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SYMMETRIES AND φ-SYMMETRIC SPACES

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1. Introduction. As is well-known, the local geodesic symmetries on a locally Riemannian symmetric space are isometries and hence they are volume-preserving local diffeomorphisms. However, there are many Riemannian manifolds all of whose geodesic symmetries are volume-preserving but which are not locally symmetric. To our knowledge it is not even known if such spaces are locally homogeneous. This last problem was considered in [12], [13], [14] and extended to a more general class of symmetries in [2], [10]. In particular, in [12] Sekigawa and the second author showed that an almost Hermitian manifold with symplectic geodesic symmetries is a locally symmetric Kähler manifold.

The main purpose of this paper is to study similar problems on almost contact metric manifolds. On such manifolds one has a preferred vector field ξ and an almost Hermitian structure on the orthogonal complement of ξ. For a general almost contact metric manifold, the behavior in the direction ξ can be quite arbitrary and hence one cannot expect a result exactly similar to that in [12]. In this paper we shall therefore consider the case where ξ generates a one-parameter group of isometries. We study a class of symmetries on these spaces, the so-called *φ-geodesic symmetries* [15], and then obtain results when the dual form η is closed or when the structure is a contact metric structure. This leads to a characterization of the so-called *φ-symmetric spaces* [15], a class of manifolds which seems to be the analogue in the almost contact metric case of the class of locally Hermitian symmetric spaces.

The paper is organized as follows. In Section 2 we give some preliminaries and in Section 3 we treat φ-symmetric spaces and φ-geodesic symmetries. In Section 4 we derive the main result giving the analogue of the already mentioned result in [12]. Finally, in Section 5 we give a complete classification of three-dimensional Sasakian spaces with volume-preserving local φ-geodesic symmetries.

2. Preliminaries. A C^∞ manifold M^{2n+1} is said to be an *almost contact manifold* if the structural group of its tangent bundle is reducible to $U(n)\times 1$. It is well-known that such a manifold admits a tensor field

φ of type (1, 1), a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1\ , \quad \varphi^2 = -I + \eta \otimes \xi\ . \tag{1}$$

These conditions imply that $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Moreover, M admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for any tangent vector fields X, Y. Note that this implies $\eta(X) = g(X, \xi)$. M together with these structure tensors is said to be an *almost contact metric manifold* and we refer to (φ, ξ, η, g) as an *almost contact metric structure*. For a general reference to these ideas see [1].

As we remarked in the introduction, on a general almost contact metric manifold, the behavior in the direction ξ can be quite general; in particular the integral curves of ξ need not be geodesics, nor does a geodesic which is initially orthogonal to ξ, necessarily remain orthogonal to ξ.

LEMMA 1. *If ξ is a Killing vector field on an almost contact metric manifold, then the integral curves of ξ are geodesics, and geodesics which are initially orthogonal to ξ remain orthogonal to ξ.*

PROOF. To see the first statement simply note that, since ξ is a unit Killing vector field, $g(\nabla_\xi \xi, X) = -g(\nabla_X \xi, \xi) = 0$. For the second statement note that for a geodesic γ, $\gamma' g(\gamma', \xi) = g(\gamma', \nabla_{\gamma'}\xi) = 0$ and hence the angle between ξ and γ' is constant.

LEMMA 2. *If on an almost contact metric manifold M, ξ is a Killing vector field and $d\eta = 0$, then M is locally the product of an almost Hermitian manifold and the real line.*

PROOF. Since $\eta(X) = g(X, \xi)$, the two conditions $d\eta = 0$ and ξ being a Killing vector field imply that ξ is parallel on M. Therefore the distribution (subbundle) orthogonal to ξ is also parallel and M is locally the product of an even-dimensional manifold N and $\boldsymbol{R}$. Now from (1) and (2) we see that φ and g restricted to N form an almost complex structure and a Hermitian metric.

Given an almost contact metric structure (φ, ξ, η, g) on a manifold M, one may define a natural almost complex structure J on $M \times \boldsymbol{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X is tangent to M, f a function on $M \times \boldsymbol{R}$ and t the coordinate on $\boldsymbol{R}$. If this almost complex structure is integrable, we say that the almost contact structure is *normal*; the integrability condition for this is the vanishing of the tensor field

$$N^{(1)} = [\varphi, \varphi] + 2d\eta \otimes \xi ,$$

where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ.

