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POSITIVE DIVISORS AND POINCARÉ SERIES ON VARIABLE RIEMANN SURFACES

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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1. Introduction. We are continuing the study of positive divisors on variable Riemann surfaces that we began in [4]. Let T_p be the Teichmüller space of closed Riemann surfaces of genus $p \geqq 2$. For any integer $n \geqq 1$ there is a fiber space $\pi_n: S_T^n(V_p) \to T_p$ whose fiber over $t \in T_p$ is the space of all positive divisors of degree n on the Riemann surface X_t represented by t. (See [4] for details.) Our goal is to find holomorphic sections of π_n. Such sections, if they exist, define on each X_t a positive divisor D_t of degree n that depends holomorphically on t.

Holomorphic sections of π_n are obtained from certain line bundles in the following standard way. Let $\pi: V_p \to T_p$ be the Teichmüller curve of genus p. For each $t \in T_p$ the fiber $\pi^{-1}(t) = X_t$ is the Riemann surface represented by t. By definition a *relative section* of the holomorphic line bundle $L \to V_p$ is a holomorphic section σ: $V_p \to L$ such that if σ vanishes identically on some fiber X_t, then σ is trivial (vanishes identically on V_p). If the relative section σ is nontrivial, then either σ has no zeros (and L is the trivial bundle over V_p) or the zeros of σ define a positive divisor D_t on X_t for each $t \in T_p$. In that case the degree n of D_t is independent of t, and the map $t \mapsto D_t$ is a holomorphic section of π_n. See [4] for details.

In this paper we shall use Poincaré series to produce relative sections of many line bundles over V_p. With their help we shall obtain holomorphic sections of π_n for every $n \geqq 2p - 2$. In fact we shall prove that every point of $S_T^n(V_p)$ lies in the range of some holomorphic section of π_n if $n \geqq 2p - 2$. For smaller values of n very little is known. Hubbard [8] showed that π_1 has no holomorphic sections unless $p = 2$. We showed in [4] that if $p = 2$, 3, or 4 then π_{p-1} has holomorphic sections such that each divisor D_t is half-canonical but that π_{p-1} has no holomorphic sections with that property if $p \geqq 5$. Bers [1] showed that π_{2p-2} has holomorphic

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sections such that each D_t is canonical, but he did not consider non-canonical divisors. Our methods are essentially the same as those of Bers, who also used Poincaré series. We obtain greater generality by using more general factors of automorphy (see § 3).

2. Statement of results. To state our first theorem we must introduce some line bundles over V_p. We shall describe them briefly here, with more details in § 3. Let Γ be the fundamental group of V_p. A *normalized character* of Γ is a homomorphism $\chi: \Gamma \to S^1$ of Γ into the multiplicative group

$$S^1 = \{z \in \boldsymbol{C}; |z| = 1\} .$$

Each such character determines a line bundle $L(\chi) \to V_p$. The *canonical line bundle* $K \to V_p$ is the determinant of the holomorphic cotangent bundle of V_p.

Our main result is

THEOREM 1. *Let $L \to V_p$ be any line bundle whose tensor power $L^{2p-2} \to V_p$ is the canonical line bundle. Choose any normalized character $\chi: \Gamma \to S^1$ and integer $n \geqq 2p - 2$. Put*

$$d = \begin{cases} n - (p - 1) & \text{if } n \geqq 2p - 1 \text{ or } n = 2p - 2 \text{ and } \chi \not\equiv 1 , \\ p & \text{if } n = 2p - 2 \text{ and } \chi \equiv 1 . \end{cases} \tag{2.1}$$

The line bundle $L^n \otimes L(\chi) \to V_p$ has d relative sections whose restrictions to each fiber X_t are linearly independent.

THEOREM 2. *If $n \geqq 2p - 2$, the map $\pi_n: S_T^n(V_p) \to T_p$ has holomorphic sections passing through any given point of $S_T^n(V_p)$.*

We shall prove Theorem 1 in §§ 4 and 5. Theorem 2 will be derived from Theorem 1 in § 6.

REMARKS. (1) If n is a multiple of $2p - 2$ and $\chi \equiv 1$, then $L^n \otimes L(\chi)$ is a power of the canonical bundle $K \to V_p$ and Theorem 1 reduces to Bers's results ([1], [2]) about holomorphic differentials on variable Riemann surfaces.

