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EINSTEIN KAEHLER SUBMANIFOLDS OF A COMPLEX LINEAR OR HYPERBOLIC SPACE

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Introduction. Einstein Kaehler submanifolds of a complex space form have been studied by several authors. In the case of codimension one, Smyth [4] and Chern [2] showed them to be either totally geodesic or certain hyperquadrics of a complex projective space. In this classification, Takahashi [5] showed that the Einstein condition can be weakened to the condition that Ricci tensor is parallel. Recently, Tsukada [6] studied the case of codimension two and obtained the same classification. In this paper we completely classify Einstein Kaehler submanifolds of a complex linear or hyperbolic space and prove the following:

THEOREM. *Every Einstein submanifold of a complex linear or hyperbolic space is always totally geodesic.*

Note that our theorem holds for any codimension.

1. Preliminaries. It is well-known that the Kaehler metric $g = 2\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ of a Kaehler n-manifold M can be locally constructed from a certain real-valued smooth function f by

$$g_{\alpha\bar{\beta}} = \partial^2 f/\partial z^\alpha \partial\bar{z}^\beta \qquad (\alpha, \beta = 1, \cdots, n),$$

where $(z^1, \cdots, z^n)$ is a local complex coordinate system. Such a function f, which is called primitive, is determined up to the real part of a holomorphic function. If the metric g is real analytic, the *diastasis* $D_M(p, q)$ is introduced (cf. [1]), which is a real analytic function defined on a neighborhood of the diagonal set $\{(p, p); p \in M\}$ of the product space $M \times M$ and satisfies the following properties:

(1) The function $D_M(p, q)$ is uniquely determined by the Kaehler metric g.

(2) $D_M(p, q) = D_M(q, p)$, and $D_M(p, p) = 0$.

(3) For $p \in M$ fixed, $D_M(p, q)$ is a primitive function of g with respect to the variable q.

EXAMPLE 1. Let $(\xi^1, \cdots, \xi^N)$ be the canonical complex coordinate system in $\boldsymbol{C}^N$. Then the diastasis of $\boldsymbol{C}^N$ is given by

$$D(p, q) = \sum_{\sigma=1}^{N} |\xi^\sigma(p) - \xi^\sigma(q)|^2 \qquad (p, q \in \boldsymbol{C}^N) ,$$

namely the square of the Euclidean distance.

EXAMPLE 2. The complex hyperbolic space $\boldsymbol{CH}^N$ of holomorphic sectional curvature -2 is a ball $\{q \in \boldsymbol{C}^N; \sum_{\sigma=1}^{N} |\xi^\sigma(q)|^2 < 1\}$, whose diastasis is given by

$$D(p, q) = -\log\left(1 - \sum_{\sigma=1}^{N} |\xi^\sigma(q)|^2\right) ,$$

where $p = (0, \cdots, 0)$.

Though the diastasis depends only on the metric, it is compatible with that of the ambient space. Using it, we can prove the following two facts:

LEMMA 1.1 ([7; Lemma 1.2]). *Let M be a Kaehler manifold, and $p \in M$ an arbitrarily fixed point. Then a neighborhood U of p is holomorphically and isometrically immersed into $\boldsymbol{C}^N$ if and only if the metric is real analytic and there exist holomorphic functions $\phi^1, \cdots, \phi^N$ defined on U such that*

$$D_M(p, q) = \sum_{\sigma=1}^{N} |\phi^\sigma(q)|^2 \qquad (q \in U) ,$$
$$\phi^\sigma(p) = 0 \qquad (\sigma = 1, \cdots, N) .$$

LEMMA 1.2 ([7; Lemma 1.3]). *Let M be a Kaehler manifold and $p \in M$ an arbitrarily fixed point. Then a neighborhood U of p is holomorphically and isometrically immersed into $\boldsymbol{CH}^N$ if and only if the metric is real analytic and there exist holomorphic functions $\phi^1, \cdots, \phi^N$ defined on U such that*

$$\exp\{-D_M(p, q)\} = 1 - \sum_{\sigma=1}^{N} |\phi^\sigma(q)|^2 \qquad (q \in U) ,$$
$$\phi^\sigma(p) = 0 \qquad (\sigma = 1, \cdots, N) .$$

Let $\Lambda(M)$ be a set of $\boldsymbol{R}$-linear combinations of real analytic functions $\{h\bar{k} + k\bar{h}$, where h and k are holomorphic functions on $M\}$. Obviously $\Lambda(M)$ is an associative algebra. In [8], the author proved the following:

LEMMA 1.3 ([8; Proposition 3.5]). *Let $\phi^1, \cdots, \phi^N$ be non-constant holomorphic functions on a complex manifold M such that $\phi^\sigma(p) = 0$ ($\sigma = 1, \cdots, N$) for a fixed point $p \in M$. Then*

(1) $\exp(\sum_{\sigma=1}^{N} |\phi^\sigma|^2) \notin \Lambda(M)$,

(2) $(1 - \sum_{\sigma=1}^{N} |\phi^\sigma|^2)^{-\alpha} \notin \Lambda(M)$ $(\alpha > 0)$.