Also for an almost contact metric structure we define its *fundamental 2-form* ϕ by

$$\phi(X, Y) = g(X, \varphi Y) .$$

If $\phi = d\eta$, we say that (φ, ξ, η, g) is a *contact metric structure*. In particular, we have $\eta \wedge (d\eta)^n \neq 0$. A normal contact metric structure is called a *Sasakian structure*. The two conditions of being normal and contact metric may be written as one, namely

$$(3) \qquad (\nabla_X \varphi) Y = g(X, Y)\xi - \eta(Y)X .$$

Note that this last condition implies that

$$(4) \qquad \nabla_X \xi = -\varphi X ,$$

from which it follows that ξ is a Killing vector field. The curvature tensor

$$R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

of a Sasakian manifold satisfies

$$(5) \qquad R_{XY}\xi = \eta(Y)X - \eta(X)Y ,$$

$$(6) \qquad R_{X\xi}Y = \eta(Y)X - g(X, Y)\xi .$$

Again, for a general reference to the above ideas, see [1].

Finally, considering a tensor field S of type $(1, 1)$ as a field of endomorphisms of tangent spaces, a tensor field P of type (p, q) is said to be *S-invariant* if for all 1-forms $\omega_1, \cdots, \omega_p$ and all vector fields $X_1, \cdots, X_q$,

$$P(\omega_1 \circ S, \cdots, \omega_p \circ S, X_1, \cdots, X_q) = P(\omega_1, \cdots, \omega_p, SX_1, \cdots, SX_q) .$$

Also, as a notational matter, we write $R(X, Y, Z, W)$ for $g(R_{XY}Z, W)$ and $(\nabla_U R)(X, Y, Z, W)$ for $g((\nabla_U R)_{XY}Z, W)$.

3. φ-geodesic symmetries and φ-symmetric spaces. Let M be an almost contact metric manifold with a Killing vector field ξ. Also we always suppose M to be connected in the rest of the paper.

A geodesic γ is said to be a *φ-geodesic* if $\eta(\gamma') = 0$. A local diffeomorphism s_m of M, $m \in M$, is said to be a *φ-geodesic symmetry* if its domain $\mathscr{U}$ is such that, for every φ-geodesic $\gamma(s)$, where $\gamma(0)$ lies in the intersection of $\mathscr{U}$ with the integral curve of ξ through m,

$$(s_m \circ \gamma)(s) = \gamma(-s) ,$$

for all s with $\gamma(\pm s) \in \mathscr{U}$, s being the arc length [15]. Since the points of the integral curve of ξ through m are fixed, we see that setting

$$S = -I + 2\eta \otimes \xi ,$$

we have, since ξ is a Killing vector field,

$$s_m = \exp_m \circ S_m \circ \exp_m^{-1} .$$

Now let M be a Sasakian manifold. Then M is said to be a *locally φ-symmetric space* if

$$\varphi^2(\nabla_V R)_{XY} Z = 0$$

for all vector fields V, X, Y, Z orthogonal to ξ. These spaces were introduced by Takahashi in [15]. We also refer to [15] for examples and some important results. In particular, the author proved:

PROPOSITION 3. *A Sasakian manifold is a locally φ-symmetric space if and only if it admits at every point a φ-geodesic symmetry, which is a local automorphism i.e., a local diffeomorphism leaving all structure tensor fields invariant.*

Also we note the following useful result proved by Tanno in [16]:

PROPOSITION 4. *Let M be a contact metric manifold with structure tensors (φ, ξ, η, g). If a diffeomorphism f of M leaves the structure tensor φ invariant, then there exists a positive constant α such that*

$$f_*\xi = \alpha\xi , \quad f^*\eta = \alpha\eta ,$$
$$(f^*g)(X, Y) = \alpha g(X, Y) + \alpha(\alpha - 1)\eta(X)\eta(Y) .$$

We also give another characterization of locally φ-symmetric spaces. This result shows how the φ-geodesic symmetries play a similar role for this class of manifolds as the geodesic symmetries do for locally symmetric spaces.

