, \dot\sigma\rangle = |P_U(\dot\sigma)|^2 - \langle\dot\sigma, U\rangle^2/|U|^2 = 1,$$

so

$$|P_U(\dot\sigma)|^2 + \langle\dot\sigma, U\rangle^2 = 1 + (1 + 1/|U|^2)\langle\dot\sigma, U\rangle^2 .$$

In a neighborhood of $\sigma(L)$, $|U|$ is bounded; therefore σ has infinite Riemannian-length if and only if $\int_0^L |\langle\dot\sigma, U\rangle| = \infty$.

Now consider any scalar function $F: M \to \boldsymbol{R}$ with ∇F timelike; σ has infinite Riemannian-length if and only if $\int_0^L |\dot\sigma F| = \infty$. However, $\int_0^L \dot\sigma F = \int_0^L (d/dt)F(\sigma(t)) = F(\sigma(L)) - F(\sigma(0))$, which is finite. Thus, by the remark made at the beginning of this proof, if σ has infinite Riemannian-length, then $(d/dt)(\dot{\sigma}F) = \dot{\sigma}\langle\dot{\sigma}, \nabla F\rangle = \langle\nabla_{\dot{\sigma}}\dot{\sigma}, \nabla F\rangle + \langle\dot{\sigma}, \nabla_{\dot{\sigma}}\nabla F\rangle = \langle\nabla_{\dot{\sigma}}\dot{\sigma}, \nabla F\rangle + H_F(\dot{\sigma}, \dot{\sigma})$ is unbounded both above and below. But since $\nabla_{\dot{\sigma}}\dot{\sigma}$ and ∇F are both timelike (or each a non-negative multiple of a timelike vector field), $\langle\nabla_{\dot{\sigma}}\dot{\sigma}, \nabla F\rangle$ has constant sign. Thus, for example, if $\nabla_{\dot{\sigma}}\dot{\sigma}$ and ∇F lie in opposite time-cones, then $H_F(\dot{\sigma}, \dot{\sigma})$ must be unbounded below. It follows that if there is a function in a neighborhood of $\sigma(L)$ with timelike gradient in the opposite time-cone as that of $\nabla_{\dot{\sigma}}\dot{\sigma}$ and with positive-definite Hessian, then σ must have finite Riemannian-length.

The remainder of the proof is devoted to constructing in a neighborhood of an arbitrary point p in a Lorentz manifold M, a function F with timelike gradient (either future- or past-directed, as needed) and positive-definite Hessian. F is the sum of a function whose Hessian is positive definite on a spacelike hyperplane in T_pM, and of a second function whose Hessian is zero on that hyperplane but positive on the vector perpendicular to it.

The first function, f, is defined by $f(x) = \langle\exp_q^{-1}(x), \exp_q^{-1}(x)\rangle$, where q is a point in the chronological past of p (i.e., $q \ll p$) that needs to be chosen appropriately. To find ∇f, consider a vector V in T_xM, $x \gg q$, with $V = (d/dv)x_v$ for some curve x_v; let $x_v = \exp_q(r_vT_v)$ with T_v unit timelike and $r_v \geqq 0$. Then $f(x_v) = -r_v^2$. Define $\beta(s, v) = \exp_q(sr_vT_v)$, so that V at x is extended by the definition to $V = \beta_*(\partial/\partial v)$; define $S = \beta_*(\partial/\partial s)$ and $T = S/|S|$. Let γ_v be the geodesic $\beta(-, v)$ from $s = 0$ to $s = 1$, so $L(\gamma_v) = |S_v| = r_v$. Then $V_xf = (d/dv)f(x_v) = -2r_v(d/dv)r_v = -2r_v(d/dv)L(\gamma_v) = -2r_v[-\langle V, T\rangle]_{s=0}^{s=1} = 2|f(x)|^{1/2}\langle V, T\rangle_x$ (first variation of timelike arc-length has been used here—see, e.g., Corollary 11.24 in [1]). Therefore,

$$\nabla f = 2|f|^{1/2}T ,$$

where T is the vector field defined by $T_x = \dot{\gamma}_x$, with γ_x the unit-speed geodesic from q to x (for $x \gg q$). Then, for any vector X at x,

$$\begin{aligned}\nabla_X\nabla f &= 2(X(-f)^{1/2})T + 2|f|^{1/2}\nabla_XT = -|f|^{-1/2}\langle X, \nabla f\rangle T + 2|f|^{1/2}\nabla_XT \\ &= -|f|^{-1/2}\langle X, 2|f|^{1/2}T\rangle T + 2|f|^{1/2}\nabla_XT = 2(|f|^{1/2}\nabla_XT - \langle X, T\rangle T) ,\end{aligned}$$

yielding

$$H_f(X, X) = \langle\nabla_X\nabla f, X\rangle = 2(|f|^{1/2}\langle\nabla_XT, X\rangle - \langle X, T\rangle^2) .$$