2. Einstein Kaehler submanifolds of a complex linear or hyperbolic space. Let M be a Kaehler n-submanifold of a Kaehler manifold of constant holomorphic sectional curvature $2c$. Then the Ricci tensor, denoted by Ric_M, satisfies

$$\mathrm{Ric}_M \leqq (n+1)cg \,, \tag{2.1}$$

where g is the Kaehler metric of M. The equality holds if and only if M is totally geodesic. This inequality is an immediate consequence of the Gauss equation (cf. [3; p. 177]). In particular, the Ricci tensor is always negative semi-definite if $c \leqq 0$.

Now we suppose that $c \leqq 0$ and M is an Einstein manifold. Then the Ricci tensor $\mathrm{Ric}_M = 2\sum_{\alpha,\beta=1}^{n} K_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ is related to the Kaehler metric $g = 2\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ by

$$K_{\alpha\bar{\beta}} = -\mu g_{\alpha\bar{\beta}} \qquad (\alpha, \beta = 1, \cdots, n) \,, \tag{2.2}$$

where $\mu \geqq 0$ is a constant. On the other hand, it is known (cf. [3; p. 158]) that the Ricci tensor is given by

$$K_{\alpha\bar{\beta}} = -\partial^2 \log G / \partial z^\alpha \partial \bar{z}^\beta \qquad (\alpha, \beta = 1, \cdots, n) \,, \tag{2.3}$$

where G denotes the determinant of the Hermitian matrix $(g_{\alpha\bar{\beta}})_{\alpha,\beta=1,\cdots,n}$. In case $\mu \neq 0$, (2.2) and (2.3) imply that $(1/\mu)\log G$ is a primitive function of g. Since the primitive function is determined up to the real part of a holomorphic function, we have

$$D_M(p, *) = (1/\mu)(h + \bar{h} + \log G) \,,$$

locally for a holomorphic function h, that is,

$$\exp\{\mu D_M(p, *)\} = |\exp(h)|^2 G \,, \tag{2.4}$$

where $p \in M$ is a fixed point. First of all we consider Einstein Kaehler submanifolds of $\boldsymbol{C}^N$.

THEOREM 2.1. *Let M be an Einstein Kaehler n-submanifold of $\boldsymbol{C}^N$ ($n \geqq 1$). Then M is totally geodesic.*

PROOF. Since M is an Einstein manifold, it satisfies (2.2) and (2.3) on a sufficiently small coordinate neighborhood $\{U; (z^1, \cdots, z^n)\}$ of a fixed point $p \in M$. If M is not totally geodesic, then (2.1) implies that $\mu > 0$. By a homothetic transformation of $\boldsymbol{C}^N$, we may sssume $\mu = 1$. By Lemma 1.1, there exist holomorphic functions $\phi^1, \cdots, \phi^N$ on U such that

$$D_M(p, q) = \sum_{\sigma=1}^{N} |\phi^\sigma(q)|^2 \qquad (q \in U) \,,$$

$$\phi^\sigma(p) = 0 \qquad (\sigma = 1, \cdots, N) \,.$$

So we have

$$g_{\alpha\bar{\beta}} = \sum_{\sigma=1}^{N} (\partial\phi^{\sigma}/\partial z^{\alpha})\overline{(\partial\phi^{\sigma}/\partial z^{\beta})} \qquad (\alpha, \beta = 1, \cdots, n) .$$

Since the matrix $(g_{\alpha\bar{\beta}})$ is Hermitian, its determinant G is real-valued. So $G \in \Lambda(U)$. On the other hand, M satisfies (2.4), that is,

$$\exp\{D_M(p, *)\} = |\exp(h)|^2 G .$$

Hence we have

$$\exp\left(\sum_{\sigma=1}^{N} |\phi^{\sigma}|^2\right) = |\exp(h)|^2 G \in \Lambda(U) .$$

But this contradicts (1) of Lemma 1.3. q.e.d.

Now we consider the hyperbolic case with $c = -1$.