THEOREM 5. *A necessary and sufficient condition for a Sasakian manifold to be a locally φ-symmetric space is that for each $m \in M$ the local φ-geodesic symmetries are isometries.*

PROOF. The necessity follows at once from Proposition 3. To prove that the condition is sufficient just note that the hypothesis implies

$$(\nabla_U R)(X, Y, Z, W) = 0$$

for U, X, Y, Z, W orthogonal to ξ.

Next, following Okumura [11] we define on a Sasakian manifold M with structure tensors (φ, ξ, η, g) a linear connection $\bar{\nabla}$ by

$$\bar{\nabla}_X Y = \nabla_X Y + T_X Y ,$$

where

$$T_X Y = d\eta(X, Y)\xi - \eta(X)\varphi Y + \eta(Y)\varphi X .$$

The torsion tensor of $\bar{\nabla}$ is $2T$ and by direct computation using (3) and (4), we have

$$\bar{\nabla}\varphi = 0 , \quad \bar{\nabla}\xi = 0 , \quad \bar{\nabla}\eta = 0 ,$$
$$\bar{\nabla}S = 0 , \quad \bar{\nabla}\phi = 0 , \quad \bar{\nabla}T = 0 .$$

Also, an easy computation using the most elementary properties (e.g., (1) and (2)) of a contact metric structure shows that η, g, ϕ and T are all S-invariant. In turn, for M Sasakian, (3) yields that $\nabla\phi$, $\nabla^2\phi$ and $\nabla^3\phi$ are S-invariant.

Let $\bar{R}$ denote the curvature tensor of $\bar{\nabla}$. Then one of the main results of [15] is the following:

THEOREM 6. *A necessary and sufficient condition for a Sasakian manifold to be locally φ-symmetric is that $\bar{\nabla}\bar{R} = 0$, or equivalently,*

$$(7) \qquad (\nabla_V R)_{XY}Z = -T_V R_{XY}Z + R_{T_V XY}Z + R_{XT_V Y}Z + R_{XY}T_V Z ,$$

for all X, Y, Z, V.

In particular, from these conditions we see that a locally φ-symmetric space is *locally homogeneous* (see, e.g., [7], [19]). Moreover, since $T_X X = 0$, it follows also that in this case, all *local geodesic symmetries are volume-preserving* (cf. [19]). Finally, the same condition $T_X X = 0$ implies that a simply connected complete locally φ-symmetric space is a *naturally reductive homogeneous space* (cf. [17]).

4. **The main result.** We now turn to our study of the φ-geodesic symmetries s_m; in particular, we study the effect of the s_m's preserving the fundamental 2-form ϕ, i.e., $s_m^*\phi = \phi$.

THEOREM 7. *Let M be an almost contact metric manifold such that ξ is a Killing vector field and that the φ-geodesic symmetries s_m are ϕ-preserving for each $m \in M$. Then we have the following:*

(1) *If $d\eta = 0$, M is locally the product of a locally symmetric Kähler manifold and the real line.*

(2) *If $\phi = d\eta$, M is a locally φ-symmetric space and is locally homogeneous. If, moreover, M is complete and simply connected, it is a naturally reductive homogeneous space.*

PROOF. By Lemma 2, if $d\eta = 0$, M is locally the product of an almost Hermitian manifold N and $\boldsymbol{R}$ with ξ tangent to the factor $\boldsymbol{R}$. Thus the φ-geodesic symmetries s_m become geodesic symmetries on N and $s_m^*\phi = \phi$ implies that all the s_m preserve the fundamental 2-form of the almost Hermitian structure, i.e., are symplectic. The result of [12] is then that N is a locally symmetric Kähler manifold.