Therefore, $H_f(X, Y) = 2(|f|^{1/2}\langle\nabla_XT, Y\rangle - \langle X, T\rangle\langle Y, T\rangle)$. For V perpendicular to T_x, the function r_v can be taken to be constant at $r = |f(x)|^{1/2}$, so $[V, T] = (1/r)[V, S] = 0$. Then

$$H_f(V, V) = 2|f|^{1/2}\langle\nabla_VT, V\rangle = |f|^{1/2}T\langle V, V\rangle ,$$

where V is a Jacobi field along γ_x with $V_q = 0$.

It remains to be shown how to choose $q \ll p$ so that H_f will be positive definite on a spacelike hyperplane at p. To this end, pick any future-directed unit-speed timelike geodesic γ with $\gamma(0) = p$; let $T = \dot{\gamma}(0)$. The basepoint q will be $\gamma(s)$ for some $s < 0$, and the hyperplane at p will be $T^\perp$. By the calculations above, $(\nabla f)_p = 2(-s)T$, and, for any U in $T^\perp$, $H_f(U, U) = (-s)T_0\langle V, V\rangle$, where $T_t = \dot{\gamma}(t)$ and V is the Jacobi field on γ defined by $V(0) = U$ and $V(s) = 0$. It will be shown that for s close enough to 0, $H_f(U, U)$ must be positive for all non-zero U in $T^\perp$.

On any finite interval of γ, the sectional curvature of any plane $X \wedge T$ containing T obeys $K(X \wedge T) \geqq -K$ for some constant $K > 0$ (X can be restricted to $T^\perp$ with $|X| = 1$, a compact set). For a given unit-length vector U in $T_0^\perp$, let $h(t) = \langle V, V\rangle_t$, V defined as above; then $h'' = (T\langle V, V\rangle)' = 2\langle \nabla_T V, V\rangle' = 2(\langle \nabla_T^2 V, V\rangle + \langle \nabla_T V, \nabla_T V\rangle) = 2(-\langle R(V, T)T, V\rangle + |\nabla_T V|^2) = 2(K(V \wedge T)|V|^2 + |\nabla_T V|^2) \geqq -2K|V|^2 = -2Kh$. Therefore,

$$h \geqq -\frac{1}{2K}h'' . \qquad (*)$$

Note that $h(s) = 0$ and $h(0) = 1$. For $-(2K)^{-1/2} < s < 0$, it can be shown that $h'(0) > 0$: There is some t_1 in $[s, 0]$ with $h'(t_1) = -1/s$. With $h'(0) \leqq 0$, there is some r_1 in $[t_1, 0]$ with $h''(r_1) = (-st_1)^{-1} \leqq (-s^2)^{-1}$. By (*), $h(r_1) \geqq (2Ks^2)^{-1}$. From this and $h(s) = 0$, we obtain some t_2 in $[s, r_1]$ with $h'(t_2) = (2Ks^2(r_1 - s))^{-1} \geqq (-2Ks^3)^{-1}$. With $h'(0) \leqq 0$, there is some r_2 in $[t_2, 0]$ with $h'(r_2) = (-2Ks^3t_2)^{-1} \leqq (-2Ks^4)^{-1}$. By (*), $h(r_2) \geqq (4K^2s^4)^{-1}$. Continuing, we obtain a sequence r_n in $[s, 0]$ with $h(r_n) \geqq (2Ks^2)^{-n}$. With s as specified, this implies that the continuous function h is unbounded on the interval $[s, 0]$, an imposibility. Thus, $q = \gamma(s)$ for such an s ensures that $H_f(U, U) = -sh'(0) > 0$ for unit-length, hence, any non-zero U in $T_p^\perp$.

To define the second function, start with the same vector T at p, but extend it differently: For any U in $T_p^\perp$, define T_x for $x = \exp_p(U)$ as the parallel translate of T_p along the geodesic from p to x; let γ_x be the geodesic $\gamma_x(s) = \exp_x(sT_x)$; and define T at $\gamma_x(s)$ to be $\dot{\gamma}_x(s)$. Define the function k by $k(\gamma_x(s)) = s$. Then $\nabla k = -T$. Since $\nabla_T T = 0$ and, at p, $\nabla_U T = 0$ for U in $T^\perp$, $H_k = 0$ at p. For any function $\phi: \boldsymbol{R} \to \boldsymbol{R}$, $\nabla(\phi\circ k) = (\phi'\circ k)\nabla k$ and $H_{\phi\circ k} = (\phi'\circ k)H_k + (\phi''\circ k)dk \otimes dk$; thus, at p, $\nabla(\phi\circ k) = -\phi'(0)T_p$ and $H_{\phi\circ k} = \phi''(0)\langle -, T_p\rangle \otimes \langle -, T_p\rangle$. Let $F = f + \phi\circ k$. Then, at p,

$$\nabla F = (-2s - \phi'(0))T_p ,$$
$$H_F = H_f + \phi''(0)\langle -, T_p\rangle \otimes \langle -, T_p\rangle .$$