LEMMA 2.2. *Let M be a complex n-manifold and $\{U; (z^1, \cdots, z^n)\}$ a complex local coordinate neighborhood of M. If $f \in \Lambda(U)$, then $f^{n+1} \det(\partial^2 \log f/\partial z^{\alpha}\partial \bar{z}^{\beta}) \in \Lambda(U)$.*

PROOF. For the sake of simplicity, we put $f_{\alpha} = \partial f/\partial z^{\alpha}$, $f_{\bar{\beta}} = \partial f/\partial \bar{z}^{\beta}$ and $f_{\alpha\bar{\beta}} = \partial^2 f/\partial z^{\alpha}\partial \bar{z}^{\beta}$ $(\alpha, \beta = 1, \cdots, n)$. Then

$$\partial^2 \log f/\partial z^{\alpha}\partial \bar{z}^{\beta} = f_{\alpha\bar{\beta}}/f - f_{\alpha}f_{\bar{\beta}}/f^2 ,$$

and we have

$$\begin{aligned} f^{n+1} \det(\partial^2 \log f/\partial z^{\alpha}\partial \bar{z}^{\beta}) &= f \det(f_{\alpha\bar{\beta}} - f_{\alpha}f_{\bar{\beta}}/f) \\ &= f \det\begin{pmatrix} f_{1\bar{1}} - f_1 f_{\bar{1}}/f & \cdots & f_{1\bar{n}} - f_1 f_{\bar{n}}/f & 0 \\ \vdots & & \vdots & \vdots \\ f_{n\bar{1}} - f_n f_{\bar{1}}/f & \cdots & f_{n\bar{n}} - f_n f_{\bar{n}}/f & 0 \\ f_{\bar{1}}/f & \cdots & f_{\bar{n}}/f & 1 \end{pmatrix} \\ &= f \det\begin{pmatrix} f_{1\bar{1}} & \cdots & f_{1\bar{n}} & f_1 \\ \vdots & & \vdots & \vdots \\ f_{n\bar{1}} & \cdots & f_{n\bar{n}} & f_n \\ f_{\bar{1}}/f & \cdots & f_{\bar{n}}/f & 1 \end{pmatrix} = \det\begin{pmatrix} f_{1\bar{1}} & \cdots & f_{1\bar{n}} & f_1 \\ \vdots & & \vdots & \vdots \\ f_{n\bar{1}} & \cdots & f_{n\bar{n}} & f_n \\ f_{\bar{1}} & \cdots & f_{\bar{n}} & f \end{pmatrix} . \end{aligned}$$

Hence $f^{n+1} \det(\partial^2 \log f/\partial z^{\alpha}\partial \bar{z}^{\beta})$ is finitely generated by holomorphic or anti-holomorphic functions on U. In addition, it is real-valued, because the matrix $(\partial^2 \log f/\partial z^{\alpha}\partial \bar{z}^{\beta})$ is Hermitian. So we conclude $f^{n+1} \det(\partial^2 \log f/\partial z^{\alpha}\partial \bar{z}^{\beta}) \in \Lambda(M)$. q.e.d.

THEOREM 2.3. *Let M be an Einstein Kaehler n-submanifold of CH^N $(n \geqq 1)$. Then M is totally geodesic.*

PROOF. By (2.1), the Ricci tensor of M is negative definite. Hence $\mu \neq 0$ and M satisfies (2.4) on a sufficiently small coordinate neighborhood $\{U; (z^1, \cdots, z^n)\}$ of a fixed point $p \in M$. By Lemma 1.2, there exist holomorphic functions $\phi^1, \cdots, \phi^N$ defined on U such that

$$D_M(p, q) = -\log\left(1 - \sum_{\sigma=1}^{N} |\phi^\sigma(q)|^2\right) \qquad (q \in U)\,, \tag{2.5}$$

$$\phi^\sigma(p) = 0 \qquad (\sigma = 1, \cdots, N)\,.$$

Now if we put $f = 1 - \sum_{\sigma=1}^{N} |\phi^\sigma|^2$, then

$$G = (-1)^n \det(\partial^2 \log f / \partial z^\alpha \partial \bar{z}^\beta)\,. \tag{2.6}$$

From (2.4), (2.5) and (2.6), we have

$$f^{-\mu} = (-1)^n |\exp(h)|^2 \det(\partial^2 \log f / \partial z^\alpha \partial \bar{z}^\beta)\,.$$

Hence

$$f^{n+1-\mu} = (-1)^n |\exp(h)|^2 \{f^{n+1} \det(\partial^2 \log f / \partial z^\alpha \partial \bar{z}^\beta)\}\,.$$

By Lemma 2.2, we obtain

$$\left(1 - \sum_{\sigma=1}^{N} |\phi^\sigma|^2\right)^{n+1-\mu} = f^{n+1-\mu} \in \Lambda(U)\,.$$

Then (2) of Lemma 1.3 implies $n + 1 - \mu \geqq 0$. On the other hand, $n + 1 - \mu \leqq 0$ by (2.1). Thus $\mu = n + 1$ and M is totally geodesic.

q.e.d.

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