To prove (2) consider the fundamental 2-form ϕ. Let B_m be a geodesic ball about $m \in M$ and $r \mapsto \exp_m ru$, $\|u\| = 1$, a geodesic emanating from m in a direction u. Then the series expansion of $\phi_{ij} = \phi(\partial/\partial x^i, \partial/\partial x^j)$, $\{x^i,\ i = 1, \cdots, 2n+1\}$ being a system of normal coordinates, is (see, e.g., [3], [4], [6])

$$\begin{aligned}(8)\qquad \phi_{ij}(\exp_m ru) &= \phi_{ij}(m) + (\nabla_u\phi)_{ij}(m)r \\ &+ \Big((\nabla^2_{uu}\phi)_{ij} + \frac{1}{3}\sum_t R_{uiut}\phi_{tj} + \frac{1}{3}\sum_t R_{ujut}\phi_{it}\Big)(m)\frac{r^2}{2} \\ &+ \Big((\nabla^3_{uuu}\phi)_{ij} + \sum_t R_{uiut}(\nabla_u\phi)_{tj} + \sum_t R_{ujut}(\nabla_u\phi)_{it} \\ &+ \frac{1}{2}\sum_t (\nabla_u R)_{uiut}\phi_{tj} + \frac{1}{2}\sum_t (\nabla_u R)_{ujut}\phi_{it}\Big)(m)\frac{r^3}{6} + 0(r^4)\ .\end{aligned}$$

If now s_m is ϕ-preserving, we must have

$$(8')\qquad \phi_{ij}(\exp_m ru) = S_i^a(m)S_j^b(m)\phi_{ab}(\exp_m rS_m u)\ .$$

Now we compare the coefficients of both series expansions in (8′). We see from the second term that $\nabla\phi$ is S-invariant and we will show first that the contact metric structure is Sasakian. Since $\nabla_\xi\phi = 0$ for any contact metric structure,

$$\begin{aligned}(\nabla_X\phi)(Y, Z) &= (\nabla_{SX}\phi)(SY, SZ) \\ &= -(\nabla_X\phi)(Y, Z) + 2\eta(Y)(\nabla_X\phi)(\xi, Z) + 2\eta(Z)(\nabla_X\phi)(Y, \xi)\ .\end{aligned}$$

Hence,

$$(9)\qquad (\nabla_X\phi)(Y, Z) = \eta(Y)(\nabla_X\phi)(\xi, Z) + \eta(Z)(\nabla_X\phi)(Y, \xi)\ .$$

On the other hand, it is well-known that if ξ is a Killing vector field on a contact metric manifold, we have $\nabla_X\xi = -\varphi X$. Therefore

$$(\nabla_X\phi)(Y, \xi) = g(Y, (\nabla_X\varphi)\xi) = g(Y, \varphi^2 X) = -g(Y, X) + \eta(X)\eta(Y)\ ,$$

and hence, (9) becomes

$$(\nabla_X \phi)(Y, Z) = \eta(Y)g(X, Z) - \eta(Z)g(X, Y) .$$

Thus, from (3), we see that M is Sasakian.

Now, we have already noted that if M is Sasakian, $\nabla^2 \phi$ and $\nabla^3 \phi$ are S-invariant. Thus from the coefficient of r^2 in (8) we have that

$$R(U, X, U, \varphi Y) - R(U, Y, U, \varphi X) \tag{10}$$

is S-invariant. We will now show that R is S-invariant. Since $\bar{\nabla}\varphi = 0$ and $\bar{\nabla}T = 0$, we have

$$0 = \bar{R}_{XY} \cdot \varphi = R_{XY} \cdot \varphi + B_{XY} \cdot \varphi , \tag{11}$$

where

$$B_{XY} = [T_Y, T_X] - T_{T_Y X - T_X Y}$$

(cf. [17, p. 15]) and $R_{XY} \cdot \varphi$ and $B_{XY} \cdot \varphi$ indicate that R_{XY} and B_{XY} are acting as derivations. Clearly $B(X, Y, Z, W) = g(B_{XY}Z, W)$ is S-invariant, since g and T are. Thus from (11) one easily has

$$\begin{aligned} &R(X, Y, \varphi Z, W) + R(X, Y, Z, \varphi W) \\ &\quad = R(SX, SY, S\varphi Z, SW) + R(SX, SY, SZ, S\varphi W) . \end{aligned} \tag{12}$$

Now, let

$$D(X, Y, Z, W) = R(X, Y, Z, W) - R(SX, SY, SZ, SW) .$$

Then D satisfies the symmetry properties of the curvature tensor. Moreover, from (12), straightforward computation using (5) gives

$$D(X, Y, \varphi Z, \varphi W) = D(X, Y, Z, W) . \tag{13}$$

Furthermore, the S-invariance of (10) gives

$$D(X, \varphi X, X, \varphi X) = 0 .$$

Now, in particular for X, Y, Z, W orthogonal to ξ, we have as in the Kähler case (see, e.g., [8, p. 166]) that $D(X, Y, Z, W) = 0$. This, together with $D(X, Y, \xi, W) = 0$ from (13) for any X, Y, W, gives $D = 0$ and hence that R is S-invariant.