For U in $T_p^\perp$, $H_F(U + aT, U + aT) = H_f(U, U) + 2aH_f(U, T) + a^2H_f(T, T) +$

$\phi''(0)\langle U + aT, T\rangle^2 = H_f(U, U) + (\phi''(0) - 2)a^2$, so

$$H_F(X, X) = H_f(X^{\perp}, X^{\perp}) + (\phi''(0) - 2)\langle X, T_p\rangle^2 ,$$

where $X^{\perp} = X + \langle X, T\rangle T$. Thus, H_F is positive-definite at p so long as $\phi''(0) > 2$, and $(\nabla F)_p$ is timelike so long as $\phi'(0) \neq -2s$: future-directed for $\phi'(0) < -2s$ and past-directed for $\phi'(0) > -2s$. These properties of the Hessian and gradient remain true in a neighborhood of p.

Taking $p = \sigma(L)$ completes the proof. □

As an application of this theorem, consider the bundle of orthonormal frames over M with the Levi-Civita connection associated with g: a horizontal lift of σ yields parallel translation along σ. If σ is a Frenet curve with a timelike principal normal vector, then the theorem below asserts that an appropriate curvature restriction on σ allows one to parallel translate the velocity vector at $\sigma(0)$ to a limit vector at $\sigma(L)$, yielding a *differentiable* end point at L. With just a little more work, we need not even assume the existence of the endpoint $\sigma(L)$, but infer its existence (first as a continuous endpoint, then as a differentiable one) from a completeness condition on M. The condition required is b-completeness ("b" for "bundle"), defined thus (see [3], p. 259 and Section 8.3): For $\sigma\colon [0, L) \to M^n$ a differentiable curve in a manifold M with a connection, any basis for $T_{\sigma(0)}M$ defines a Riemannian metric in the tangent spaces along σ by being parallel-translated all along σ and being regarded as an orthonormal basis at each point. This determines a length for σ in terms of this metric, called the *Schmidt length* of σ relative to the initial basis at $\sigma(0)$. Whether a Schmidt length for a given curve σ is finite or infinite is independent of the choice of initial basis. M is called *b-complete* if any differentiable curve σ: $[0, L) \to M$ of finite Schmidt length can be continuously extended to L.

THEOREM 2. *Let M be a b-complete Lorentz manifold, and let $\sigma\colon [0, L) \to M$ be a unit-speed spacelike curve obeying $\nabla_{\dot\sigma}\dot\sigma = \kappa N$ with N a unit-timelike vector defined over σ and κ a non-negative scalar defined over σ. If $L = L(\sigma)$ is finite and $|\nabla_{\dot\sigma}N|$ is bounded, then σ is differentiably extendible to (and past) L.*

PROOF. Let $\tau_s^t\colon T_{\sigma(t)}M \to T_{\sigma(s)}M$ be parallel translation along σ. Define $E(t) = \tau_t^0 N(0)$. Let $T = \dot\sigma$ and $S = T + \langle T, E\rangle E$, the component of T perpendicular to E. Let $'$ denote $\nabla_{\dot\sigma}$. The main burden of the proof is to show that with L finite and $|N'|$ bounded, $\langle T, E\rangle'$ and $|S'|$ are bounded also (S', being perpendicular to E, is spacelike). From this it immediately follows that $\langle T, E\rangle$ is bounded, as well as $|S| = (1 + \langle T, E\rangle^2)^{1/2}$. The

Schmidt length of σ, relative to an orthonormal basis at $\sigma(0)$ containing E_0, is $\int_0^L (\langle T, E\rangle^2 + |S|^2)^{1/2} dt$, which is therefore finite: this yields the (continuous) endpoint $\sigma(L)$. For differentiability, consider $X_t = \tau_0^t S_t$: This vector always lies in the spacelike subspace perpendicular to E_0; furthermore, $|X_t'| = |S_t'|$ is bounded. Therefore X_t has a limit X_L. Similarly, $\langle T, E\rangle_t$ has a limit r, so $\tau_0^t T_t = X_t - \langle T, E\rangle_t E_0$ has a limit $X_L - rE_0$. By Theorem 1, τ_L^t is defined. Let $E_L = \tau_L^0 E_0$. Then we have $\tau_L^t \dot{\sigma}(t) = \tau_L^0 \tau_0^t T_t$ has a limit $\tau_L^0(X_L - rE_0) = \tau_L^0 X_L - rE_L$. It follows that $\dot{\sigma}(t)$ approaches $\tau_L^0 X_L - rE_L$.