(2) Andrew Sommese has communicated to us the following short proof of Theorem 1. The bundle $\omega: L^n \otimes L(\chi) \to V_p$ has the property that for each $t \in T_p$, the dimension of the space of holomorphic sections of the restricted line bundle $\omega^{-1}(X_t) \to X_t$ is the number d in (2.1). Since that number is independent of t and the space T_p is contractible and Stein, Grauert's semicontinuity theorem (see [7]) implies that $L^n \otimes L(\chi) \to V_p$ has d relative sections that restrict to a basis for the sections over each

X_t. Our proof, using Poincaré series, is both more elementary and more concrete.

(3) The motivation for this investigation was the study of Prym differentials. We wanted to construct a basis for the Prym differentials that varied holomorphically with moduli. Theorem 1 with $n = 2p - 2$ yields such a basis.

(4) Theorem 1 for $n > 2p - 2$ can also be obtained by studying the mapping of $S^n_T(V_p)$ into the universal Jacobian variety $J(V_p)$. In this context see Gunning [5] and Earle [3].

3. Some factors of automorphy on the Bers fiber space. Let Γ be a Fuchsian group acting on the open unit disk Δ so that the quotient map $\Delta \to \Delta/\Gamma$ is a universal covering of a closed surface of genus $p \geqq 2$. Consider the set of all quasiconformal maps w of the plane onto itself such that

(i) $w \circ \gamma \circ w^{-1} = \gamma^w$ is a Möbius transformation for all $\gamma \in \Gamma$, and

(ii) w is conformal in the exterior of Δ with behavior

$$w(z) = z + O(z^{-1}) , \quad z \to \infty .$$

Call two such mappings *equivalent* if they agree on $\partial\Delta$. The set of all equivalence classes $[w]$ is the *Teichmüller space* T_p. It is a complex manifold of dimension $3p - 3$ and can be embedded in $\boldsymbol{C}^{3p-3}$ as a bounded contractible domain of holomorphy. We choose such an embedding.

The *Bers fiber space* F_p over T_p is the subregion

$$F_p = \{([w], z); [w] \in T_p \quad \text{and} \quad z \in w(\Delta)\}$$

of $T_p \times \boldsymbol{C} \subset \boldsymbol{C}^{3p-2}$. It is a bounded contractible domain of holomorphy in $\boldsymbol{C}^{3p-2}$. The group Γ acts properly discontinuously and freely on F_p as a group of biholomorphic maps

$$\gamma([w], z) = ([w], \gamma^w(z)) \quad \text{for all} \quad \gamma \in \Gamma \text{ and } ([w], z) \in F_p . \tag{3.1}$$

(Note that the Möbius transformation γ^w depends only on the equivalence class of w.) The projection $([w], z) \mapsto [w]$ of F_p onto T_p induces a holomorphic map π from the quotient manifold $V_p = F_p/\Gamma$ onto T_p, and $\pi: V_p \to T_p$ is the Teichmüller curve of genus p.

Since F_p is contractible and Stein, Γ is the fundamental group of V_p and all line bundles over V_p are determined by factors of automorphy on $\Gamma \times F_p$ (see Gunning [6], pp. 14–16). By definition, a factor of automorphy is a map $\xi: \Gamma \times F_p \to \boldsymbol{C}$ such that $\xi(\gamma, \cdot)$ is a nowhere vanishing holomorphic function on F_p for each $\gamma \in \Gamma$, and

$$\xi(\gamma_1\gamma_2, \zeta) = \xi(\gamma_1, \gamma_2(\zeta))\xi(\gamma_2, \zeta)$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $\zeta = ([w], z) \in F_p$. The holomorphic sections of the line bundle determined by ξ are given by the ξ-automorphic functions on F_p. These are the holomorphic functions $f\colon F_p \to \boldsymbol{C}$ such that

$$f(\gamma(\zeta)) = \xi(\gamma, \zeta)f(\zeta) \quad \text{for all} \quad \gamma \in \Gamma \text{ and } \zeta = ([w], z) \in F_p . \tag{3.2}$$

The ξ-automorphic function f determines a nontrivial relative section if the function $f([w], \cdot)$ never vanishes identically on $w(\Delta)$.