Turning now to the coefficient of r^3 in (8), we see that

$$(\nabla_U R)(U, X, U, \varphi Y) - (\nabla_U R)(U, Y, U, \varphi X)$$

is S-invariant. Therefore

$$(\nabla_U R)(U, \varphi U, U, \varphi U) - (\nabla_{SU} R)(SU, S\varphi U, SU, S\varphi U) = 0$$

and moreover, the same is true for $\bar{\nabla}$, since T and R are S-invariant. Thus setting

$$P(U, X, Y, Z, W) = (\bar{\nabla}_U R)(X, Y, Z, W) - (\bar{\nabla}_{SU} R)(SX, SY, SZ, SW)$$

we have

$$P(U, U, \varphi U, U, \varphi U) = 0 \,. \tag{14}$$

Also, since $\bar{\nabla} T = 0$, $\bar{\nabla}\bar{R} = \bar{\nabla} R$ (see, e.g., [17, p. 15].) This implies, since $\bar{\nabla}\varphi = 0$ and $\bar{\nabla}\xi = 0$, that

$$P(U, X, Y, \varphi Z, \varphi W) = P(U, X, Y, Z, W) \,. \tag{15}$$

Note that P also satisfies the second Bianchi identity. Now, in (14) write U as $\alpha Y + \beta Z$, α, β arbitrary and Y, Z orthogonal to ξ. Taking the coefficient of $\alpha\beta^4$, the curvature identities and (15) yield

$$P(Y, Z, \varphi Z, Z, \varphi Z) + 4P(Z, Y, \varphi Z, Z, \varphi Z) = 0 \,. \tag{16}$$

Using the second Bianchi identity on the second term we have

$$5P(Y, Z, \varphi Z, Z, \varphi Z) - 4P(\varphi Z, Z, Y, Z, \varphi Z) = 0 \,.$$

Replacing Z by φZ in this and comparing with (16), we have

$$P(Y, Z, \varphi Z, Z, \varphi Z) = 0 \tag{17}$$

for Y, Z orthogonal to ξ. Next, in (17) replace Z by $\alpha V + \beta Z$, with V and Z orthogonal to ξ, and consider the coefficient of $\alpha\beta^3$. Proceeding as before, we have

$$P(Y, V, \varphi Z, Z, \varphi Z) + 3P(Y, Z, \varphi V, Z, \varphi Z) = 0 \,.$$

Replacing Z by φZ and V by φV and comparing, we obtain

$$P(Y, V, \varphi Z, Z, \varphi Z) = 0 \,. \tag{18}$$

In (18) replace Z by $\alpha W + \beta Z$, with W and Z orthogonal to ξ, and consider the coefficient of $\alpha\beta^2$. Setting $V = W$ and using the first Bianchi identity, we have

$$3P(Y, W, \varphi Z, W, \varphi Z) + P(Y, W, Z, W, Z) = 0 \,,$$

from which by replacing Z by φZ we have

$$P(Y, U, Z, U, Z) = 0 \,.$$

This now implies that P restricted to vectors orthogonal to ξ vanishes (cf. [3], [5], [18]). However, if any of the vectors in (7) is equal to ξ, (7) is automatically satisfied on a Sasakian manifold as can be easily checked using (5) and (6). Thus $P \equiv 0$ and hence $\bar{\nabla} R = \bar{\nabla}\bar{R} = 0$, giving the result.