To show the boundedness of $\langle T, E\rangle'$ and $|S'|$, first we note that $\kappa = \langle N', T\rangle$. At each point $x = \sigma(t)$, define $\pi: T_xM \to T_xM$ to be projection onto the (spacelike) subspace perpendicular to both N and T, i.e., $\pi Y = Y + \langle Y, N\rangle N - \langle Y, T\rangle T$. Then $|\pi E|^2 = -1 + \langle E, N\rangle^2 - \langle E, T\rangle^2$, or

$$\langle N, E\rangle = \pm(1 + \langle T, E\rangle^2 + |\pi E|^2)^{1/2} . \tag{1}$$

We thus have $\langle T, E\rangle' = \kappa\langle N, E\rangle = \pm\langle N', T\rangle(1 + \langle T, E\rangle^2 + |\pi E|^2)^{1/2}$, so

$$\langle T, E\rangle^{2\prime} = \pm 2\langle N', T\rangle\langle T, E\rangle(1 + \langle T, E\rangle^2 + |\pi E|^2)^{1/2} . \tag{2}$$

Furthermore, using the fact that $\langle \pi X, Y\rangle = \langle X, \pi Y\rangle$, we also have

$$\begin{aligned} |\pi E|^{2\prime} &= 2(\langle E, N\rangle\langle E, N\rangle' - \langle E, T\rangle\langle E, \kappa N\rangle) \\ &= 2\langle N, E\rangle\langle N' - \langle N', T\rangle T, E\rangle \\ &= 2\langle N, E\rangle\langle \pi N', E\rangle = 2\langle N', \pi E\rangle\langle N, E\rangle \\ &= \pm 2\langle N', \pi E\rangle(1 + \langle T, E\rangle^2 + |\pi E|^2)^{1/2} . \end{aligned} \tag{3}$$

Let $x = \langle T, E\rangle^2$ and $y = |\pi E|^2$. Suppose that $|N'| \leqq C$, a constant. Then, since N', T, and πE all lie in the subspace perpendicular to N, we have, from equations (2) and (3)

$$|x'| \leqq 2Cx^{1/2}(1 + x + y)^{1/2} \leqq 2C(1 + x + y) ,$$

and

$$|y'| \leqq 2Cy^{1/2}(1 + x + y)^{1/2} \leqq 2C(1 + x + y) .$$

Let $z = \ln(x + y)$; then

$$|z'| \leqq (|x'| + |y'|)/(x + y) \leqq 4C(1/(x + y) + 1) = 4C(e^{-z} + 1) \tag{4}$$

If $\limsup(z) = \infty$ as $t \to L$, then (since $L < \infty$) there is a sequence $\{t_i\}$ with $z(t_i) \geqq i$ and $z'(t_i) \geqq i$, which contradicts inequality (4). Therefore, z is bounded above, so x and y, i.e., $\langle T, E\rangle^2$ and $|\pi E|^2$, must be also. From equation (1), it follows that $\langle N, E\rangle$ is bounded. Therefore, $\langle T, E\rangle' = \kappa\langle N, E\rangle = \langle N', T\rangle\langle N, E\rangle$ is bounded, as is $|S'| = |\kappa|\,|N + \langle N, E\rangle E| = |\langle N', T\rangle|(\langle N, E\rangle^2 - 1)^{1/2}$. □

Note that if σ is a geodesic in a spacelike hypersurface in M, then it satisfies $\nabla_{\dot\sigma}\dot\sigma = \kappa N$, with N the normal vector to the hypersurface. Theorem 2 is used in this context in [2] to show that in a b-complete Lorentz manifold, a closed spacelike hypersurface with bounded principal curvatures must be complete.

REMARK. If the timelike quality of the acceleration vector for σ is removed from the hypotheses of Theorem 1, then it is possible to construct counter-examples. For instance, let $\sigma: [\pi, \infty) \to \boldsymbol{R}^3$ be defined by $\sigma(t) = (4t^{-1/2}, t^{-1}\sin(t), \int_\pi^t s^{-1}\cos(s)ds)$. This has a continuous endpoint at $t = \infty$. With metric $dx^2 + dy^2 - dz^2$, it is spacelike and has finite length, but its Euclidean length is infinite. If the metric used is $e^{\rho(x)}(dx^2 + dy^2 - dz^2)$ for some function $\rho: \boldsymbol{R} \to \boldsymbol{R}$, then the Lorentz length is still finite. If ρ is appropriately chosen, then parallel translation along σ can be precisely calculated, and there is a choice of ρ under which parallel translation along σ fails to have a limit as $t \to \infty$.

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