The canonical line bundle $K \to V_p$ is determined by the factor of automorphy

$$\xi(\gamma, ([w], z)) = \frac{\partial \gamma}{\partial z}([w], z)^{-1} = (\gamma^w)'(z)^{-1} .$$

Since there are line bundles $L \to V_p$ such that $L^{2p-2} = K$ (see Sipe [11] and the remark at the end of this section), there are factors of automorphy ξ_1 such that

$$\xi_1(\gamma, ([w], z))^{2p-2} = (\gamma^w)'(z)^{-1} \quad \text{for all} \quad \gamma \in \Gamma \text{ and } ([w], z) \in F_p . \tag{3.3}$$

We choose once and for all such a ξ_1 and the line bundle $L \to V_p$ it determines.

The normalized character $\chi\colon \Gamma \to S^1$ determines the "flat" factor of automorphy

$$\xi(\gamma, ([w], z)) = \chi(\gamma)$$

and corresponding line bundle $L(\chi) \to V_p$. The line bundles $L^n \otimes L(\chi)$ in Theorem 1 are determined by the factors of automorphy

$$\xi(\gamma, ([w], z)) = \chi(\gamma)\xi_1(\gamma, ([w], z))^n \quad \text{for all} \quad \gamma \in \Gamma \text{ and } ([w], z) \in F_p . \tag{3.4}$$

By (3.1), (3.2) and (3.3), the ξ-automorphic functions $f\colon F_p \to \boldsymbol{C}$ for these factors of automorphy satisfy

$$f([w], z) = f([w], \gamma^w(z))(\gamma^w)'(z)^{n/(2p-2)}\chi(\gamma)^{-1} .$$

We shall put $q = n(2p-2)^{-1}$ and write that equation in the more familiar form

$$f([w], z) = f([w], \gamma^w(z))(\gamma^w)'(z)^q\chi(\gamma)^{-1} \quad \text{for all} \quad \gamma \in \Gamma \text{ and } ([w], z) \in F_p . \tag{3.5}$$

In (3.5) we must remember that $q(2p-2)$ is an integer and that by definition

$$(\gamma^w)'(z)^q = \xi_1(\gamma, ([w], z))^{q(2p-2)} .$$

REMARK. For the reader's convenience we outline a proof that there is a factor of automorphy ξ_1 satisfying (3.3). Without loss of generality we assume that the Riemann surface Δ/Γ is hyperelliptic. Thus Δ/Γ has an abelian differential of the first kind with a single zero, of order $2p-2$. Let f be a $(2p-2)$-th root of its lift to Δ. It can be verified that there is a unique factor of automorphy ξ_1 that satisfies (3.3) and has the property

$$\xi_1(\gamma, ([I], z)) = \frac{f(\gamma z)}{f(z)} \quad \text{for all} \quad \gamma \in \Gamma \text{ and } z \in \Delta .$$

(Here I denotes the identity map on Δ.)

4. **Proof of Theorem 1 for** $q \geqq 2$. For any $q = n(2p-2)^{-1}$, $n \geqq 2p-2$, the holomorphic sections of the line bundle $\omega\colon L^n \otimes L(\chi) \to V_p$ in Theorem 1 are defined by the holomorphic functions f on F_p that satisfy (3.5). If $[w] = t \in T_p$, the sections of the restricted bundle $\omega^{-1}(X_t) \to X_t$ over $X_t = \pi^{-1}(t)$ are defined by the holomorphic functions f on $w(\Delta)$ such that

$$f(z) = f(\gamma^w(z))(\gamma^w)'(z)^q \chi(\gamma)^{-1} \quad \text{for all} \quad \gamma \in \Gamma \text{ and } z \in w(\Delta) .$$

These functions on $w(\Delta)$ form a vector space $A_q(\Gamma, \chi^{-1}, t)$ whose dimension, by the Riemann-Roch theorem, is the number d defined by (2.1).

Let ρ be the natural projection from

$$A_q(T_p, \chi^{-1}) = \bigcup_{t \in T_p} A_q(\Gamma, \chi^{-1}, t)$$

to T_p, which maps $A_q(\Gamma, \chi^{-1}, t)$ to t for each $t \in T_p$. We shall prove Theorem 1 by defining an appropriate vector bundle structure on $\rho\colon A_q(T_p, \chi^{-1}) \to T_p$.