REMARK. First note that, since $\phi = d\eta$ on a Sasakian manifold, when the s_m preserve η, they also preserve ϕ. Secondly, suppose that the s_m preserve φ. Then, it follows from Proposition 4 and the orthogonality of S that $\alpha = 1$. Hence, the s_m preserve η. From this and Theorem 5 we obtain:

THEOREM 8. *Let M be a Sasakian manifold. Then M is a locally φ-symmetric space if and only if all the local φ-geodesic symmetries are*
(a) *ϕ-preserving, or*
(b) *φ-preserving, or*
(c) *η-preserving, or*
(d) *g-preserving.*

5. Three-dimensional manifolds. In this final section we consider three-dimensional Sasakian manifolds such that all the local φ-geodesic symmetries are assumed only to be volume-preserving. Therefore, let $m \in M$ and let θ_m denote the volume density function of the exponential map at m. (We always work in a geodesic ball B_m with center m and sufficiently small radius r.) Then the local φ-geodesic symmetry s_m is volume-preserving if and only if for any unit vector $u \in T_mM$ we have

$$\theta_m(\exp_m rS_m u) = \theta_m(\exp_m ru) . \tag{19}$$

Next, we state a result of [19].

PROPOSITION 9. *Let M be a three-dimensional connected Sasakian space with constant scalar curvature. Then M is a locally φ-symmetric space.*

Note that the converse is also true since a locally φ-symmetric space is locally homogeneous.

Now we prove:

THEOREM 10. *Let M be a three-dimensional connected Sasakian manifold such that all local φ-geodesic symmetries are volume-preserving. Then M is locally φ-symmetric.*

PROOF. According to Proposition 9 we have only to prove that the scalar curvature is constant on M. To do so we use the expansion for θ_m (see, e.g., [3], [4], [6]):

$$\theta_m(\exp_m ru) = 1 - \frac{r^2}{3}\rho(u, u)(m) - \frac{r^3}{6}(\nabla_u\rho)(u, u)(m) + 0(r^4) , \tag{20}$$

where ρ denotes the Ricci tensor.

Using (20), the criterion (19) implies

$$(\nabla_u\rho)(u, u) = (\nabla_{Su}\rho)(Su, Su)$$

for any $u \in T_m M$ and all $m \in M$. This is equivalent to

$$\mathfrak{S}_{X,Y,Z}\{(\nabla_X \rho)(Y, Z) - (\nabla_{SX}\rho)(SY, SZ)\} = 0 \tag{21}$$

where $\mathfrak{S}$ denotes the cyclic sum. Further, put $Y = Z = e_i$ at m, where $\{e_i, i = 1, \cdots, 2n+1\}$ is an arbitrary orthonormal basis of $T_m M$. Summing up with respect to i, we get for the scalar curvature τ

$$\nabla_{(I-S)X}\tau = 0$$

for any tangent vector field X. Using the expression for S, we obtain

$$\nabla_X \tau = \eta(X)\nabla_\xi \tau .$$

Now, since ξ is a Killing vector field, $\nabla_\xi \tau = 0$ and hence $\nabla_X \tau = 0$. So, we see that even for general n, τ is constant. For $n = 1$ the result follows at once.

Finally, the result of Theorem 10 shows that M is in fact a Sasakian space form (see [1], [20]). Note that, for arbitrary dimension, a Sasakian space form is always locally φ-symmetric. For a globally φ-symmetric space M (that is, a simply connected complete locally φ-symmetric space) we noted already that M is a naturally reductive homogeneous space. Using the explicit classification of naturally reductive homogeneous spaces in dimension 3 (cf. [9], [17]), we obtain finally:

THEOREM 11. *Let M be a three-dimensional connected simply connected complete Sasakian manifold. Then all the φ-geodesic symmetries are volume-preserving if and only if M is isometric to one of the following spaces*:

(a) *the unit sphere S^3 in $\boldsymbol{R}^4$*;

(b) *$SU(2)$, the universal covering space $SL(2, \boldsymbol{R})^\sim$ of $SL(2, \boldsymbol{R})$ or the Heisenberg group H, each with a special left invariant metric.*

The case (a) corresponds to the symmetric Sasakian manifold. $SU(2)$ corresponds to the case $c + 3 > 0$, $SL(2, \boldsymbol{R})^\sim$ to $c + 3 < 0$ and H to $c + 3 = 0$, where c denotes the φ-sectional curvature. See again [1], [20].

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