First we assume $q \geqq 2$. In that case, if $P(z)$ is any polynomial, the Poincaré series

$$(\Theta P)([w], z) = \sum_{\gamma \in \Gamma} P(\gamma^w(z))(\gamma^w)'(z)^q \chi(\gamma)^{-1}$$

converges uniformly on compact sets in F_p to a holomorphic function that satisfies (3.5). Now fix any point $[w_0] = t_0 \in T_p$. By Theorem 3 of Knopp [9] there are polynomials $P_1(z), \cdots, P_d(z)$ such that the functions $(\Theta P_j)(t_0, \cdot)$, $1 \leqq j \leqq d$, are a basis for $A_q(\Gamma, \chi^{-1}, t_0)$. For each $[w] = t \in T_p$, let $W(t, \cdot)$ be the Wronskian of the d functions $(\Theta P_j)(t, \cdot)$ on $w(\Delta)$. Then $W(t, z)$ is a holomorphic function on F_p. By construction $W(t_0, \cdot)$ does not vanish identically in $w_0(\Delta)$. It follows that t_0 has an open neighborhood $D \subset T_p$ such that $W(t, \cdot)$ does not vanish identically in $w(\Delta)$ if $[w] = t \in D$. Therefore the functions $(\Theta P_j)(t, \cdot)$, $1 \leqq j \leqq d$, are a basis for $A_q(\Gamma, \chi^{-1}, [w])$ whenever $[w] = t \in D$. The bijective map

$$(t, c) \mapsto \sum_{j=1}^{d} c_j (\Theta P_j)(t, \cdot)$$

from $D \times \boldsymbol{C}^d$ to $\rho^{-1}(D)$ defines a local trivialization of $A_q(T_p, \chi^{-1})$ over D.

We must show that two such trivializations over the same set $D \subset T_p$ are compatible. Suppose the polynomials $P_1, \cdots, P_d$ and $Q_1, \cdots, Q_d$ define trivializations over D as above. For each $t \in D$, there is a matrix $A(t) \in GL(d, \boldsymbol{C})$ such that

$$\begin{bmatrix} \Theta P_1(t, z) \\ \vdots \\ \Theta P_d(t, z) \end{bmatrix} = A(t) \begin{bmatrix} \Theta Q_1(t, z) \\ \vdots \\ \Theta Q_d(t, z) \end{bmatrix}$$

for all $z \in w(\Delta)$, $[w] = t$. We must show that $A(t)$ depends holomorphically on t. This is a local problem. Fix $t_0 = [w_0] \in D$ and choose points $z_1, \cdots, z_d \in w_0(\Delta)$ so that the linear functionals $f \mapsto f(z_j)$, $1 \leqq j \leqq d$, on $A_q(\Gamma, \chi^{-1}, t_0)$ are linearly independent. Then the matrices

$$B(t) = (\Theta P_i(t, z_j)) \ , \quad C(t) = (\Theta Q_i(t, z_j)) \ , \quad 1 \leqq i, j \leqq d \ ,$$

are nonsingular when $t = t_0$, hence for t in a neighborhood of t_0. In that neighborhood $A(t) = B(t)C(t)^{-1}$ is a holomorphic function of t, as required.

We have shown that $\rho: A_q(T_p, \chi^{-1}) \to T_p$ is a holomorphic vector bundle. Since T_p is a contractible domain of holomorphy, a theorem of Grauert implies that this vector bundle is trivial. It therefore has holomorphic sections $s_1, \cdots, s_d$ such that the functions $s_j(t)$, $1 \leqq j \leqq d$, are linearly independent in $A_q(\Gamma, \chi^{-1}, t)$ for every $t \in T_p$. The functions

$$f_j(t, z) = s_j(t)(z) \quad \text{for all} \quad (t, z) \in F_p$$

are the relative sections required in Theorem 1.

REMARK. If $q \geqq 2$ is an integer and $\chi \equiv 1$, the main theorem of Kra [10] shows that Γ may be chosen so that $A_q(T_p, \chi^{-1})$ is the set of functions $(\Theta P)(t, \cdot)$, $t \in T_p$ and P a polynomial of degree $\leqq d - 1$. These functions obviously form a trivial vector bundle over T_p.

5. Proof of Theorem 1 for $1 \leqq q < 2$. It remains to define the vector bundle structure on $\rho: A_q(T_p, \chi^{-1}) \to T_p$ when $1 \leqq q < 2$. Let q be given; if $q = 1$, assume $\chi \not\equiv 1$. Fix $t_0 = [w_0] \in T_p$. The functions whose zeros are all simple form a dense open set in $A_2(\Gamma, 1, t_0)$. Applying Theorem 1 with $q = 2$ and $\chi \equiv 1$, we obtain a holomorphic function $f(t, z)$ on F_p such that $f(t, \cdot) \in A_2(\Gamma, 1, t)$ for all t and the zeros of $f(t_0, \cdot)$ are all simple. Let $z_1, \cdots, z_{4p-4} \in w_0(\Delta)$ be Γ-inequivalent zeros of $f(t_0, \cdot)$. There are holomorphic functions $z_j(t)$, $1 \leqq j \leqq 4p - 4$, defined in a neighborhood D of t_0, such that $z_j(t_0) = z_j$ and $f(t, z_j(t)) = 0$ for all $t \in D$. If $t \in D$, $z_1(t), \cdots, z_{4p-4}(t)$ are a complete set of Γ-inequivalent zeros of $f(t, \cdot)$, and all zeros of $f(t, \cdot)$ are simple.

Since the vector bundle $A_{q+2}(T_p, \chi^{-1}) \to T_p$ is trivial, there are holomorphic functions h_i on F_p, $1 \leqq i \leqq (2q+3)(p-1) = l$, such that for each $t \in T_p$ the functions $h_i(t, \cdot)$ are a basis for $A_{q+2}(\Gamma, \chi^{-1}, t)$. For $t \in D$ consider

$$h = \sum_{i=1}^{l} c_i h_i$$

and $g = h/f$. Then $g(t, \cdot) \in A_q(\Gamma, \chi^{-1}, t)$ if and only if $c = (c_1, \cdots, c_l)$ is chosen so that

$$h(t, z_j(t)) = 0 , \qquad 1 \leqq j \leqq 4p-4 . \tag{5.1}$$

Therefore, for each given t the space of solutions c of (5.1) has dimension $d = \dim A_q(\Gamma, \chi^{-1}, t)$. Since $l - d = 4p - 4$, we can apply the implicit function theorem to obtain holomorphic functions $h_i^*(t, z)$, $1 \leqq i \leqq d$, defined for $t = [w]$ in a neighborhood D_0 of t_0 and $z \in w(\Delta)$, such that

$$h_i^*(t, z_j(t)) = 0 , \qquad 1 \leqq i \leqq d \quad \text{and} \quad 1 \leqq j \leqq 4p-4 ,$$

and the functions $h_i^*(t, \cdot) \in A_{q+2}(\Gamma, \chi^{-1}, t)$ are linearly independent for each $t \in D_0$. The functions $g_i = h_i^*/f$ are holomorphic and give a basis $g_i(t, \cdot)$, $1 \leqq i \leqq d$, of $A_q(\Gamma, \chi^{-1}, t)$ for each $t \in D_0$. The bijective map

$$(t, c) \mapsto \sum_{i=1}^{d} c_i g_i(t, \cdot)$$

from $D_0 \times \boldsymbol{C}^d$ to $\rho^{-1}(D_0)$ is the required local trivialization of $A_q(T_p, \chi^{-1})$ near t_0.

The proof is completed exactly as before by showing that all such trivializations are compatible and by choosing sections of $\rho\colon A_q(T_p, \chi^{-1}) \to T_p$. The details are unchanged.

The case $q = 1$ and $\chi \equiv 1$ must be treated slightly differently because $l - d < 4p - 4$. We omit the details because this case of Theorem 1 was already proved by Bers [1].

6. Proof of Theorem 2. Fix any point $t \in T_p$ and any positive divisor D_t on $X_t = \pi^{-1}(t)$. There is a holomorphic line bundle $L_t \to X_t$ with a holomorphic section $s_t\colon X_t \to L_t$ whose divisor is D_t. Let $\deg(D_t) = n \geqq 2p - 2$. There is a unique normalized character $\chi\colon \Gamma \to S^1$ such that $L_t \to X_t$ is (isomorphic to) the restriction $\omega^{-1}(X_t) \to X_t$ of the line bundle $\omega\colon L^n \otimes L(\chi) \to V_p$. By Theorem 1, there is a nontrivial relative section $s\colon V_p \to L^n \otimes L(\chi)$ whose restriction to X_t is s_t. The divisor of s provides a holomorphic section $\sigma\colon T_p \to S_T^n(V_p)$ such that $\sigma(t) = D_t$.

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